Solving Recurrences I
COMS10007 2020, Lecture 13

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(substituting for Dr. Christian Konrad)

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Divide-and-conquer algorithms

Many algorithms in this course (and in general!) follow the **divide-and-conquer** approach:

1. **Divide** the problem into smaller instances of the same problem.
2. **Conquer** the subproblems by solving them, either recursively or directly.
3. **Combine** the solutions to the subproblems into a solution for the original problem.

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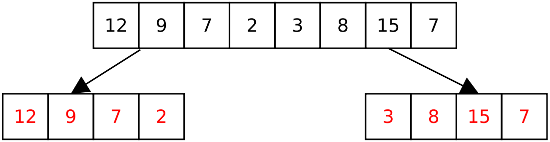
For example:

- Mergesort.
- Quicksort.
- The maximum subarray algorithm.
- Binary search.
- FAST-PEAK-FINDING.

Example: Merge sort

Recall: Merge Sort

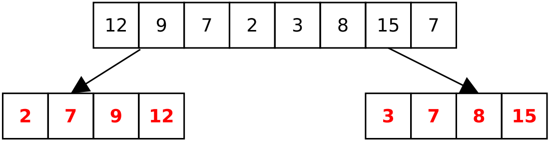
1. **Divide**
   Split input array A of length n into subarrays $A_1 = A[0, \lfloor n/2 \rfloor]$ and $A_2 = A[\lfloor n/2 \rfloor + 1, n-1]$



Example: Merge sort

Recall: Merge Sort

1. **Divide** $A \to A_1$ and A_2
2. **Conquer**
   Sort A_1 and A_2 recursively using the same algorithm



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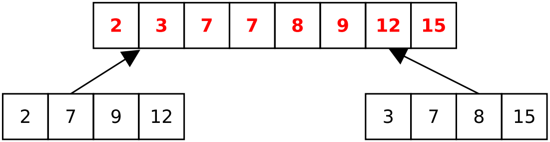
   Combine sorted subarrays A_1 and A_2 and obtain sorted array A



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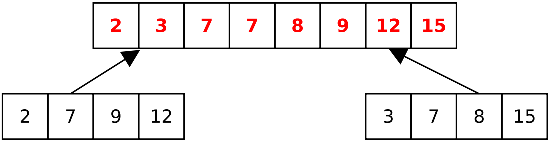

Runtime: (assuming that n is a power of 2)

$$
\begin{aligned}
T(1) &= O(1) \\
T(n) &= 2T(n/2) + O(n)
\end{aligned}
$$

How to solve recurrences?

Recurrences

- Divide-and-conquer algorithms naturally lead to recurrences (or "recurrence relations") like that one.
- How can we solve them? Or at least get a decent upper bound?

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Methods for solving recurrences

- Recursion-tree method (as used for mergesort and max subarray).
  Often has too many awkward details (e.g. floors and ceilings, pivots), but great for getting intuition.
- Substitution method (this lecture).
  Very powerful, but needs a reasonable initial guess.
- The "Master Theorem".
  Only applies to recurrences of the form $T(n) = aT(n/b) + f(n)$, but makes things trivial when it does apply. Not covered in this course.

Generally: use recursion-tree to get a guess for substitution!

The substitution method

1. Remove the O-notation from the recurrence.
2. Guess a partial form of the solution (with some unknown constants).
3. Use mathematical induction to show the solution works for the right choice of constants.

Dealing with O-notation can introduce some added complications...

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$$\begin{aligned} T(1) &= O(1), & \longrightarrow\quad & T(n) \leq c_1 & \text{for all } n \leq n_0,\\ T(n) &= 2T(n/2) + O(n). & & T(n) \leq 2T(n/2) + c_2 n & \text{for all } n > n_0. \end{aligned}$$

Step 1: Replace the O-notation by constants. Remember, $f(n) \in O(g(n))$ means that there exist C and n_0 such that for all $n \geq n_0$, $f(n) \leq Cg(n)$.

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For mergesort specifically, we can take $n_0 = 1$.

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$$T(n) \leq 2T(n/2) + c_2 n \text{ for all } n > 1.$$

Step 2: Guess a bound. Here, guess $T(n) \leq Cn \log n$ for some $C > 0$.

Step 3: Prove it works by induction.

Base case $n = 1$: $T(1) \leq c_1$, and $C \cdot 1 \log(1) = 0 > c_1$.

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But it's fine! We're only trying to prove $T(n) = O(n \log n)$, which means we need $T(n) \le Cn \log n$ for all $n \ge n_0$ (for some C, n_0 of our choice).

We **don't** need $T(1) \le C \cdot 1 \log 1$. We can just take $n_0 = 2$.

Key point: Since we're only going for asymptotic results, not exact results, we can choose any base case we want.

$$T(1) \le c_1,$$
$$T(n) \le 2T(n/2) + c_2 n \text{ for all } n > 1.$$

Step 3: Prove by induction that $T(n) \le Cn \log n$ **for all $n \ge 2$**.

Note that we haven't fixed a value for C yet — we'll see what values work over the course of the proof.

