Advanced Algorithms – COMS31900

Probability recap.

Raphaël Clifford

Slides by Markus Jalsenius

Randomness and probability

Probability

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Amount of money you can win when playing some lottery:

$$S = \{\pounds 0, \pounds 10, \pounds 100, \pounds 1000, \pounds 10,000, \pounds 100,000\}.$$

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$$S = \{£0, £10, £100, £1000, £10,000, £100,000\}.$$

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$$\Pr(\pounds 0) = 0.9,\ \Pr(\pounds 10) = 0.08,\ \ldots,\ \Pr(\pounds 100,000) = 0.0001.$$

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For each $x \in S$, $\Pr(x) = \frac{1}{8}$

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$$\Pr(V) = \Pr(\text{HHH}) + \Pr(\text{HTH}) + \Pr(\text{THT}) + \Pr(\text{TTT}) = 4 \times \frac{1}{8} = \frac{1}{2}.$$

Random variable

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$$\mathbb{E}(Y) = \left(2 \cdot \tfrac{1}{2}\right) + \left(1 \cdot \tfrac{1}{4}\right) + \left(5 \cdot \tfrac{1}{4}\right) = \tfrac{7}{2}$$

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Theorem (Linearity of expectation)

Let $Y_1, Y_2, \ldots, Y_k$ be k random variables. Then

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$$\mathbb{E}(Y_1) = \mathbb{E}(Y_2) = 3.5$$

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THEOREM (Linearity of expectation)

Let $Y_1, Y_2, \ldots, Y_k$ be k random variables. Then

$$\mathbb{E}\left(\sum_{i=1}^{k} Y_i\right) = \sum_{i=1}^{k} \mathbb{E}(Y_i)$$

Linearity of expectation **always** holds,

(regardless of whether the random variables are independent or not.)

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so $\mathbb{E}(Y) = \mathbb{E}(Y_1 + Y_2) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) = 7$

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From the example above:

- $\Pr($speed of a random car ≥ 120 mph$) \leq \frac{60}{120} = \frac{1}{2}$,
- $\Pr($speed of a random car ≥ 90mph$) \leq \frac{60}{90} = \frac{2}{3}$.

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In fact, here it can be shown that as $n \to \infty$, the probability that at least one person leaves with their own hat is $1 - \frac{1}{e} \approx 0.632$.

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COROLLARY

If X is a non-negative r.v. that only takes integer values, then

$$\Pr(X > 0) = \Pr(X \geq 1) \leq \mathbb{E}(X).$$

For an indicator r.v. I, the bound is tight ($=$), as $\Pr(I > 0) = \mathbb{E}(I)$.

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Theorem (union bound)

Let $V_1, \ldots, V_k$ be k events. Then

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We define two events:
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$$V_2 = \{1, 2, 3\}$$

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Union bound

Theorem (union bound)

Let $V_1, \ldots, V_k$ be k events. Then

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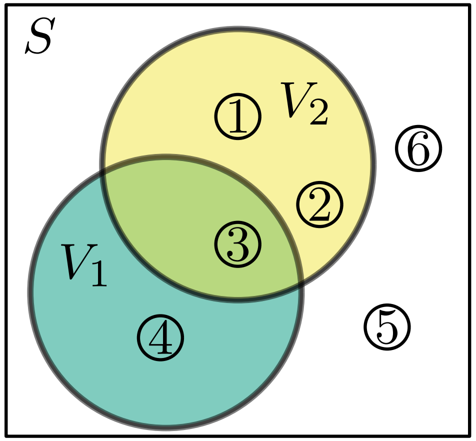

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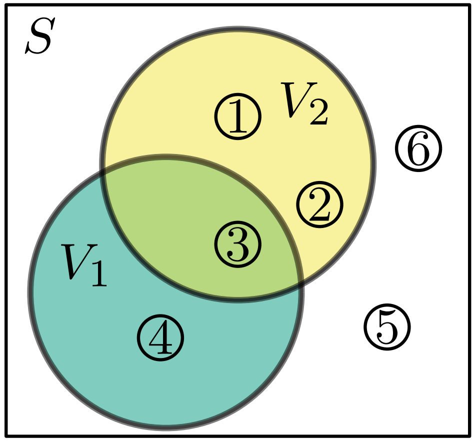

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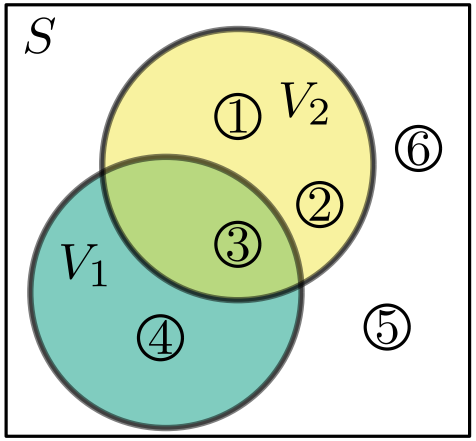

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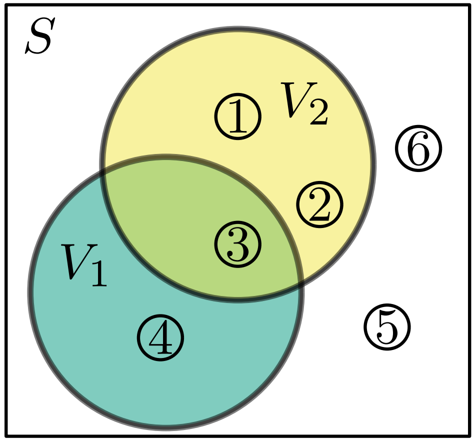

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Typically the union bound is used when each $\Pr(V_i)$ is *much* smaller than k.

Summary

The **sample space** S is the set of *outcomes* of an experiment.

For $x \in S$, the **probability** of x, written $\Pr(x)$,is a real number between 0 and 1,

such that $\sum_{x \in S} \Pr(x) = 1$.

An **event** is a subset V of the sample space S, $\Pr(V) = \sum_{x \in V} \Pr(x)$

A **random variable** (r.v.) Y is a function which maps $x \in S$ to $S(x) \in \mathbb{R}$

The probability of Y taking value y is $\Pr(Y = y) = \sum_{\{x \in S \text{ st. } Y(x) = y\}} \Pr(x)$.

The **expected value** (the mean) of Y is $\mathbb{E}(Y) = \sum_{x \in S} Y(x) \cdot \Pr(x)$.

An **indicator random variable** is a r.v. that can only be 0 or 1.

Fact: $\mathbb{E}(I) = \Pr(I = 1)$.

THEOREM (Linearity of expectation)

Let $Y_1, Y_2, \ldots, Y_k$ be k random variables then,

$$\mathbb{E}\left(\sum_{i=1}^{k} Y_i\right) = \sum_{i=1}^{k} \mathbb{E}(Y_i)$$

THEOREM (union bound)

Let $V_1, \ldots, V_k$ be k events then,

$$\Pr\left(\bigcup_{i=1}^{k} V_i\right) \leq \sum_{i=1}^{k} \Pr(V_i).$$

THEOREM (Markov's inequality)

If X is a non-negative r.v., then for all $a > 0$,

$$\Pr(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$