Lecture 1: Introduction - Peak Finding

COMS10007 - Algorithms

Dr. Christian Konrad

27.01.2020

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Bursaries available for eligible students

Info session: **Wednesday 5th February 2pm, Senate House 5.10**
Application deadline: **Wednesday 12th February**

Find out more: bristol.ac.uk/summer-abroad

Algorithms?

A procedure that solves a *computational problem*

Computational Problem?

- Sort an array of n numbers
- How often does "Juliet" appear in Shakespeare's "Romeo And Juliet"?
- How do we factorize a large number?
- Shortest/fastest way to travel from Bristol to Glasgow?
- How to execute a database query?
- Is it possible to partition the set $\{17, 8, 4, 22, 9, 28, 2\}$ into two sets s.t. their sums are equal? $\{8, 9, 28\}$, $\{2, 4, 17, 22\}$

What we want and how we work

Efficiency

- The faster the better: **Runtime analysis**
- Use as little memory as possible: **Space complexity**

Mathematics

- We will prove that algorithms run fast and use little memory
- We will prove that algorithms are correct
- **Tools:** Induction, algebra, sums, . . . , rigorous arguments

Theoretical Computer Science

No implementations in this unit!

What you get out of this unit

Goals

- First steps towards becoming an algorithms designer
- Learn techniques that help you design & analyze algorithms
- Understand a set of well-known algorithms

Systematic Approach to Problem/Puzzle Solving

- Study a problem at hand, discover structure within problem, exploit structure and design algorithms
- Useful in all areas of Computer Science
- Interview questions, Google, Facebook, Amazon, etc.

My Goals

My Goals

- Get you excited about Algorithms
- Shape new generation of Algorithm Designers at Bristol

Algorithms in Bristol

- 1st year: Algorithms (Algorithms 1)
- 2nd year: Data Structures and Algorithms (Algorithms 2)
- 3rd year: Advanced Algorithms (Algorithms 3)
- 4th year: *in progress* (Algorithms 4)

Projects, Theses, PhD students, Seminars

Unit Structure

Teaching Units

- Lectures: Mondays 2-3pm, Wednesdays 10-11am, PUGSLEY, Instructor: Dr. Christian Konrad
- Exercise classes/in-class tests: Tuesdays 1pm-2pm (A-L) and 2pm-3pm (M-Z), Room MVB 1.11

Assessment

- Exam: Counts 90%
- One In-class test: Counts 10% (Extra time? let me know as soon as possible)
- You pass the unit if your final grade is at least 40%

Teaching Staff and Office Hours

Teaching Staff

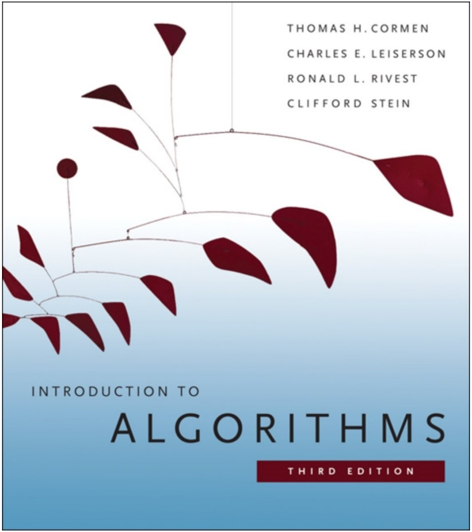
- Unit Director: Christian Konrad
- TAs: Lidiya Binti Khalil, Emil Centiu, Igor Dolecki, Daniel Jones, Joseph MacManus, Mutalib Mohammed, Yuhang Ming, Kar Hor Yap

Optional Drop-in Session

- Thursdays 10-11am, MVB 4.01
- **OPTIONAL!**

My Office Hours Wednesdays 1-2pm in MVB 3.06

Book


THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN
INTRODUCTION TO
ALGORITHMS
THIRD EDITION

How to Succeed in this Unit

How to succeed

- Make sure you understand the material
- **Work on provided exercises!**
- Come to our drop in sessions
- **Work on provided exercises!!**
- Piazza for discussions and questions
- **Work on provided exercises!!!**
- Come to my office hours

Unit webpage

`http://people.cs.bris.ac.uk/~konrad/courses/2019_2020_COMS10007/coms10007.html`

- News, announcements
- Download slides, exercises, etc.

