Solving Recurrences II
COMS10007 2020, Lecture 14

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(substituting for Dr. Christian Konrad)

March 18th 2020

The recursion tree method

This lecture we'll ignore O-notation and divisibility issues, and focus on examples of the recurrences themselves. Consider the recurrence

$$T(1) = 1, \qquad T(n) = 3T(n/4) + n/2, \qquad n \text{ is a power of } 4.$$

Step 1: Use the recursion tree method to get a good guess at a solution.

As with the mergesort analysis, we view this as a tree.

Each recursive invocation corresponds to a child node — so the root has three children, each of which has three children, and so on.

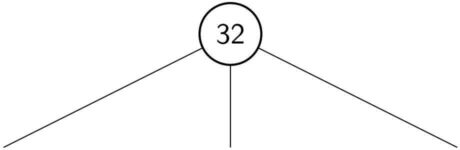
Each node gets labelled with the non-recursive running time at that step, and then $T(n)$ is the sum of all the labels in the tree.

A concrete example

$$T(1) = 1, \qquad T(n) = 3T(n/4) + n/2, \qquad n \text{ is a power of } 4.$$

$n = 64$

$n = 16$



$$T(64) = 3T(16) + 32$$

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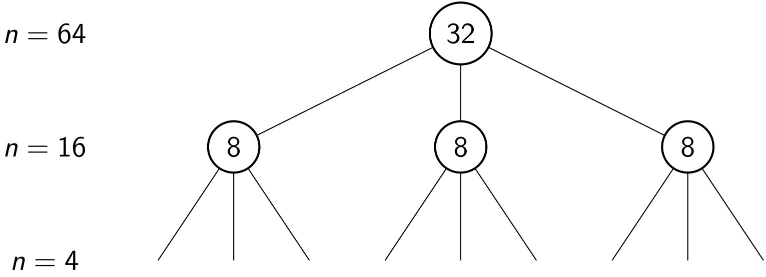

$$T(64) = 3T(16) + 32 = \mathbf{9T(4)} + \mathbf{3 \cdot 8} + 32$$

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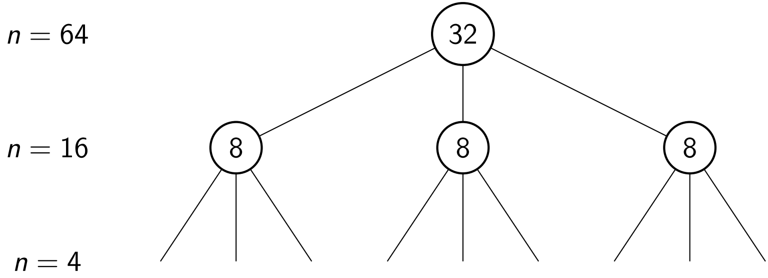

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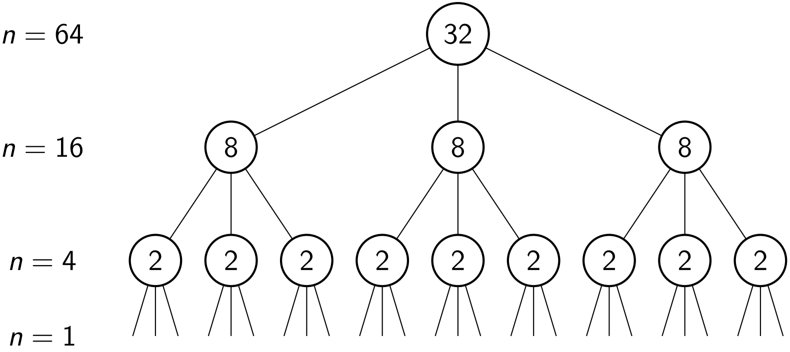

