Lecture 2: O-notation (Why Constants Matter Less)

COMS10007 - Algorithms

Dr. Christian Konrad

29.01.2019

Runtime of Algorithms

Runtime of an Algorithm

- Function that maps the input length n to the number of simple/unit/elementary operations (worst case, best case, average case, runtime on a specific input, ...)
- The number of array accesses in PEAK FINDING represents the number of unit operations very well

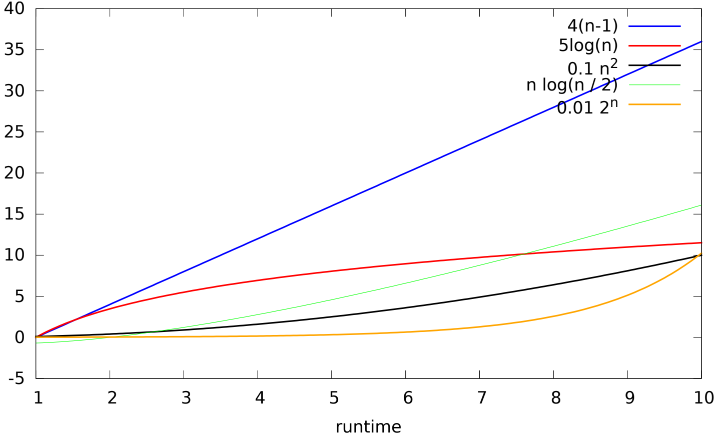
Which runtime is better?

- $4(n-1)$ (simple peak finding algorithm)
- $5 \log n$ (fast peak finding algorithm)
- $0.1n^2$
- $n \log(0.5n)$
- $0.01 \cdot 2^n$

Answer:

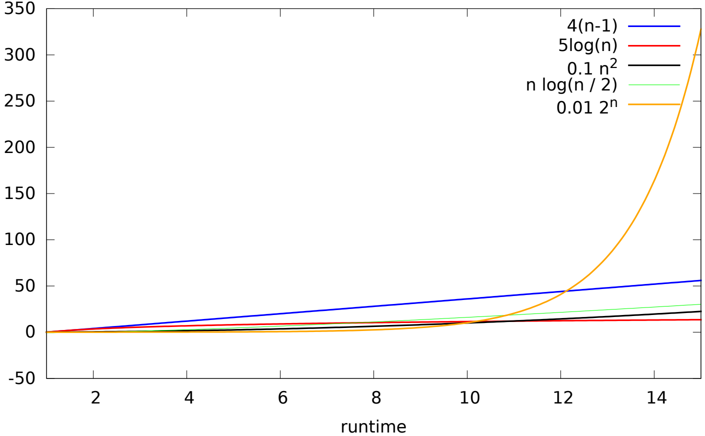
It depends... But there is a favourite

Runtime Comparisons



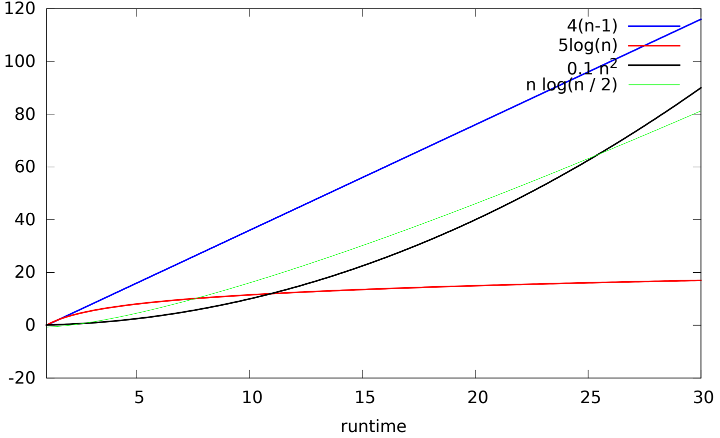
$$0.1n^2 \leq \color{orange}{0.01 \cdot 2^n} \leq \color{red}{5 \log n} \leq \color{green}{n \log(n/2)} \leq \color{blue}{4(n-1)}$$
$$(n = 10)$$

