Topics in TCS

Frequency estimation via sketching

Raphaël Clifford

Frequent items via sketching

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CountSketch

The sketch is a 2D-array C with t rows and k columns. All hash functions are chosen from a pairwise independent family.

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```
stream ⟨a_1, ..., a_m⟩, a_i ∈ [n]
initialise C[1...t][1...k] = 0
choose hash functions h_1, ... h_t : [n] → [k]
choose hash function g_1, ..., g_t : [n] → {−1, 1}

CountSketch(a_i)
for each j ∈ [t]
      C[j, h_j(a_i)] += c_i g_j(a_i)

return f̂_{a_i} = median{g_j(a_i) C[j, h_j(a_i)]}
```

c_i is the number of instances of a_i. In the turnstile model this can be either positive of negative.

CountSketch - worked example

	1	2	3
h_1			
h_2			

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
	$2, -$	$3, +$

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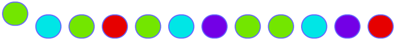

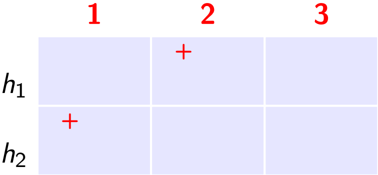

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
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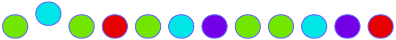

	1	2	3
h_1		+	−
h_2	+	+	

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
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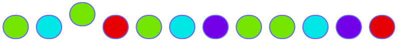

	1	2	3
h_1		++	−
h_2	++	+	

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
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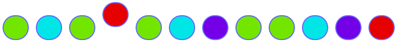

	1	2	3
h_1		$++-$	$-$
h_2	$++$	$+$	$+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
	$2, -$	$3, +$

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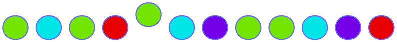

	1	2	3
h_1		$++-+$	$-$
h_2	$+++$	$+$	$+$

	h_1, g_1	h_2, g_2
(green)	$2, +$	$1, +$
(cyan)	$3, -$	$2, +$
(purple)	$1, +$	$3, -$
(red)	$2, -$	$3, +$

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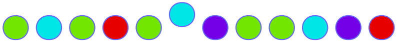

	1	2	3
h_1		$++-+$	$--$
h_2	$+++$	$++$	$+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
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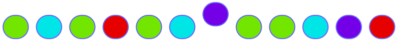

	1	2	3
h_1	$+$	$++-+$	$--$
h_2	$+++$	$++$	$+-$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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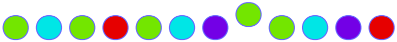

	1	2	3
h_1	$+$	$++-++$	$--$
h_2	$++++$	$++$	$+-$

	h_1, g_1	h_2, g_2
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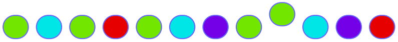

	1	2	3
h_1	$+$	$++-++$ $+$	$--$
h_2	$+++++$	$++$	$+-$

	h_1, g_1	h_2, g_2
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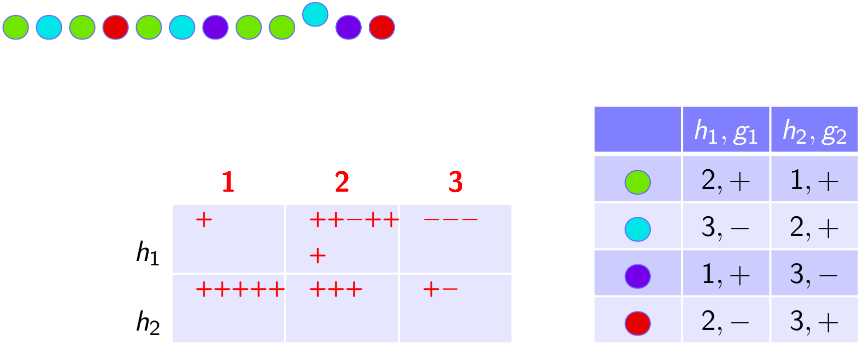

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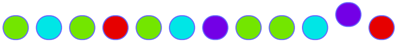

	1	2	3
h_1	$++$	$++-++$ $+$	$---$
h_2	$+++++$	$+++$	$+--$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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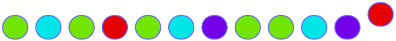

	1	2	3
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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	1	2	3
h_1	++	++−++ +−	−−−
h_2	+++++	+++	+−−+

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
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`return` $\hat{f}_{a_i} =$ `median`$\{g_j(a_i) C[j, h_j(a_i)]\}$

CountSketch - worked example

	1	2	3
h_1	++	++−++ +−	−−−
h_2	+++++	+++	+−−+

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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$\hat{f}$ $= \text{median}(g_1($$)C[1, h_1($$)], g_2($$)C[2, h_2($$)]) = \text{median}(1 \cdot 3, 1 \cdot 5)$

CountSketch - worked example

	1	2	3
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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	1	2	3
h_1	++	++−++ +−	−−−
h_2	+++++	+++	+−−+

	h_1, g_1	h_2, g_2
	2, +	1, +
	3, −	2, +
	1, +	3, −
	2, −	3, +

`return` $\hat{f}_{a_i} = \texttt{median}\{g_j(a_i)C[j, h_j(a_i)]\}$

$\hat{f}_{\text{green}} = \text{median}(g_1(\text{green})C[1, h_1(\text{green})], g_2(\text{green})C[2, h_2(\text{green})]) = \text{median}(1 \cdot 3, 1 \cdot 5)$