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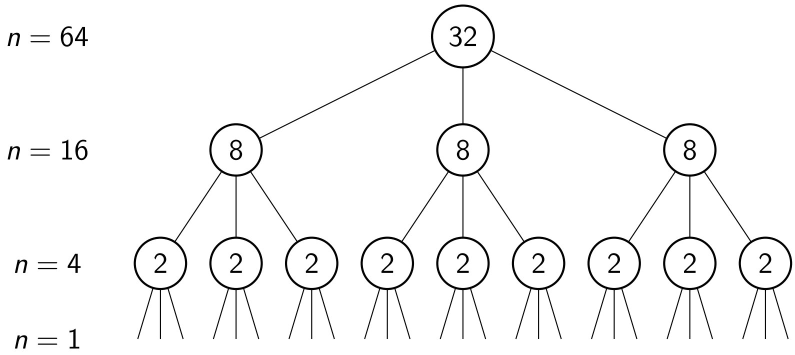

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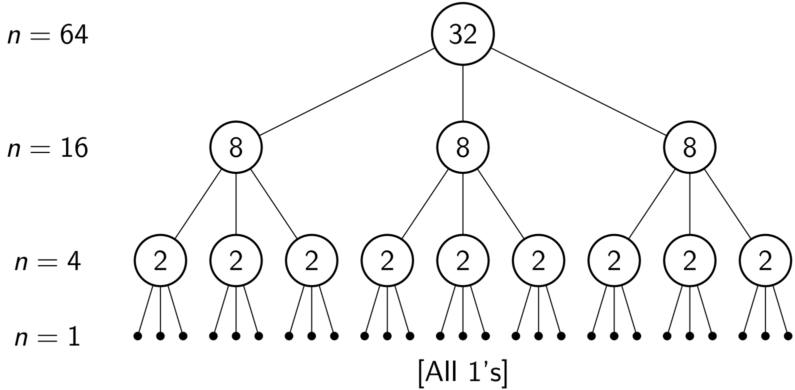

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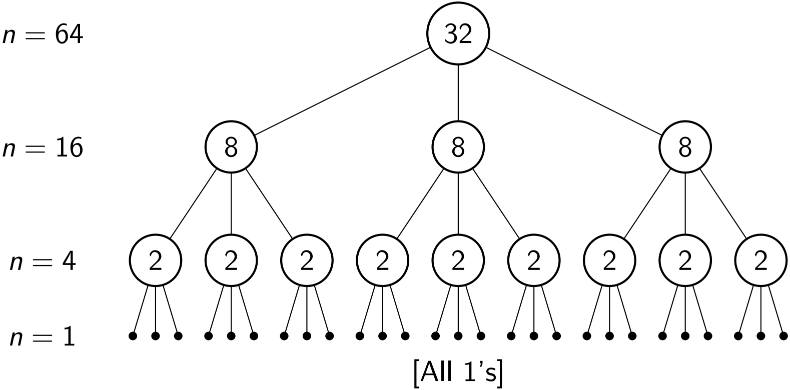

$$T(64) = 27 \cdot 1 + 9 \cdot 2 + 3 \cdot 8 + 32 = 101.$$

The general case

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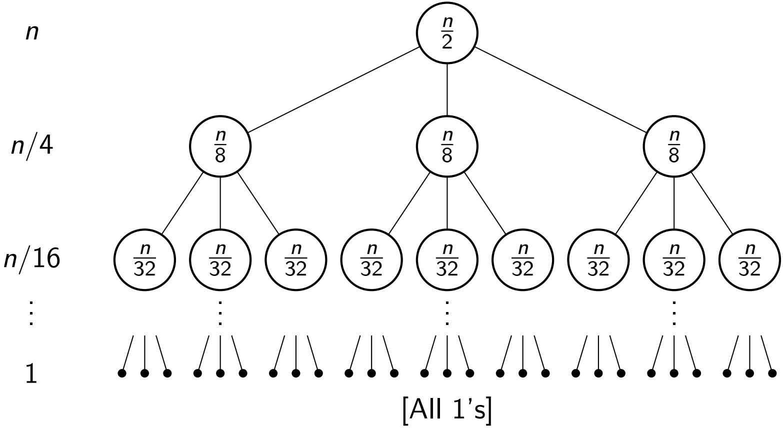

