# Topics in TCS

## Sparse recovery

Raphaël Clifford

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Imagine that at some point in a stream, there is exactly one token with non-zero frequency. How can we recover it?

# 1-sparse recovery - simple case

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  3. We know the frequency of token $z/\ell$ is $\ell$.
- If we don't know if the stream is 1-sparse it is much harder and we will have to use randomisation.

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- Sketches should have the property that if two vectors are distinct then with high probability the sketches are distinct too. If two vectors are equal then our sketches will always be equal.
- We will use a polynomial sketch.

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- If there is exactly one non-zero frequency then $z/\ell$ is the identity of the token as before and $p = \ell r^{z/\ell}$ as all the other coefficients will be zero.
- Otherwise, we show that $p \neq \ell r^{z/\ell}$ with high probability.

# 1-sparse recovery algorithm

Define $\boldsymbol{e}_i$ to be an all zero vector except for a 1 at index $i$.

```
initialise (ℓ, z, p) = (0, 0, 0)
choose r to be a uniform random element of 𝔽_q

1-SPARSE(j, c)                                  # token, count
set ℓ = ℓ + c
set z = z + cj
set p = p + cr^j                                # fingerprint

OUTPUT
if ℓ = z = p = 0 return 0
     return f = 0
     else if z/ℓ ∉ [n]
          return ‖f‖_0 > 1
     else if p ≠ ℓr^(z/ℓ)
          return ‖f‖_0 > 1
     else
          return f = ℓe_(z/ℓ)
```

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- Output: $\ell \boldsymbol{e}_{z/\ell} = (0, 1)$

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- How likely are we to get false-positives?

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- Now we have that a false positive occurs only when $r$ is a root of the polynomial $q(x) - \ell x^i$.
- But $q(x) - \ell x^{z/\ell}$ has degree at most $n$ and hence at most $n$ roots and so

$$\Pr(r \text{ is a root of } q(x) - \ell x^{z/\ell}) \le \frac{n}{|\mathbb{F}_q|} \in O\left(\frac{1}{n^2}\right)$$

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# 1-sparse recovery - summary and time/space

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- Each $(j, c)$ pair is processed in constant time so the total running time is $O(m)$.

- If $M$ is the largest frequency of any item, the total space is $O(\log n + \log M)$ bits. This is because $|\ell| \leq nM$ and $|z| \leq n^2 M$ and $p, r \in \mathbb{F}_q$ with $q \leq 2n^3$.

# $s$-sparse recovery

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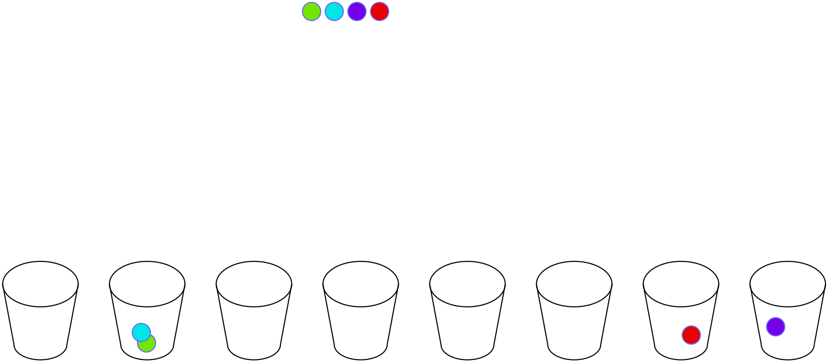
- We run our 1-sparse detection and recovery algorithm on each stream.
- We repeat the whole process to decrease the error probability.

# Throwing $s$ balls in $2s$ bins

What happens when you throw $s$ balls into $2s$ bins?

Attempt 1:



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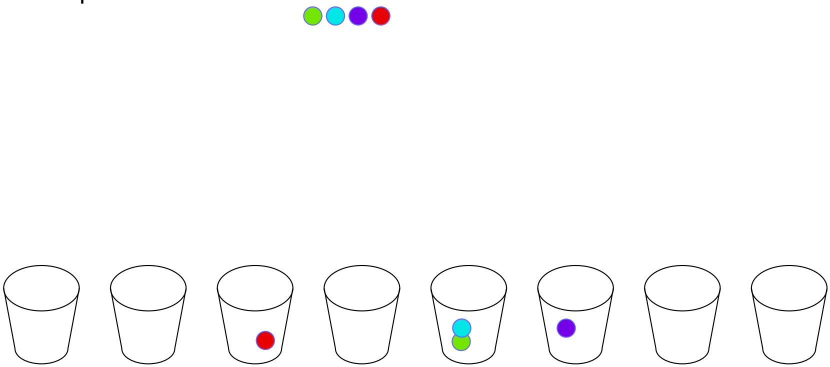
Attempt 2:



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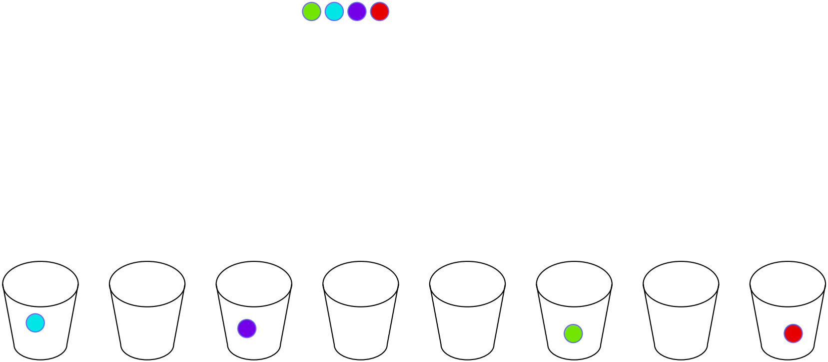

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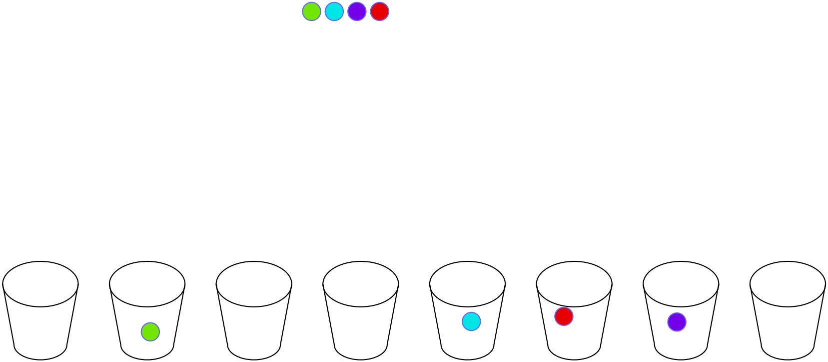

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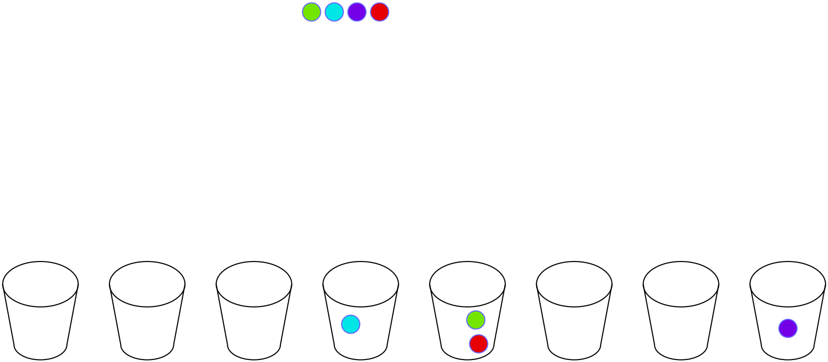
Attempt 5:



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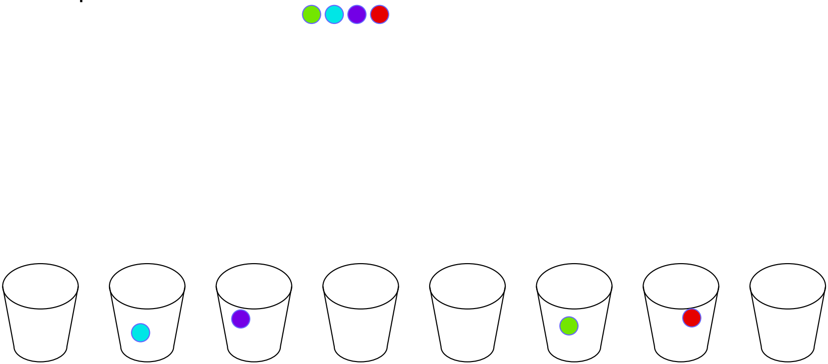

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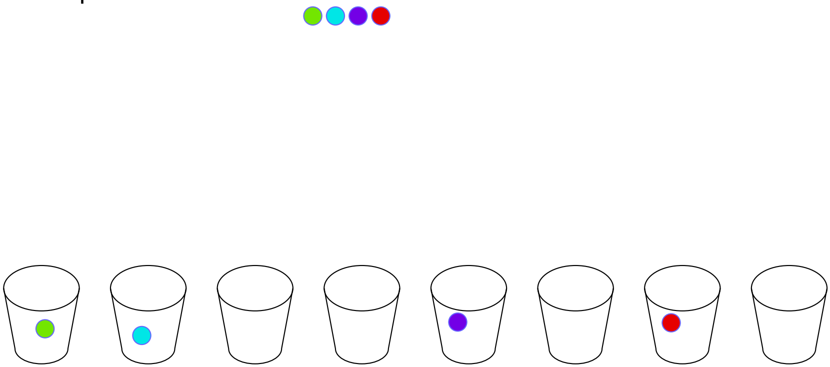