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Base case $n = 2$: We have

$$T(2) \le 2T(1) + c_2 \cdot 2 \le 2(c_1 + c_2),$$

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T(2) \le 2T(1) + c_2 \cdot 2 &\le 2(c_1 + c_2), \\
C \cdot 2 \log 2 &= 2C.
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So $T(2) \le C \cdot 2 \log 2$ as long as we choose $C \ge c_1 + c_2$. ✓

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By the induction hypothesis,

$$T(n) \le 2T(n/2) + c_2 n$$

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$$T(n) \le 2T(n/2) + c_2 n \le 2C \cdot \frac{n}{2} \log(n/2) + c_2 n$$

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$$\begin{aligned} T(n) \le 2T(n/2) + c_2 n &\le 2C \cdot \frac{n}{2} \log(n/2) + c_2 n \\ &= Cn(\log(n) - 1) + c_2 n \end{aligned}$$

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$$= Cn(\log(n) - 1) + c_2 n = Cn\log(n) + (c_2 - C)n.$$

This is at most $Cn\log n$ as long as we choose $\boldsymbol{C \ge c_2}$. ✓

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So we have proved $T(n) \leq (c_1 + c_2) \log n$ for all $n \geq 2$.

This implies $T(n) = O(n \log n)$, as we were hoping. □

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But what if n isn't a power of 2?

For a back-of-the-envelope calculation, we'd just say $T(n) \le T(N)$ where N is the nearest power of two. But sometimes this might be false...

Dealing with floors and ceilings

The "real" recurrence for mergesort is

$$\begin{aligned}&T(1) \leq c_1,\\ &T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2 n \text{ for all } n \geq 2.\end{aligned}$$

To deal with floors and ceilings, our guess needs an **additive** term. Let's try to find C and a such that $T(n) \leq Cn\log(n) + a$ for all $n \geq 2$.

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Base case $n = 2$:

As before, $T(2) \le 2T(1) + 2c_2 \le 2(c_1 + c_2)$.
Also, we have $C \cdot 2\log(2) + a = 2C + a$.

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So the base case works whenever $2C + a \ge 2(c_1 + c_2)$. ✓

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Goal: Prove by induction that for all $n \geq 2$, $T(n) \leq Cn\log(n) + a$.

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$$\begin{aligned} T(n) &\le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2 n \\ &\le C\Big(\Big\lfloor \frac{n}{2} \Big\rfloor \log(\lfloor n/2 \rfloor) + \Big\lceil \frac{n}{2} \Big\rceil \log(\lceil n/2 \rceil)\Big) + 2a + c_2 n. \end{aligned}$$

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Goal: Prove by induction that for all $n \ge 2$, $T(n) \le Cn\log(n) + a$.

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To deal with floors and ceilings, we normally use these bounds:

$\lfloor x \rfloor \le x$ for all $x \in \mathbb{R}$, $\lceil x \rceil \le x+1$ for all $x \in \mathbb{R}$, $\lceil x \rceil \le 2x$ for all $x \ge 1$.

Dealing with floors and ceilings

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Goal: Prove by induction that for all $n \geq 2$, $T(n) \leq Cn\log(n) + a$.

Base case $n = 2$: Requires $2C + a \geq 2(c_1 + c_2)$. ✓

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Using the "right" bounds in the "right" expressions:

$$T(n) \leq C\Big(\frac{n}{2}\log(n/2) + \Big(\frac{n}{2} + 1\Big)\log(n)\Big) + 2a + c_2 n.$$

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Base case $n = 2$: Requires $2C + a \ge 2(c_1 + c_2)$. ✓

Inductive step: Suppose that for all $2 \le n' < n$, $T(n') \le Cn' \log n' + a$. Then we must prove $T(n) \le Cn \log n + a$. We showed

$$T(n) \le C\Big(\frac{n}{2}\log(n/2) + \Big(\frac{n}{2} + 1\Big)\log(n)\Big) + 2a + c_2 n.$$

Dealing with floors and ceilings

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We also bound $\log(n/2) \leq \log(n)$ to make the algebra a bit easier. Then rearranging gives:

$$T(n) \leq Cn\log(n) + \log(n) + 2a + c_2 n$$

This is at most $Cn\log(n)$ as long as we take $a \leq -(\log(n) + c_2 n)/2$. ✓

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So all that's left is to pick C and a that work.

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If we take $a(n) = -(\log(n) + c_2 n)/2$, then the inductive step works and $a(2) = -\frac{1}{2} - c_2$.

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So all that's left is to pick C and a that work.

If we take $a(n) = -(\log(n) + c_2 n)/2$, then the inductive step works and $a(2) = -\frac{1}{2} - c_2$.

So to make the base case work, we take

$$C = c_1 + c_2 - \frac{a}{2} = c_1 + \frac{3}{2}c_2 + \frac{1}{4} > 0.$$

(Note we do need $C > 0$ here!)

Dealing with floors and ceilings

$$T(1) \leq c_1,$$
$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2 n \text{ for all } n \geq 2.$$

We proved: Let $C = c_1 + \frac{3}{2}c_2 + \frac{1}{4}$, and let $a(n) = -\frac{1}{2}(c_2 n + \log(n))$.
Then $T(n) \leq Cn\log(n) + a(n)$ for all $n \geq 2$. $\square$

Dealing with floors and ceilings

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We proved: Let $C = c_1 + \frac{3}{2}c_2 + \frac{1}{4}$, and let $a(n) = -\frac{1}{2}(c_2 n + \log(n))$.
Then $T(n) \leq Cn\log(n) + a(n)$ for all $n \geq 2$. □

In particular, this implies $T(n) = O(n \log n)$ as before. Phew!

Note we proved something **stronger** than $T(n) \leq Cn\log(n)$ for all $n \geq 2$. And yet, if we'd tried the proof with $a(n) = 0$, it wouldn't have worked!

It's counterintuitive, but if you're having trouble with an induction, strengthening your inductive hypothesis can be very helpful.

Next time: More examples!

(Lecture to be given online...)