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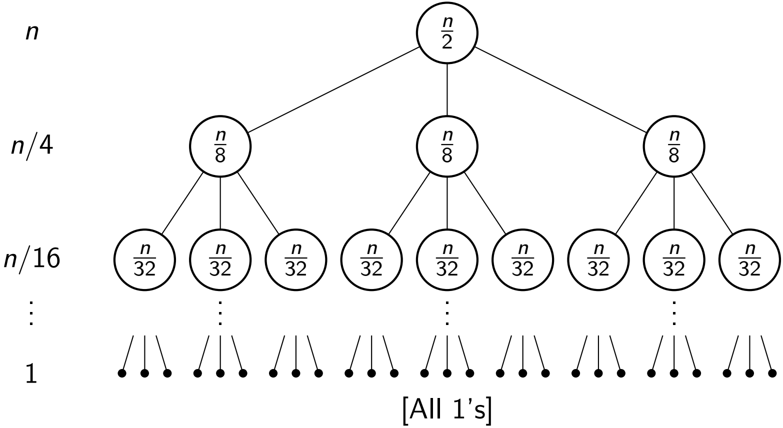

Except for the bottom, level i has 3^{i-1} nodes, each with cost $n/(2 \cdot 4^{i-1})$.
Let $\#(\text{levels}) = t$. Then $n/4^{t-1} = 1$, so we have $t = 1 + \log_4 n$.

$$T(1) = 1, \qquad T(n) = 3T(n/4) + n/2, \qquad n \text{ is a power of } 4.$$

There are $1 + \log_4 n$ levels in total.
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$T(n)$ is the total time cost over the whole tree, so

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So overall, we expect $T(n) = O(n)$. In other words, **the root dominates**.

Formal proof via substitution

$$T(1) = 1, \qquad T(n) = 3T(n/4) + n/2, \qquad n \text{ is a power of } 4.$$

Guess: $T(n) \leq Cn$ for all $n \geq 1$ (C to be determined).

Now that we have a good guess, proving it formally is a standard induction.

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We have proved $T(n) \le 2n$ for all $n \ge 1$, and hence $T(n) = O(n)$. □

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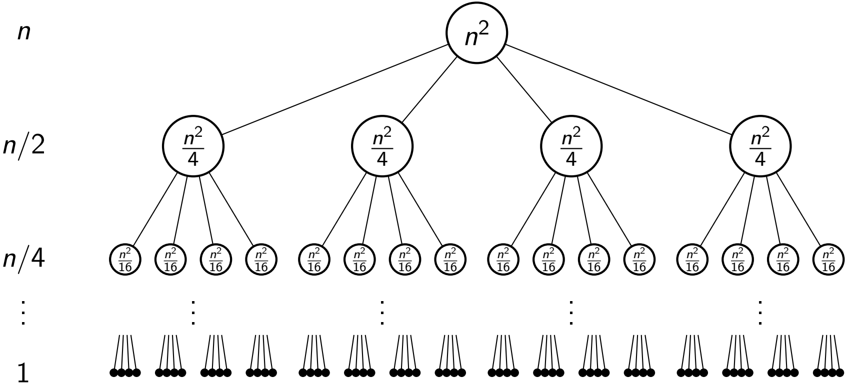

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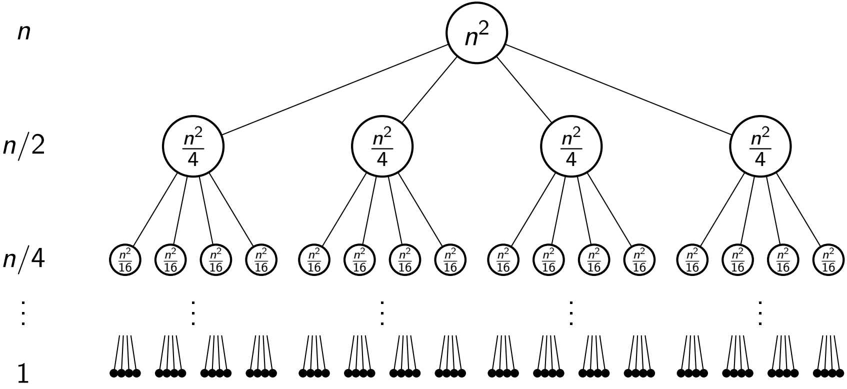