$\hat{f}_{\text{cyan}} = \text{median}(g_1(\text{cyan})C[1, h_1(\text{cyan})], g_2(\text{cyan})C[2, h_2(\text{cyan})]) = \text{median}(-1 \cdot -3, 1 \cdot 3)$

$\hat{f}_{\text{purple}} = \text{median}(g_1(\text{purple})C[1, h_1(\text{purple})], g_2(\text{purple})C[2, h_2(\text{purple})]) = \text{median}(1 \cdot 2, -1 \cdot 0)$

CountSketch - worked example

	1	**2**	**3**
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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	$1, +$	$3, -$
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To start, let us look just at an arbitrary row of C. We will show that for each row CountSketch gives an unbiased estimate. Define $C[x] = C[1, x]$.

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Therefore

$$X = g(a) \sum_{j=1}^{n} f_j g(j) Y_j = f_a + \sum_{j \in [n] \setminus \{a\}} f_j g(a) g(j) Y_j$$

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As g and h are independent and g is from a pairwise independent family,

$$\mathbb{E}[g(a)g(j)Y_j] = \mathbb{E}(g(a)) \cdot \mathbb{E}(g(j)) \cdot \mathbb{E}(Y_j) = 0 \cdot 0 \cdot \mathbb{E}(Y_j) = 0$$

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By linearity of expectation

$$\mathbb{E}(X) = f_a + \sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(a)g(j)Y_j] = f_a$$

COUNTSKETCH - Analysis IIa

We will now derive the variance of our estimator $X = \hat{f}$. Recall $Y_j = 1$ iff $h(j) = h(a)$.

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We will now derive the variance of our estimator $X = \hat{f}$. Recall $Y_j = 1$ iff $h(j) = h(a)$.

$$\mathrm{var}(X) = 0 + \mathrm{var}\left[g(a)\sum_{j\in[n]\setminus\{a\}} f_j \cdot g(j)Y_j\right]$$

$$= \mathbb{E}\left[g(a)^2 \sum_{j\in[n]\setminus\{a\}} f_j^2 Y_j^2 + \sum_{\substack{j\in[n]\setminus\{a\}\\ i\neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] -$$

$$\left[\sum_{j\in[n]\setminus\{a\}} f_j \mathbb{E}[g(a)g(j)Y_j]\right]^2$$

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$$= \mathbb{E}\left[g(a)^2 \sum_{j \in [n] \setminus \{a\}} f_j^2 Y_j^2 + \sum_{\substack{j \in [n] \setminus \{a\} \\ i \neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] -$$

$$\left[\sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(a) g(j) Y_j]\right]^2$$

We will need two facts to simplify these terms.

CountSketch - Analysis IIb

$$\mathrm{var}(X) = \mathbb{E}\left[g(a)^2 \sum_{j \in [n] \setminus \{a\}} f_j^2 Y_j^2 + \sum_{\substack{j \in [n] \setminus \{a\} \\ i \neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] - \left[\sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(a) g(j) Y_j]\right]^2$$

CountSketch - Analysis IIb

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$$= \frac{\|\boldsymbol{f}\|_2^2 - f_a^2}{k} \quad \text{where } \boldsymbol{f} \text{ is the array of frequencies}$$

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&\leq \frac{\mathrm{var}(X)}{\epsilon^2(\|\boldsymbol{f}\|_2^2 - f_a^2)} \\
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For an arbitrary token a, the probability of being further than $\epsilon \|\boldsymbol{f}_{-a}\|_2$ from the correct frequency is at most $\exp(-t/36)$.

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With $k = \lceil 3/\epsilon^2 \rceil$ and $t = \lceil \ln 1/\delta \rceil$, this equals

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Running time: one-pass and $O(t)$ time per token.

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CountSketch is a one-pass randomised algorithm to estimate the frequency of the tokens in a stream.

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Assuming we set $k = 3/\epsilon^2$, for an arbitrary token a, the probability that CountSketch's estimate is further than $\epsilon \|\boldsymbol{f}_{-a}\|_2$ from the correct frequency is at most $\exp(-t/36)$.

Count-Min sketch

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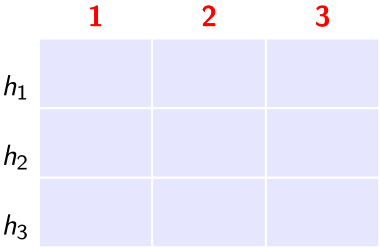
```
stream ⟨a_1, ..., a_m⟩, a_i ∈ [n]
initialise C[1...t][1...k] = 0
choose hash functions h_1, ... h_t : [n] → [k]