Runtime Comparisons



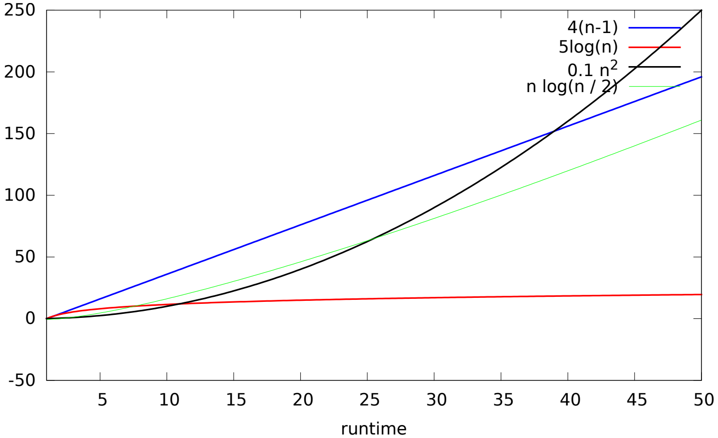
$$5 \log n \leq 0.1 n^2 \leq n \log(n/2) \leq 4(n-1) \leq 0.01 \cdot 2^n$$
$$(n = 15)$$

Runtime Comparisons



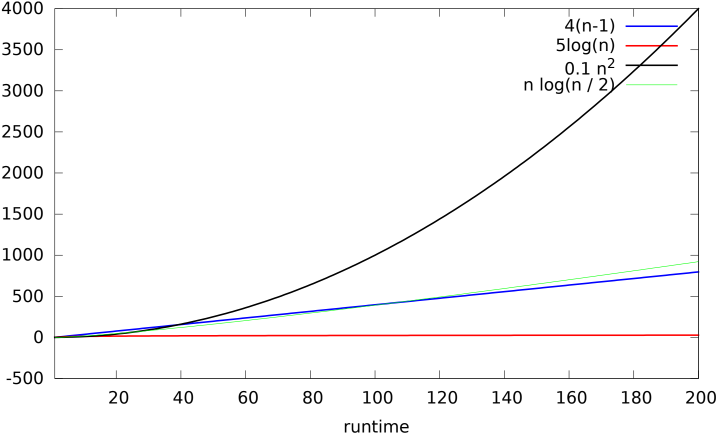
$$\color{red}{5\log n} \leq \color{green}{n\log(n/2)} \leq 0.1n^2 \leq \color{blue}{4(n-1)}$$
$$(n = 30)$$

Runtime Comparisons



$$5 \log n \leq n \log(n/2) \leq 4(n-1) \leq 0.1n^2$$
$$(n = 50)$$

Runtime Comparisons



$$5 \log n \leq 4(n-1) \leq n \log(n/2) \leq 0.1n^2$$
$$(n = 200)$$

Order Functions Disregarding Constants

Aim: We would like to sort algorithms according to their runtime

Is algorithm A faster than algorithm B?

Asymptotic Complexity

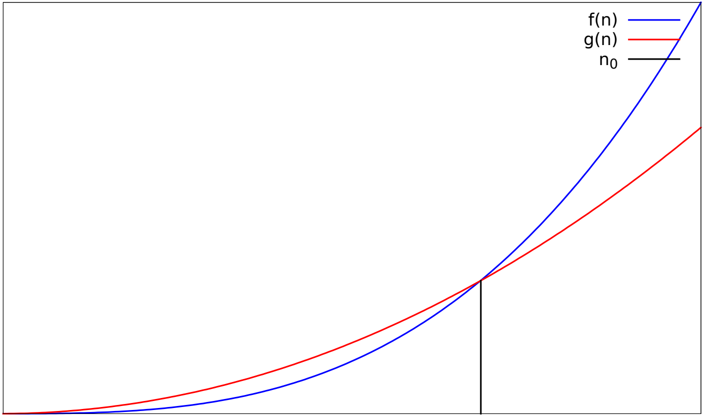
- For large enough n, constants seem to matter less
- For small values of n, most algorithms are fast anyway (not always true!)

