In-class Test
COMS10007 Algorithms 2019/2020

10.03.2020

Reminder: $\log n$ denotes the binary logarithm, i.e., $\log n = \log_2 n$. We also write $\log^c n$ as an abbreviation for $(\log n)^c$.

Make sure to put your name on every piece of paper that you hand in!

1 O-notation

1. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Define the set $O(f(n))$.
2. Give a formal proof of the statement:
$$2n^2 + 5n \in O(n^2) \ .$$
3. For each of the following statements, indicate whether it is true of false: (no justification needed)
   (a) $10n \in O(n^{\log n})$
   (b) $\log^2 n \in O(n^3)$
   (c) $\frac{1}{2} \log n \in O(\sqrt{\log n})$
   (d) $n! \in O(\sum_{i=0}^{n} 2^i)$
   (e) $2^{\sqrt{\log \log n}} \in O(\log^2 n)$
   (f) $f(n) \in \Theta(g(n))$ implies $g(n) \in \Omega(f(n))$
   (g) $f(n) \in O(g(n))$ implies $2^{f(n)} \in O(2^{g(n)})$

2 Sorting

1. What is the runtime of Insertionsort in Θ-notation (our aim is to sort the input in increasing order) on the following array of length n: (no justification needed)
$$A[i] = 1, \text{ if } 0 \leq i \leq \lfloor \tfrac{n}{2} \rfloor, \text{ and } A[i] = 0 \text{ otherwise}$$
2. Heapsort interprets an array of length n as a binary tree. What is the runtime of the Build-Heap() operation that transforms the initial binary tree into a heap? (no justification needed)
3. Suppose that Mergesort is executed on an array of length 2^k, for some integer k. What is the height of the corresponding recursion tree of this execution in Θ-notation? (no justification needed)

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3 Loop Invariant

Consider the following algorithm: (it operates on an array A of length n of positive integers)

Algorithm 1

Require: A is an array of n positive integers

1: $x \leftarrow \frac{A[0]}{A[1]}$
2: **for** $i \leftarrow 1, \ldots, n-2$ **do**
3: $\quad x \leftarrow x \cdot \frac{A[i]}{A[i+1]}$
4: **end for**
5: **return** x

Consider the following loop invariant:

> At the beginning of iteration i (i.e., after i is updated in Line 2 and before the code in Line 3 is executed) the following property holds:
>
> $$x = \frac{A[0]}{A[i]} \; .$$

1. *Initialization:* Argue that at the beginning of the first iteration, i.e. when $i = 1$, the loop invariant holds.

2. *Maintenance:* Suppose that the loop invariant holds at the beginning of iteration i. Argue that the loop invariant then also holds at the beginning of iteration $i + 1$.

3. *Termination:* What does the algorithm compute? Argue that this follows from the loop invariant.