Lecture 10: Quicksort

COMS10007 - Algorithms

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Sorting Algorithms seen so far:

	Worst case	Average case	stable?	in place?
Insertion Sort	$O(n^2)$	$O(n^2)$	yes	yes
Mergesort	$O(n \log n)$	$O(n \log n)$	yes	no
Heapsort	$O(n \log n)$	$O(n \log n)$	no	yes
Quicksort	$O(n^2)$	$O(n \log n)$	no	yes

Quicksort

- Very efficient in practice!
- *In place version of Mergesort:*

$A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \textsc{MergeSort}(A[0, \lfloor \frac{n}{2} \rfloor])$
$A[\lfloor \frac{n}{2} \rfloor + 1, n-1] \leftarrow \textsc{MergeSort}(A[\lfloor \frac{n}{2} \rfloor, n-1])$
$A \leftarrow \textsc{Merge}(A)$
return A

recursive calls in mergesort

Merge Sort versus Quick Sort

Mergesort versus Quicksort

- *Mergesort:* First solve subproblems recursively, then merge their solutions
- *Quicksort:* First partition problem into two subproblems in a clever way so that no extra work is needed when combining the solutions to the subproblems, then solve subproblems recursively

Quicksort

Divide and Conquer Algorithm:

- **Divide:** Chose a good *pivot* $A[q]$. Rearrange A such that every element $\leq A[q]$ is left of $A[q]$ in the resulting ordering and every element $> A[q]$ is right of $A[q]$ in the resulting ordering. Let p be the new position of $A[q]$.
- **Conquer:** Sort $A[0, p-1]$ and $A[p+1, n-1]$ recursively.

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				7						

- **Combine:** No work needed

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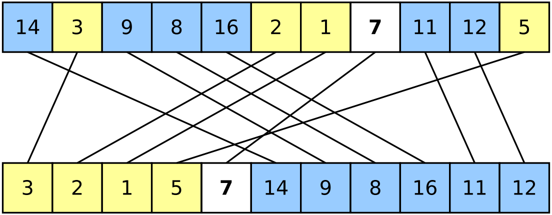

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Quicksort

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1	2	3	5	**7**	8	9	11	12	14	16

- **Combine:** No work needed

Quicksort (2)

We need to address:

1. We need to be able to rearrange the elements around the pivot in $O(n)$ time
2. What is a good pivot? Ideally we would like to obtain subproblems of equal/similar sizes

The Partition Step

Partition Step:

- **Input:** Array A of length n
- **Output:** Partitioning around pivot $A[n-1]$

```
Require: Array A of length n
  x ← A[n − 1]
  i ← −1
  for j ← 0 . . . n − 1 do
    if A[j] ≤ x then
      i ← i + 1
      exchange A[i] with A[j]
  return i
```

PARTITION

Pivot: Algorithm assumes pivot is $A[n-1]$. Why is this okay?

Example

$x \leftarrow A[n-1]$
$i \leftarrow -1$
for $j \leftarrow 0 \ldots n-1$ **do**
    if $A[j] \leq x$ **then**
        $i \leftarrow i + 1$
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x: 7

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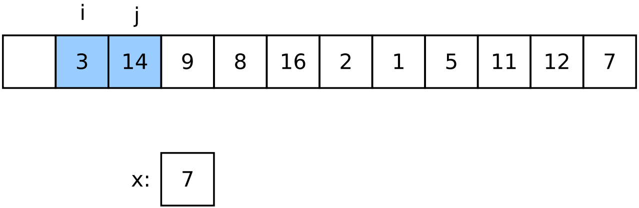

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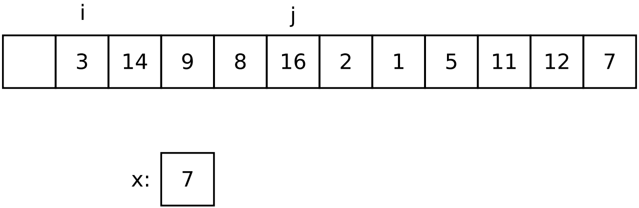