Peak Finding

Let $A = a_0, a_1, \ldots, a_{n-1}$ be an array of integers of length n

0	1	2	3	4	5	6	7	8	9
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9

Definition: (Peak)
Integer a_i is a *peak* if adjacent integers are not larger than a_i

Example:

4	3	9	10	14	8	7	2	2	2

Peak Finding

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Peak Finding: Simple Algorithm

Problem PEAK FINDING: Write algorithm with properties:

1. **Input:** An integer array of length n
2. **Output:** A position $0 \leq i \leq n-1$ such that a_i is a peak

```
int peak(int *A, int len) {
    if(A[0] >= A[1])
        return 0;
    if(A[len-1] >= A[len-2])
        return len-1;

    for(int i=1; i < len-1; i=i+1) {
        if(A[i] >= A[i-1] && A[i] >= A[i+1])
            return i;
        }
    return -1;
}
```

C++ code

Peak Finding: Simple Algorithm

Problem PEAK FINDING: Write algorithm with properties:

1. **Input:** An integer array of length n
2. **Output:** A position $0 \leq i \leq n-1$ such that a_i is a peak

Require: Integer array A of length n
  if $A[0] \geq A[1]$ **then**
    return 0
  if $A[n-1] \geq A[n-2]$ **then**
    return $n-1$
  for $i = 1 \dots n-2$ **do**
    if $A[i] \geq A[i-1]$ **and** $A[i] \geq A[i+1]$ **then**
      return i
  return -1

Pseudo code

Peak Finding: Problem well-defined?

Is Peak Finding well defined? Does every array have a peak?

Lemma

Every integer array has at least one peak.

Proof.
Let A be an integer array of length n. Suppose for the sake of a contradiction that A does not have a peak. Then $a_1 > a_0$ since otherwise a_0 is a peak. But then $a_2 > a_1$ since otherwise a_1 is a peak. Continuing, for the same reason, $a_i > a_{i-1}$ since otherwise a_{i-1} is a peak, for every $i \leq n-1$. But this implies $a_{n-1} > a_{n-2}$ and hence a_{n-1} is a peak. A contradiction. Hence, every array has a peak. $\square$

0	1	2	3	4	5	6
a_0	a_1	a_2	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	a_2	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	$> a_1$	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	$> a_1$	$> a_2$	$> a_3$	$> a_4$	$> a_5$

Peak Finding: Problem well-defined?

Is Peak Finding well defined? Does every array have a peak?

Lemma

Every integer array has at least one peak.

Proof.

Every maximum is a peak. (Shorter and immediately convincing!) □

Peak Finding: How fast is the Simple Algorithm?

How fast is our Algorithm?

Require: Integer array A of length n
  if $A[0] \geq A[1]$ **then**
    return 0
  if $A[n-1] \geq A[n-2]$ **then**
    return $n-1$
  for $i = 1 \dots n-2$ **do**
    if $A[i] \geq A[i-1]$ **and** $A[i] \geq A[i+1]$ **then**
      return i
  return -1

How often do we look at the array elements? (worst case!)

- $A[0]$ and $A[n-1]$: twice
- $A[1] \dots A[n-2]$: 4 times
- Overall: $2 + 2 + (n-2) \cdot 4 = 4(n-1)$

Can we do better?!

Peak Finding: An even faster Algorithm

Finding Peaks even Faster: FAST-PEAK-FINDING

1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return FAST-PEAK-FINDING$(A[0, \lfloor n/2 \rfloor - 1])$
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$
   FAST-PEAK-FINDING$(A[\lfloor n/2 \rfloor + 1, n - 1])$

Comments:

- FAST-PEAK-FINDING is *recursive* (it calls itself)
- $\lfloor x \rfloor$ is the floor function ($\lceil x \rceil$: ceiling)

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 16/2 \rfloor] = A[8]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[7] \geq A[8]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,7])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 8

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 8/2 \rfloor] = A[4]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[3] \geq A[4]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,3])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 4

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 4/2 \rfloor] = A[2]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[1] \geq A[2]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,1])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Else **return** FAST-PEAK-FINDING($A[3]$), which returns 3

How often does the Algorithm look at the array elements?

- Without the recursive calls, the algorithm looks at the array elements at most 5 times
- Let $R(n)$ be the number of calls to FAST-PEAK-FINDING when the input array is of length n. Then:

$$\begin{aligned} R(1) &= R(2) = 1 \\ R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \text{ , for } n \geq 3 \text{ .} \end{aligned}$$

- Solving the recurrence (see lecture on recurrences):

$$\begin{aligned} R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \leq R(n/2) + 1 = R(\lfloor n/4 \rfloor) + 2 \\ &\leq R(n/4) + 2 = \dots \leq \lceil \log n \rceil \text{ .} \end{aligned}$$

- Hence, we look at most at $5\lceil \log n \rceil$ array elements!

