Lecture 16: Dynamic Programming - Pole Cutting

COMS10007 - Algorithms

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Pole Cutting

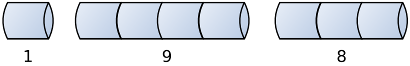
Pole-cutting:

- Given is a pole of length n

- The pole can be cut into multiple pieces of integral lengths
- A pole of length i is sold for price $p(i)$, for some function p

Example:

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

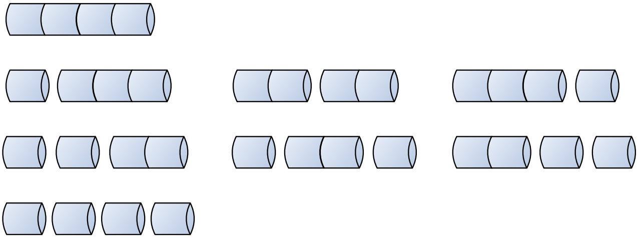


Pole Cutting (2)

Problem: POLE-CUTTING

1. **Input:** Price table p_i, for every $i \geq 1$, length n of initial pole
2. **Output:** Maximum revenue r_n obtainable by cutting pole into smaller pieces

How many ways of cutting the pole are there?



Pole Cutting (3)

$\boxed{\text{There are } 2^{n-1} \text{ ways to cut a pole of length } n.}$

Proof.

There are $n-1$ positions where the pole can be cut. For each position we either cut or we don't. This gives 2^{n-1} possibilities. □

Problem:

- Find best out of 2^{n-1} possibilities
- We could disregard similar cuts, but we would still have an exponential number of possibilities
- A fast algorithm cannot try out all possibilities

Pole Cutting (4)

Notation

$$7 = 2 + 2 + 3$$

means we cut a pole of length 7 into pieces of lengths $2, 2$ and 3

Optimal Cut

- Suppose the optimal cut uses k pieces

$$n = i_1 + i_2 + \cdots + i_k$$

- Optimal revenue r_n:

$$r_n = p(i_1) + p(i_2) + \cdots + p(i_k)$$

Pole Cutting (5)

What are the optimal revenues r_i?

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$
\begin{aligned}
r_1 &= 1 & 1 &= 1 \\
r_2 &= 5 & 2 &= 2 \\
r_3 &= 8 & 3 &= 3 \\
r_4 &= 10 & 4 &= 2+2 \\
r_5 &= 13 & 5 &= 2+3 \\
r_6 &= 17 & 6 &= 6 \\
r_7 &= 18 & 7 &= 2+2+3 \\
r_8 &= 22 & 8 &= 2+6 \\
r_9 &= 25 & 9 &= 3+6 \\
r_{10} &= 30 & 10 &= 10
\end{aligned}
$$

Optimal Substructure

Optimal Substructure

- Consider an optimal solution to input length n

$$n = i_1 + i_2 + \cdots + i_k \text{ for some } k$$

- Then:

$$n - i_1 = i_2 + \cdots + i_k$$

  is an optimal solution to the problem of size $n - i_1$

Computing Optimal Revenue r_n:

$$r_n = \max\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\}$$

- p_n corresponds to the situation of no cut at all
- $r_i + r_{n-i}$: initial cut into two pieces of sizes i and $n - i$

Pole Cutting: Dynamic Programming Formulation

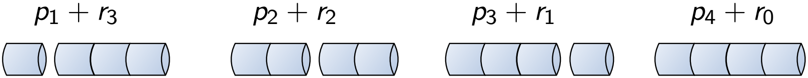
Simpler Recursive Formulation: Let $r_0 = 0$

$$r_n = \max_{1 \le i \le n} \left(p_i + r_{n-i} \right) .$$

Observe: Only one subproblem in this formulation

Example: $n = 4$

$$r_n = \max\{p_1 + r_3, p_2 + r_2, p_3 + r_1, p_4 + r_0\}$$



Recursive Top-down Implementation

Recall:

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \text{ and } r_0 = 0 \; .$$

Direct Implementation:

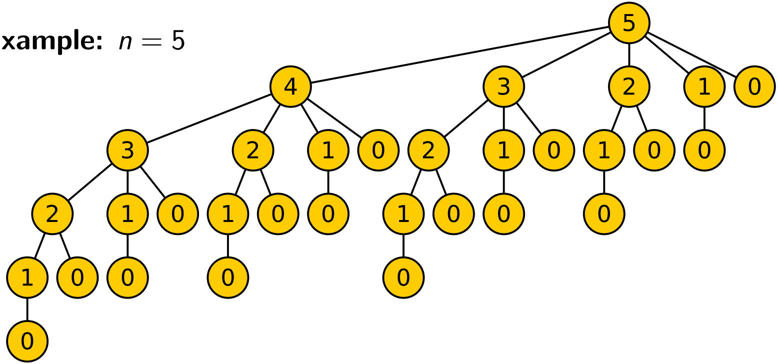
Require: Integer n, Array p of length n with prices
  if $n = 0$ **then**
    return 0
  $q \leftarrow -\infty$
  for $i = 1 \dots n$ **do**
    $q \leftarrow \max\{q, p[i] + \textsc{Cut-Pole}(p, n-i)\}$
  return q

Algorithm $\textsc{Cut-Pole}(p, n)$

How efficient is this algorithm?

Recursion Tree for CUT-POLE

Example: $n = 5$



Number Recursive Calls: $T(n)$

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) \text{ and } T(0) = 1$$

Solving Recurrence

How to Solve this Recurrence?

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) \text{ and } T(0) = 1$$

- Substitution Method: Using guess $T(n) = O(c^n)$, for some c
- **Trick:** compute $T(n) - T(n-1)$

$$\begin{aligned} T(n) - T(n-1) &= 1 + \sum_{j=0}^{n-1} T(j) - \left(1 + \sum_{j=0}^{n-2} T(j)\right) \\ &= T(n-1) \text{ , hence:} \\ T(n) &= 2T(n-1) \text{ .} \end{aligned}$$

This implies $T(i) = 2^i$.

Discussion

Runtime of Cut-Pole

- Recursion tree has 2^n nodes
- Each function call takes time $O(n)$ (for-loop)
- Runtime of CUT-POLE is therefore $O(n2^n)$. ($O(2^n)$ can also be argued)

What can we do better?

- Observe: We compute solutions to subproblems many times
- Avoid this by storing solutions to subproblems in a table!
- This is a key feature of dynamic programming

Implementing the Dynamic Programming Approach

Top-down with memoization

- When computing r_i, store r_i in a table T (of size n)
- Before computing r_i again, check in T whether r_i has previously been computed

Bottom-up

- Fill table T from smallest to largest index
- No recursive calls are needed for this

Top-down Approach

Require: Integer n, Array p of length n with prices
  Let $r[0 \dots n]$ be a new array
  for $i = 0 \dots n$ **do**
    $r[i] \leftarrow -\infty$
  return $\textsc{Memoized-Cut-Pole-Aux}(p, n, r)$

Algorithm $\textsc{Memoized-Cut-Pole}(p,\ n)$

- Prepare a table r of size n
- Initialize all elements of r with $-\infty$
- Actual work is done in $\textsc{Memoized-Cut-Pole-Aux}$, table r is passed on to $\textsc{Memoized-Cut-Pole-Aux}$

Top-down Approach (2)

Require: Integer n, array p of length n with prices, array r of revenues
  if $r[n] \geq 0$ **then**
    return $r[n]$
  if $n = 0$ **then**
    $q \leftarrow 0$
  else
    $q \leftarrow -\infty$
    for $i = 1 \ldots n$ **do**
      $q \leftarrow \max\{q, p[i] + \textsc{Memoized-Cut-Pole-Aux}(p, n - i, r)\}$
  $r[n] \leftarrow q$
  return q

Algorithm $\textsc{Memoized-Cut-Pole-Aux}(p,\ n,\ r)$

Observe: If $r[n] \geq 0$ then $r[n]$ has been computed previously

Bottom-up Approach

Require: Integer n, array p of length n with prices
  Let $r[0 \dots n]$ be a new array
  $r[0] \leftarrow 0$
  for $j = 1 \dots n$ **do**
    $q \leftarrow -\infty$
    for $i = 1 \dots j$ **do**
      $q \leftarrow \max\{q, p[i] + r[j-i]\}$
    $r[j] \leftarrow q$
  return $r[n]$

Algorithm $\textsc{Bottom-Up-Cut-Pole}(p, n)$

Runtime: Two nested for-loops

$$\sum_{j=1}^{n} \sum_{i=1}^{j} O(1) = O(1) \sum_{j=1}^{n} \sum_{i=1}^{j} 1 = O(1) \sum_{j=1}^{n} j = O(1) \frac{n(n+1)}{2} = O(n^2) \ .$$

Conclusion

Runtime of Top-down Approach $O(n^2)$

(please think about this!)

Dynamic Programming

- Solves a problem by combining subproblems
- Subproblems are solved at most once, store solutions in table
- If a problem exhibits *optimal substructure* then dynamic programming is often the right choice
- Top-down and bottom-up approaches have the same runtime