# Lecture 10: Quicksort

## COMS10007 - Algorithms

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# Quicksort

**Sorting Algorithms seen so far:**

| | Worst case | Average case | stable? | in place? |
|---|---|---|---|---|
| Insertion Sort | $O(n^2)$ | $O(n^2)$ | yes | yes |
| Mergesort | $O(n \log n)$ | $O(n \log n)$ | yes | no |
| Heapsort | $O(n \log n)$ | $O(n \log n)$ | no | yes |
| Quicksort | $O(n^2)$ | $O(n \log n)$ | no | yes |

**Quicksort**

- Very efficient in practice!
- *In place version of Mergesort:*

$A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \textsc{MergeSort}(A[0, \lfloor \frac{n}{2} \rfloor])$
$A[\lfloor \frac{n}{2} \rfloor + 1, n-1] \leftarrow \textsc{MergeSort}(A[\lfloor \frac{n}{2} \rfloor, n-1])$
$A \leftarrow \textsc{Merge}(A)$
**return** $A$

recursive calls in mergesort

# Merge Sort versus Quick Sort

**Mergesort versus Quicksort**

- *Mergesort:* First solve subproblems recursively, then merge their solutions
- *Quicksort:* First partition problem into two subproblems in a clever way so that no extra work is needed when combining the solutions to the subproblems, then solve subproblems recursively

**Divide and Conquer Algorithm:**

- **Divide:** Chose a good *pivot* $A[q]$. Rearrange $A$ such that every element $\leq A[q]$ is left of $A[q]$ in the resulting ordering and every element $> A[q]$ is right of $A[q]$ in the resulting ordering. Let $p$ be the new position of $A[q]$.
- **Conquer:** Sort $A[0, p-1]$ and $A[p+1, n-1]$ recursively.

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| | | | | | | | | | | |
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- **Combine:** No work needed

# Quicksort

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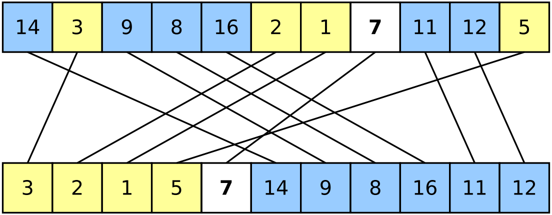


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# Quicksort

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- **Combine:** No work needed

# Quicksort (2)

**We need to address:**

1. We need to be able to rearrange the elements around the pivot in $O(n)$ time
2. What is a good pivot? Ideally we would like to obtain subproblems of equal/similar sizes

**Partition Step:**

- **Input:** Array $A$ of length $n$
- **Output:** Partitioning around pivot $A[n-1]$

```
Require: Array A of length n
   x ← A[n − 1]
   i ← −1
   for j ← 0 ... n − 1 do
      if A[j] ≤ x then
         i ← i + 1
         exchange A[i] with A[j]
   return i
```

PARTITION

**Pivot:** Algorithm assumes pivot is $A[n-1]$. Why is this okay?

# Example

$x \leftarrow A[n-1]$
$i \leftarrow -1$
**for** $j \leftarrow 0 \ldots n-1$ **do**
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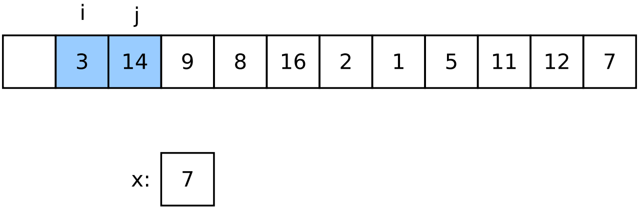

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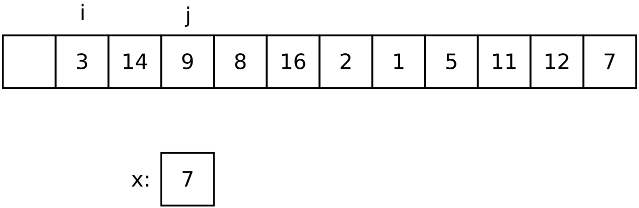

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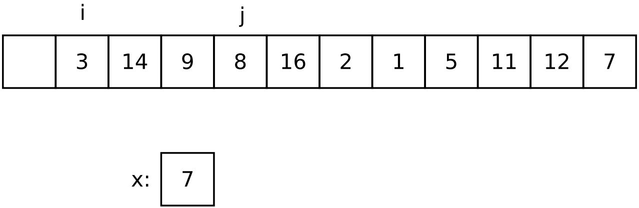