Peak Finding: Correctness

Why is the Algorithm correct?!

Steps 1,2,3 are clearly correct

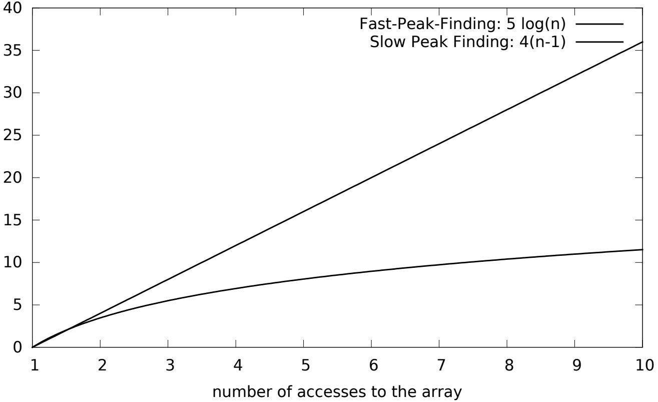
1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return Fast-Peak-Finding($A[0, \lfloor n/2 \rfloor - 1]$)
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$
   Fast-Peak-Finding($A[\lfloor n/2 \rfloor + 1, n - 1]$)

Why is step 4 correct? (step 5 is similar)

- Need to prove: peak in $A[0, \lfloor n/2 \rfloor - 1]$ is a peak in A
- Critical case: $\lfloor n/2 \rfloor - 1$ is a peak in $A[0, \lfloor n/2 \rfloor - 1]$
- Condition in step 4 guarantees $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ and hence $\lfloor n/2 \rfloor - 1$ is a peak in A as well (very important!) $\square$

Peak Finding: Runtime Comparison

$4(n-1)$ **versus** $5 \log n$



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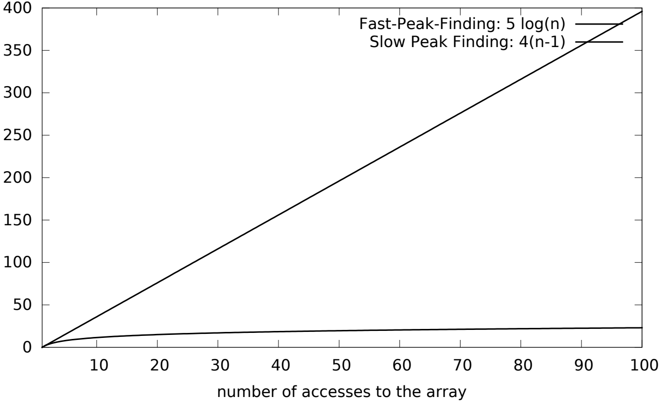

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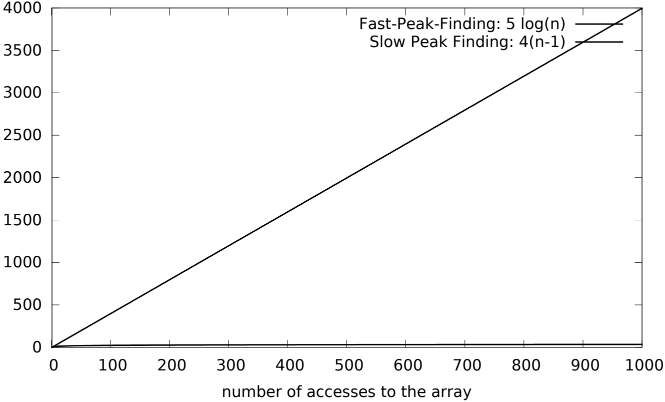

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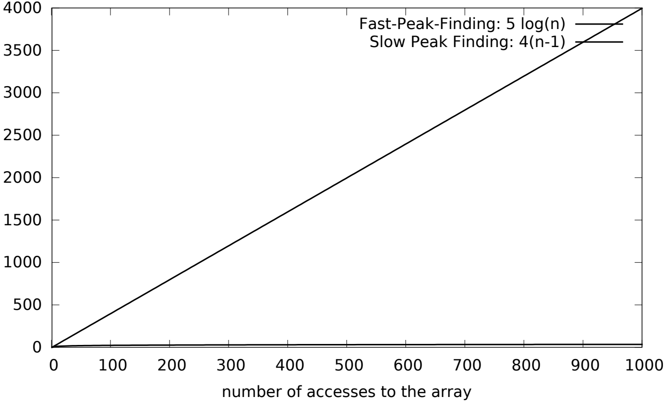

Conclusion: $5 \log n$ is so much better than $4(n-1)$!