Count-Min(a_i)
for each j ∈ [t]
      C[j, h_j(a_i)] += c_i



return f̂_a = min_{1≤i≤t} C[i, h_i(a)]
```

c_i is the number of instances of a_i. In the turnstile model this can be either positive of negative.

Count-Min - worked example



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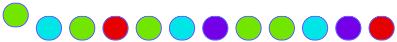

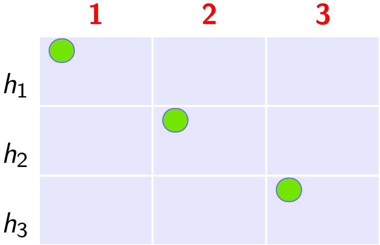

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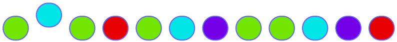

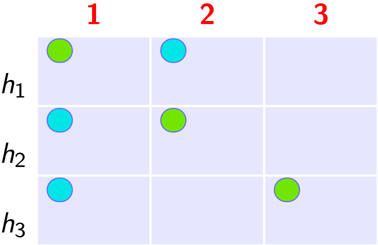

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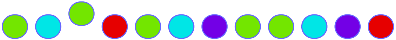

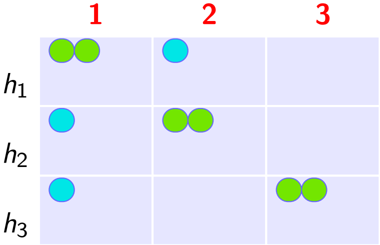

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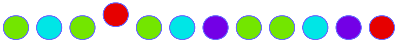

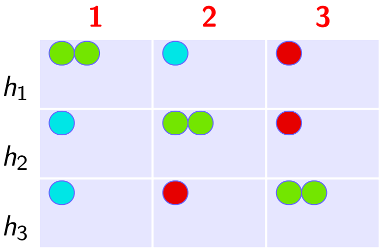

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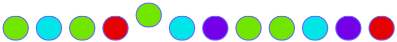

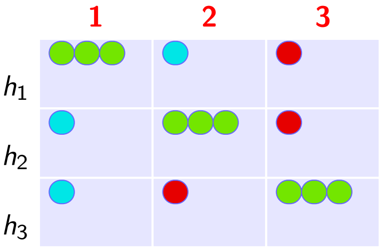

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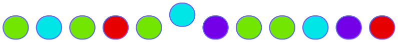

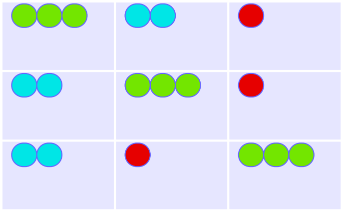

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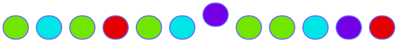

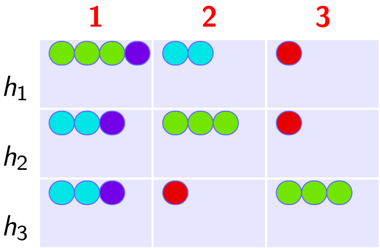

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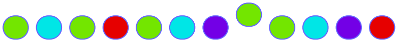

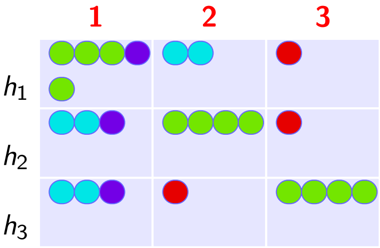

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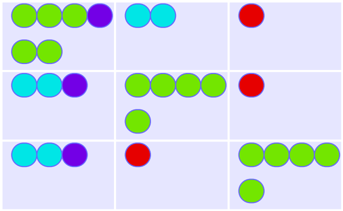

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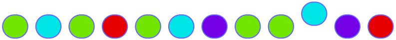

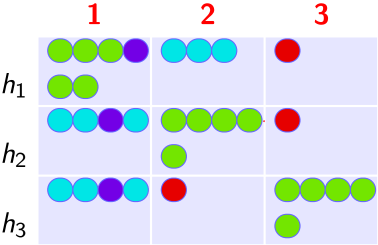

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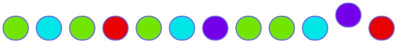

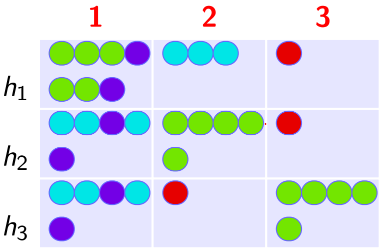

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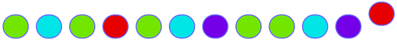

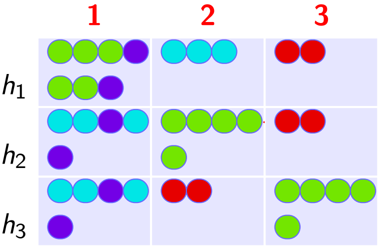

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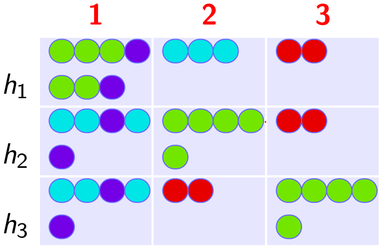