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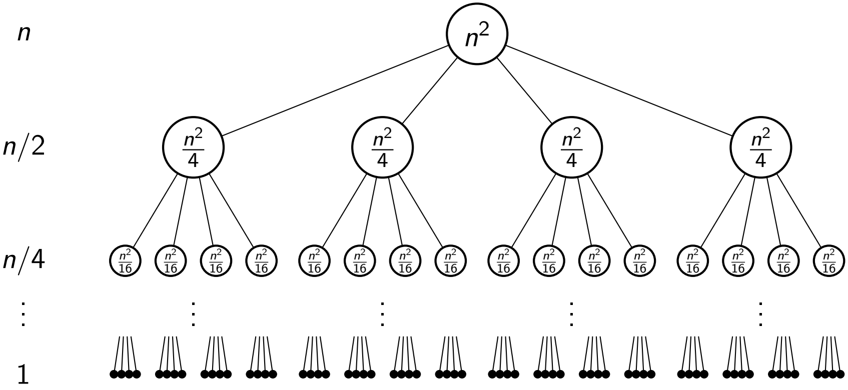

Except for the bottom, level i has 4^{i-1} nodes, each with cost $(n/2^{i-1})^2$. Let $\#(\text{levels}) = t$. Then $n/2^{t-1} = 1$, so we have $t = 1 + \log n$.

Analysing the tree

$$T(1) = 1, \qquad T(n) = 4T(n/2) + n^2, \qquad n \text{ is a power of 2.}$$

There are $1 + \log n$ levels in total.
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$$T(n) = \sum_{i=1}^{\log n} 4^{i-1} \cdot \left(\frac{n}{2^{i-1}}\right)^2 + n^2$$

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In other words, **every level costs the same**.

Formal proof by substitution

$$T(1) = 1, \qquad T(n) = 4T(n/2) + n^2, \qquad n \text{ is a power of } 2.$$

Guess: $T(n) \le Cn^2 \log n$ for all $n \ge \mathbf{2}$ (C to be determined).

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We have proved $T(n) \leq 2n^2 \log n$ for all $n \geq 2$, so $T(n) = O(n^2 \log n)$. □

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Now consider the recurrence

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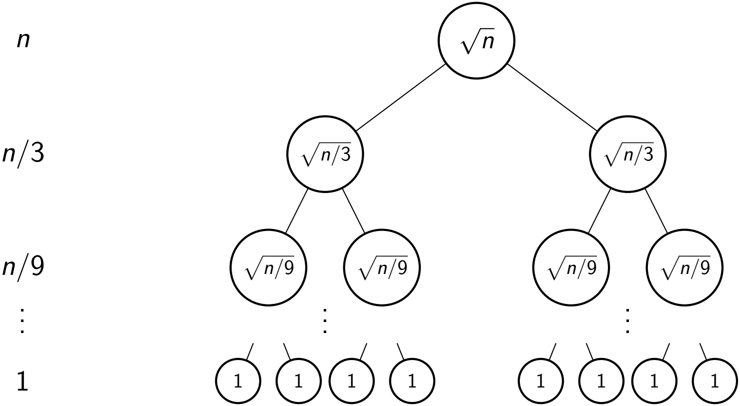

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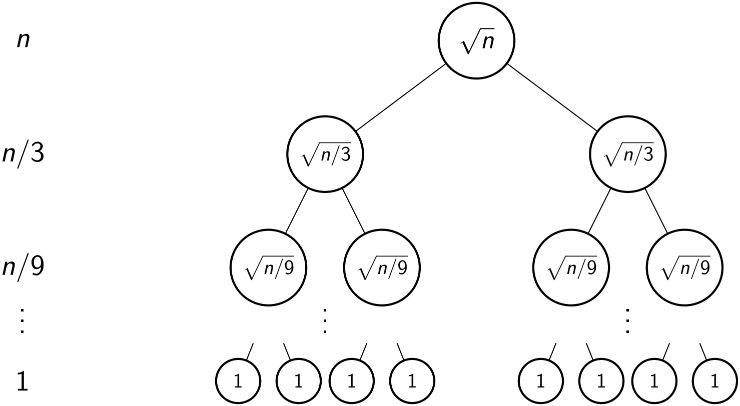