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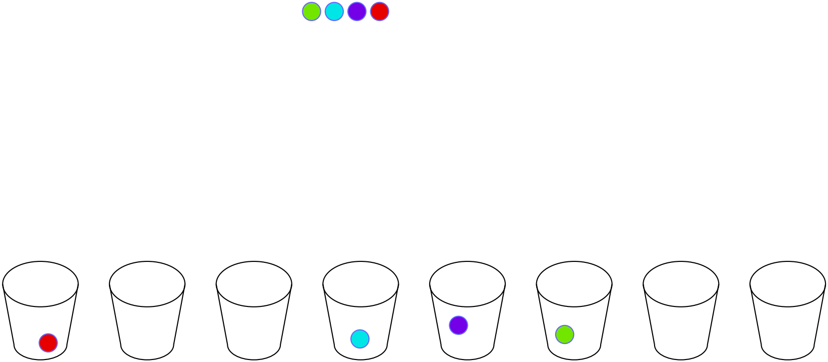

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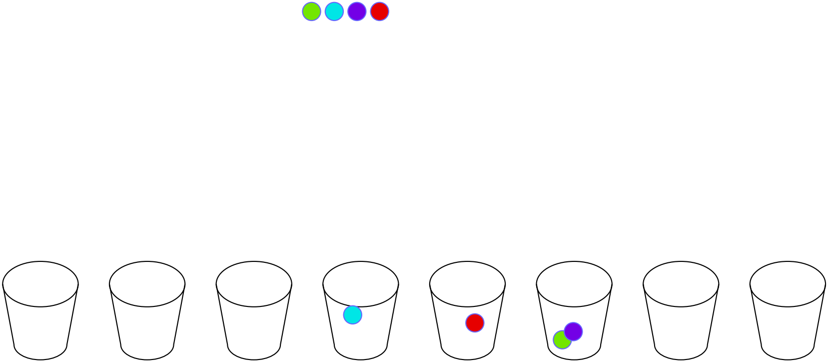

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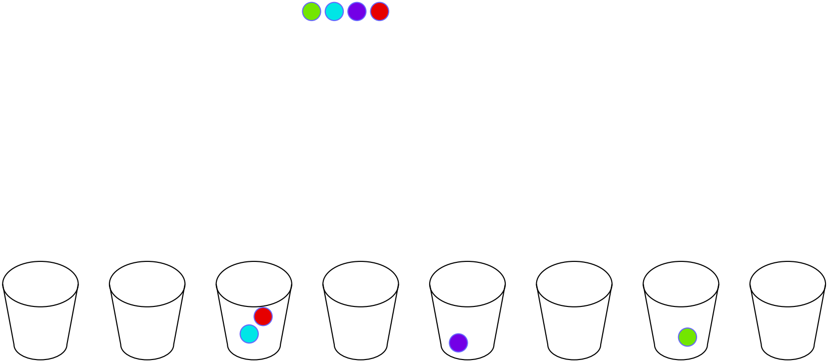

# $s$-sparse recovery

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initialise t = ⌈log(s/δ)⌉
initialise D[1...t][1...2s] = 0
choose t independent hash functions h_1,...,h_t : [n] → [2s]

s-SPARSE(j, c)                              # c can be negative
for each i ∈ [t]
     update D[i, h_i(j)] with token, count pair (j, c)
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- `update` runs one step of the 1-sparse recovery and detection algorithm with the new incoming token pair.

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- The $t$ rows of $D$ are for the $t$ independent hash functions.

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- Output all frequency/token pairs inferred from the stored information.

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Proof for SR$_1$. Consider a particular item $j \in \operatorname{supp} \boldsymbol{f}$. For each $i \in [t]$, let $\boldsymbol{\sigma}_i(j)$ be the substream generated by $h_i$ containing elements of the form $(j, c)$. Note that $f_j \neq 0$ necessarily for this stream.

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$\Pr(\text{SR}_1 \text{ fails for item } j) = \prod_{i=1}^{t} \Pr(\boldsymbol{\sigma}_i(j) \text{ is not 1-sparse}) \leq \left(\frac{1}{2}\right)^t \leq \frac{\delta}{s}$

# End of $s$-sparse proof

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- By another union bound over the entries of $D$,

$$\Pr(\mathrm{SR}_2 \text{ fails}) \leq 2st \cdot O\left(\frac{1}{n^2}\right) \in o(1),$$

because $st \leq n$. By yet another union bound the probability that the recovery fails is at most $\delta + o(1)$.

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- How can we check if the stream really was $s$-sparse?