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x ← A[n − 1]
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for j ← 0 ... n − 1 do
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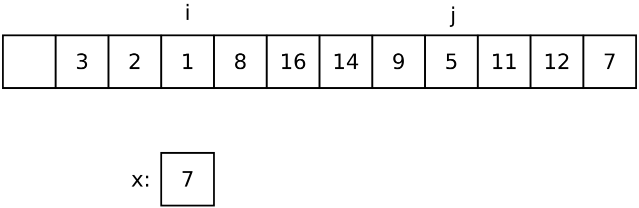

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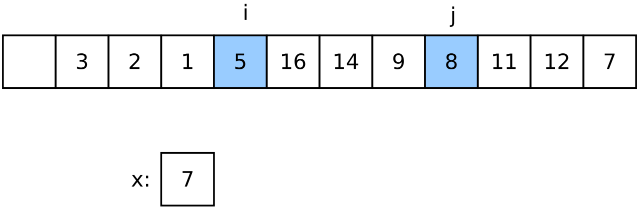

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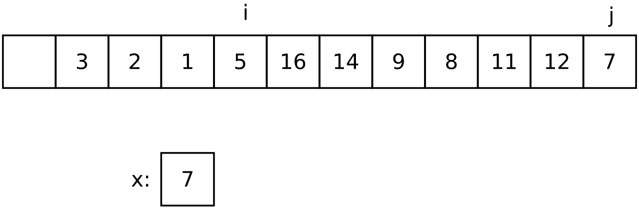

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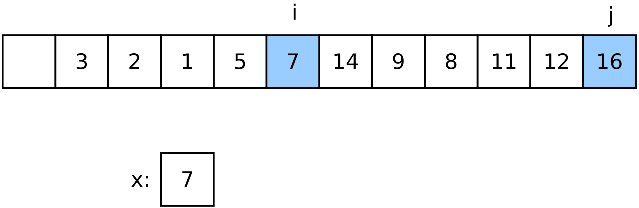

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					i						j
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Loop Invariant

Invariant: At the beginning of the for loop, the following holds:

1. Elements left of i (including i) are smaller or equal to x:

$$\text{For } 0 \leq k \leq i : A[k] \leq x$$

2. Elements right of i (excluding i) and left of j (excluding j) are larger than x:

$$\text{For } i+1 \leq k \leq j-1 : A[k] > x$$

Proof of Loop Invariant

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

> $x \leftarrow A[n-1]$
> $i \leftarrow -1$
> **for** $j \leftarrow 0 \dots n-1$ **do**
>   **if** $A[j] \leq x$ **then**
> $i \leftarrow i + 1$
>   exchange $A[i]$ with $A[j]$

Initialization: $i = -1, j = 0$

i	j												
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Maintenance: Two cases:

1. $A[j] > x$: Then j is incremented

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Proof of Loop Invariant (2)

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Proof of Loop Invariant (3)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

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x ← A[n − 1]
i ← −1
for j ← 0 ... n − 1 do
    if A[j] ≤ x then
        i ← i + 1
        exchange A[i] with A[j]
```

Termination: (useful property showing that algo. is correct)

- $A[i]$ contains pivot element x that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to x
- All elements right of $A[i]$ are larger than x

Proof of Loop Invariant (3)

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Quicksort

Require: array A of length n
  if $n \leq 10$ **then**
    Sort A using your favourite sorting algorithm
  else
    $i \leftarrow$ Partition(A)
    $\textsc{Quicksort}(A[0, i-1])$
    $\textsc{Quicksort}(A[i+1, n-1])$

Algorithm Quicksort

Termination Condition

Observe that $n \leq 10$ is arbitrary (any constant would do)

What is the runtime of Quicksort?