Solution: Consider asymptotic behavior of functions

An increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ grows *asymptotically at least as fast as* an increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ if there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds:

$$f(n) \geq g(n) \ .$$

Example: f grows at least as fast as g



Example with Proof

Example: $f(n) = 2n^3$, $g(n) = \frac{1}{2} \cdot 2^n$

Then $g(n)$ grows asymptotically at least as fast as $f(n)$ since for every $n \geq 16$ we have $g(n) \geq f(n)$

Proof: Find values of n for which the following holds:

$$\begin{aligned}
\frac{1}{2} \cdot 2^n &\geq 2n^3 \\
2^{n-1} &\geq 2^{3\log n + 1} \quad \text{(using } n = 2^{\log n}\text{)} \\
n - 1 &\geq 3\log n + 1 \\
n &\geq 3\log n + 2
\end{aligned}$$

This holds for every $n \geq 16$ (which follows from the *racetrack principle*). Thus, we chose any $n_0 \geq 16$. $\square$

The Racetrack Principle

Racetrack Principle: Let f, g be functions, k an integer and suppose that the following holds:

1. $f(k) \geq g(k)$ and
2. $f'(n) \geq g'(n)$ for every $n \geq k$.

Then for every $n \geq k$, it holds that $f(n) \geq g(n)$.

Example: $n \geq 3 \log n + 2$ holds for every $n \geq 16$

- $n \geq 3 \log n + 2$ holds for $n = 16$
- We have: $(n)' = 1$ and $(3 \log n + 2)' = \frac{3}{n \ln 2} < \frac{1}{2}$ for every $n \geq 16$. The result follows.

Order Functions by Asymptotic Growth

If $\leq$ means *grows asymptotically at least as fast as* then we get:

$$5 \log n \leq 4(n-1) \leq n \log(n/2) \leq 0.1n^2 \ \leq 0.01 \cdot 2^n$$

Observe:

"polynomial of logarithms" $\leq$ "polynomial" $\leq$ "exponential"

Big O Notation

Definition: O-notation ("Big O")

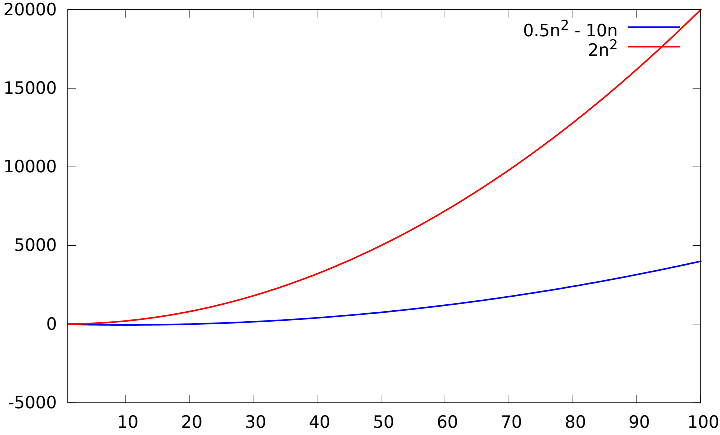
Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Then $O(g(n))$ is the set of functions:

$$O(g(n)) = \{f(n) : \text{There exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Meaning: $f(n) \in O(g(n))$: "g grows asymptotically at least as fast as f up to constants"

O-Notation: Example

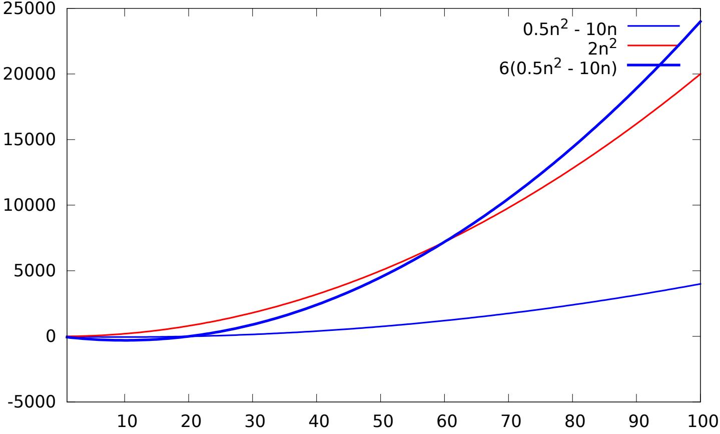
Example: $f(n) = \frac{1}{2}n^2 - 10n$ and $g(n) = 2n^2$



