Reminder: $\log n$ denotes the binary logarithm, i.e., $\log n = \log_2 n$.

1  Proofs by Induction

Prove the following statements by induction:

1. For every integer $n \geq 0$, the following holds:
   \[
   \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
   \]

2. For every $n \geq 1$, the following holds:
   \[11^n - 6\] is divisible by 5.

3. Consider the following sequence: $s_1 = 1$, $s_2 = 2$, $s_3 = 3$, and $s_n = s_{n-1} + s_{n-2} + s_{n-3}$, for every $n \geq 4$. Prove that the following holds:
   \[s_n \leq 2^n.\]

2  Loop Invariant

Prove that the stated invariant holds throughout the execution of the loop (using the Initialization, Maintenance, Termination approach discussed in the lectures):

\begin{algorithm}
\caption{Algorithm $A$}
\begin{algorithmic}[1]
\Require Array $A$ of length $n$ ($n \geq 2$)
2: for $i \leftarrow 1 \ldots n-2$ do
3: \quad $S \leftarrow S + A[i] - A[i+1]$
4: end for
5: \Return $S$
\end{algorithmic}
\end{algorithm}

\textbf{Invariant:}

\begin{quote}
At the beginning of iteration $i$, $S = A[0] - A[i]$ holds.
\end{quote}

What does the algorithm compute?
3 Insertionsort

What is the runtime (in $\Theta$-notation) of Insertionsort when executed on the following arrays of lengths $n$:

1. $1, 2, 3, 4, \ldots, n-1, n$
2. $n, n-1, n-2, \ldots, 2, 1$
3. The array $A$ such that $A[i] = 1$ if $i \in \{1, 2, 4, 8, 16, \ldots\}$ (i.e., when $i$ is a power of two) and $A[i] = i$ otherwise.

4 Runtime Analysis

Algorithm 2

**Require:** Integer $n \geq 2$

$x \leftarrow 0$

$i \leftarrow n$

while $i \geq 2$ do

$j \leftarrow \lceil n^{1/4} \rceil \cdot i$

while $j \geq i$ do

$x \leftarrow x + 1$

$j \leftarrow j - 10$

end while

$i \leftarrow \lfloor i/\sqrt{n} \rfloor$

end while

return $x$

Determine the runtime of Algorithm 3 in $\Theta$-notation.