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$\hat{f}$ $= \min(C[1, h_1($$)], C[2, h_2($$)], C[3, h_3($$)]) = \min(7, 5, 5) = 5\checkmark$

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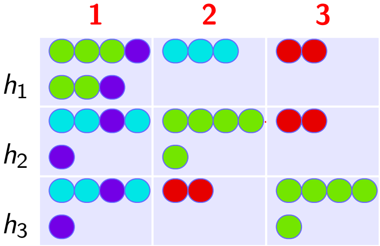

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$$\hat{f}_{\bullet} = \min(C[1, h_1(\bullet)], C[2, h_2(\bullet)], C[3, h_3(\bullet)]) = \min(7, 5, 5) = 5 \checkmark$$

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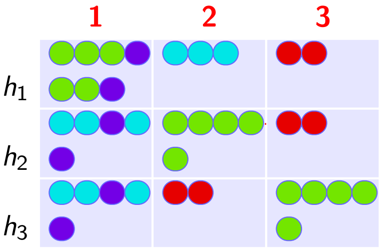
COUNT-MIN - worked example



	h_1	h_2	h_3
	1	2	3
	2	1	1
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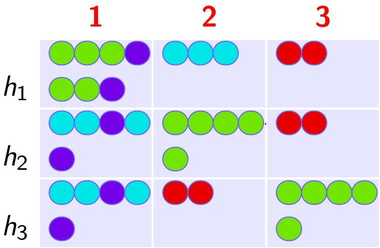

	h_1	h_2	h_3
$\color{green}{\bullet}$	1	2	3
$\color{cyan}{\bullet}$	2	1	1
$\color{violet}{\bullet}$	1	1	1
$\color{red}{\bullet}$	3	3	2

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$\hat{f}_{\color{cyan}{\bullet}} = \min(C[1, h_1(\color{cyan}{\bullet})], C[2, h_2(\color{cyan}{\bullet})], C[3, h_3(\color{cyan}{\bullet}]])) = \min(3,5,5) = 3\checkmark$

$\hat{f}_{\color{violet}{\bullet}} = \min(C[1, h_1(\color{violet}{\bullet})], C[2, h_2(\color{violet}{\bullet})], C[3, h_3(\color{violet}{\bullet}]])) = \min(7,5,5) = 5$

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By Markov's inequality

$$\Pr(X_i \geq \epsilon \|\boldsymbol{f}_{-a}\|_1) \leq \frac{\|\boldsymbol{f}_{-a}\|_1}{k\epsilon \|\boldsymbol{f}_{-a}\|_1} = \frac{1}{2} \qquad \text{set } k = 2/\epsilon$$

COUNT-MIN - Analysis II

We have a bound for a single counter. Over t counters the reported excess is the minimum over all X_i. We can now derive the probability that all the excesses are at least $\epsilon \left\| \boldsymbol{f}_{-a} \right\|_1$ directly.

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Misra-Gries gives a lower bound on frequency where Count-Min/CountSketch give upper bounds.

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MISRA-GRIES uses $O((1/\epsilon)(\log m + \log n)$ bits but does not work in the turnstile model (with deletions).