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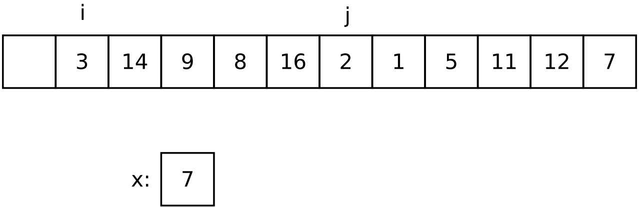

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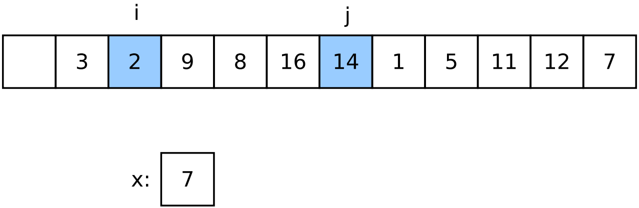

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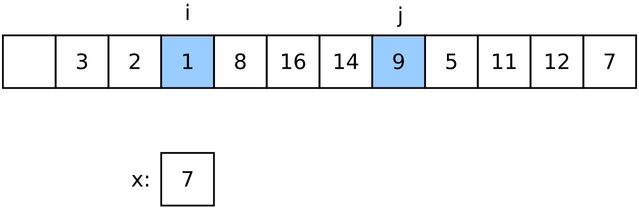

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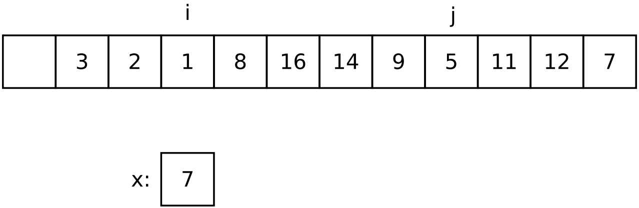

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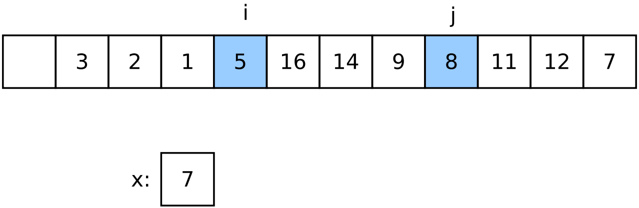

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# Example

**Pseudocode**

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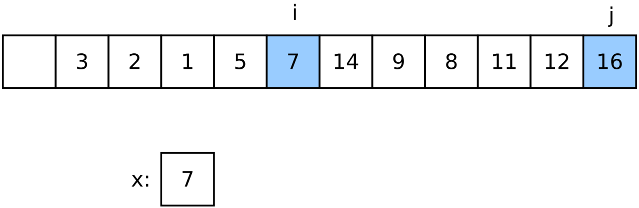

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# Loop Invariant

**Invariant:** At the beginning of the for loop, the following holds:

1. Elements left of $i$ (including $i$) are smaller or equal to $x$:

$$\text{For } 0 \leq k \leq i : A[k] \leq x$$

2. Elements right of $i$ (excluding $i$) and left of $j$ (excluding $j$) are larger than $x$:

$$\text{For } i+1 \leq k \leq j-1 : A[k] > x$$

# Proof of Loop Invariant

1. Left of $i$ (including $i$): smaller equal to $x$
2. Right of $i$ and left of $j$ (excl. $j$): larger than $x$

$x \leftarrow A[n-1]$
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**Initialization:** $i = -1, j = 0$

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# Proof of Loop Invariant (2)

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**Maintenance:** Two cases:

1. $A[j] > x$: Then $j$ is incremented

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x ← A[n − 1]
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```

**Termination:** (useful property showing that algo. is correct)

- $A[i]$ contains pivot element $x$ that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to $x$
- All elements right of $A[i]$ are larger than $x$

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# Quicksort

**Require:** array $A$ of length $n$
  **if** $n \leq 10$ **then**
    Sort $A$ using your favourite sorting algorithm
  **else**
    $i \leftarrow$ Partition($A$)
    $\textsc{Quicksort}(A[0, i-1])$
    $\textsc{Quicksort}(A[i+1, n-1])$

Algorithm $\textsc{Quicksort}$

**Termination Condition**

Observe that $n \leq 10$ is arbitrary (any constant would do)

**What is the runtime of Quicksort?**