Then: $g(n) \in O(f(n))$, since $6f(n) \geq g(n)$, for every $n \geq n_0 = 60$

O-Notation: Example

Example: $f(n) = \frac{1}{2}n^2 - 10n$ and $g(n) = 2n^2$



Then: $g(n) \in O(f(n))$, since $6f(n) \geq g(n)$, for every $n \geq n_0 = 60$

More Examples/Exercises

Recall:

$$O(g(n)) = \{f(n) : \text{There exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Exercises:

- $100n \stackrel{?}{\in} O(n)$ Yes, chose $c = 100, n_0 = 1$
- $0.5n \stackrel{?}{\in} O(n/\log n)$ No: Suppose that such constants c and n_0 exist. Then, for every $n \geq n_0$:

$$\begin{aligned} 0.5n &\leq cn/\log n \\ \log n &\leq 2c \\ n &\leq 2^{2c} \text{ , a contradiction,} \end{aligned}$$

since this does not hold for every $n > 2^{2c}$.

Properties

Recipe

- To prove $f \in O(g)$: We need to find constants c, n_0 as in the statement of the definition
- To prove $f \notin O(g)$: We assume that constants c, n_0 exist and derive a contradiction

Constants $100 \overset{?}{\in} O(1)$ yes, every constant is in $O(1)$

Lemma (Sum of Two Functions)

Suppose that $f, g \in O(h)$. Then: $f + g \in O(h)$.

Proof. Let c, n_0 be such that $f(n) \leq ch(n)$, for every $n \geq n_0$. Let c', n_0' be such that $g(n) \leq c'h(n)$, for every $n \geq n_0'$.
Let $C = c + c'$ and let $N_0 = \max\{n_0, n_0'\}$. Then:

$$f(n) + g(n) \leq ch(n) + c'h(n) = Ch(n) \text{ for every } n \geq N_0 \ . \quad \square$$

Further Properties

Lemma (Polynomials)

Let $f(n) = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + \cdots + c_k n^k$, for some integer k that is independent of n. Then: $f(n) \in O(n^k)$.

Proof: Apply statement on last slide $O(1)$ times (k times) $\square$

Attention: Wrong proof of $n^2 \in O(n)$: (this is clearly wrong)

$$\begin{aligned} n^2 &= n + n + \underbrace{n + \ldots n}_{n-2 \text{ times}} = O(n) + O(n) + \underbrace{n + \ldots n}_{n-2 \text{ times}} \\ &= O(n) + \underbrace{n + \ldots n}_{n-2 \text{ times}} = O(n) + O(n) + \underbrace{n + \ldots n}_{n-3 \text{ times}} = \\ &= O(n) + \underbrace{n + \ldots n}_{n-3 \text{ times}} = \cdots = O(n) \ . \end{aligned}$$

Application of statement on last slide n times! (only allowed to apply statement $O(1)$ times!)

Runtime of Algorithms

Tool for the Analysis of Algorithms

- We will express the runtime of algorithms using O-notation
- This allows us to compare the runtimes of algorithms
- **Important:** Find the slowest growing function f such that our runtime is in $O(f)$ (most algorithms have a runtime of $O(2^n)$)

Important Properties for the Analysis of Algorithms

- Composition of instructions:

$$f \in O(h_1), g \in O(h_2) \text{ then } f + g \in O(h_1 + h_2)$$

- Loops: (repetition of instructions)

$$f \in O(h_1), g \in O(h_2) \text{ then } f \cdot g \in O(h_1 \cdot h_2)$$

Hierachy

Rough incomplete Hierachy

- Constant time: $O(1)$ (individual operations)
- Sub-logarithmic time: e.g., $O(\log \log n)$
- Logarithmic time: $O(\log n)$ (Fast-Peak-Finding)
- Poly-logarithmic time: e.g., $O(\log^2 n), O(\log^{10} n), \dots$
- Linear time: $O(n)$ (e.g., time to read the input)
- Quadratic time: $O(n^2)$ (potentially slow on big inputs)
- Polynomial time: $O(n^c)$ (used to be considered efficient)
- Exponential time: $O(2^n)$ (works only on very small inputs)
- Super-exponential time: e.g. $O(2^{2^n})$ (big trouble...)