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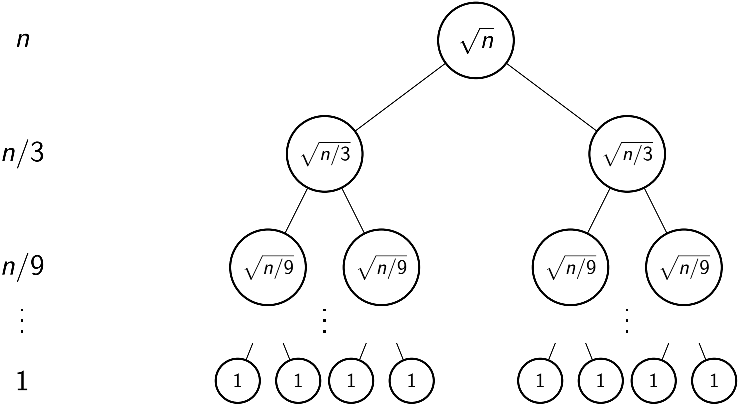

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So $T(n) = O(2^{\log_3 n}) = O(n^{1/\log(3)})$, and **the leaves dominate**.

Formal proof by substitution

$$T(1) = 1, \qquad T(n) = 2T(n/3) + \sqrt{n}, \qquad n \text{ is a power of 3.}$$

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We have $3^{1/\log(3)} = 2^{\log(3)/\log(3)} = 2$, so $T(n) \leq Cn^{1/\log(3)} + \sqrt{n}$.

Formal proof by substitution

$$T(1) = 1, \qquad T(n) = 2T(n/3) + \sqrt{n}, \qquad n \text{ is a power of } 3.$$

Guess: $T(n) \leq Cn^{1/\log(3)}$ for all $n \geq 1$ (C to be determined).

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which isn't quite good enough.

But we know how to deal with this: add a correction term!

Formal proof by substitution (attempt 2)

$$T(1) = 1, \qquad T(n) = 2T(n/3) + \sqrt{n}, \qquad n \text{ is a power of 3.}$$

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We have proved $T(n) \le 2n^{1/\log(3)} - \sqrt{n}$ for all n, so $T(n) = O(n^{1/\log(3)})$. □

Recap

These examples fell into three categories:

1. For $T(n) = 2T(n/3) + \sqrt{n}$, the leaves dominated.
2. For $T(n) = 4T(n/2) + n^2$, the levels were all equal.
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Wouldn't it be nice if we could put **any** recurrence relation of the form $T(n) = aT(n/b) + f(n)$ into one of those three categories?

Then we could just write the answer down without having to solve it...

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Actually, we can!

The Master Theorem (non-examinable!)

The Master Theorem: Suppose $T(1) = O(1)$ and, for $n > 1$, $T(n) = aT(n/b) + f(n)$ for some constants $a, b > 0$ and some function $f : \mathbb{N} \to \mathbb{R}$. Let $\xi = \log_b a$ be the **critical exponent**. Then:

1. If $f(n) = O(n^{\xi-\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^\xi)$.
   In other words, **the leaves dominate**.
2. If $f(n) = \Theta(n^\xi)$, then $T(n) = \Theta(n^\xi \log n)$.
   In other words, **the levels are roughly equal**.
3. If $f(n) = \Omega(n^{\xi+\varepsilon})$ for some $\varepsilon > 0$ **and** $af(n/b) = O(f(n))$, then $T(n) = \Theta(f(n))$. In other words, **the root dominates**.

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3. If $f(n) = \Omega(n^{\xi+\varepsilon})$ for some $\varepsilon > 0$ **and** $af(n/b) = O(f(n))$, then $T(n) = \Theta(f(n))$. In other words, **the root dominates**.

The condition $af(n/b) = O(f(n))$ in the last case always holds when f is a polynomial in n, but rules out weird cases like

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n^2 & \text{if } n \text{ is even.} \end{cases}$$

Next time:
Christian's triumphant return!*

*Return may not be triumphant or even physical, rules and restrictions apply.