Lecture 1: Introduction - Peak Finding

COMS10007 - Algorithms

Dr. Christian Konrad

28.01.2019

Algorithms?

A procedure that solves a *computational problem*

Computational Problem?

- Sort an array of n numbers
- Find the median of an array
- How often does "Juliet" appear in Shakespeare's "Romeo And Juliet"?
- How do we factorize a large number?
- Shortest/fastest way to travel from Bristol to Glasgow?
- Is it possible to partition the set $\{17, 8, 4, 22, 9, 28, 2\}$ into two sets s.t. their sums are equal? $\{8, 9, 28\}$, $\{2, 4, 17, 22\}$

What we want and how we work

Efficiency

The faster the better (runtime)
Use as little memory as possible (space complexity)

Mathematics

We will prove that algorithms run fast and use little memory
We will prove that algorithms are correct
Tools: Induction, algebra, sums, . . . , rigorous arguments

Theoretical Computer Science

No implementations in this course. But please go ahead and write code...!

What you get out of this course

Goals

- First steps towards becoming an algorithms designer
- Learn techniques that help you design & analyze algorithms
- Understand a set of well-known algorithms

Systematic Approach to Problem/Puzzle Solving

- Study a problem at hand, discover structure within problem, exploit structure and design algorithms
- Useful in all areas of Computer Science
- Interview questions, Google, Facebook, Amazon, etc.

My Goals

My Goals

- Get you excited about Algorithms
- Shape new generation of Algorithm Designers at Bristol *(who solve all my open problems...)*

Algorithms in Bristol

- 1st year: Algorithms (Algorithms 1)
- 2nd year: Data Structures and Algorithms (Algorithms 2)
- 3rd year: Advanced Algorithms (Algorithms 3)
- 4th year: *in progress* (Algorithms 4)

Projects, Theses, PhD students, Seminars

Course Structure

Teaching Units

- Lectures: Mondays 10-11am, Tuesdays 2-3pm, Room PHYS BLDG G42 POWELL, Instructor: Dr. Christian Konrad
- Exercise classes/in-class tests: Tuesdays 3pm-4pm (every fortnight), Room MVB 1.11

Assessment

- Exam: Counts 90%
- One In-class test: Counts 10% (March, 12th) (Extra time? let me know as soon as possible)
- You pass the course if your final grade is at least 40%

Teaching Staff and Office Hours

Teaching Staff

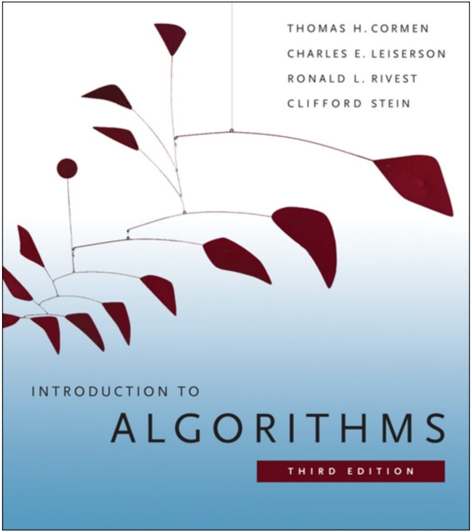
- Unit Director: Christian Konrad
- TAs: Thomas Delaney, Igor Dolecki, Nazaal Ibrahim, David Manda, Perla Jazmin Mayo Diaz de Leon, Matthew Owusu, Theano Xirouchaki

Drop-in Sessions

- Thursdays 5-6pm, MVB 3.44
- Fridays 1-2pm, MVB 3.44

My Office Hours Tuesdays 4-5pm in MVB 3.06 (to be confirmed!)

Book


THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN
INTRODUCTION TO
ALGORITHMS
THIRD EDITION

How to Succeed in this Course

Advice

- Make sure you understand the course material
- **Work on provided exercises!**
- Come to our drop in sessions
- **Work on provided exercises!!**
- Piazza for discussions and questions
- **Work on provided exercises!!!**
- Come to my office hours
- Course material has changed significantly from last year

Course webpage

`http://people.cs.bris.ac.uk/~konrad/courses/COMS10007/coms10007.html`

- News, announcements
- Download slides, exercises, etc.

Peak Finding

Let $A = a_0, a_1, \ldots, a_{n-1}$ be an array of integers of length n

0	1	2	3	4	5	6	7	8	9
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9

Definition: (Peak)
Integer a_i is a *peak* if adjacent integers are not larger than a_i

Example:

Peak Finding

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Example:

4	3	9	10	14	8	7	2	2	2

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Peak Finding: Simple Algorithm

Problem PEAK FINDING: Write algorithm with properties:

1. **Input:** An integer array of length n
2. **Output:** A position $0 \leq i \leq n - 1$ such that a_i is a peak

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```
int peak(int *A, int len) {
    if(A[0] >= A[1])
        return 0;
    if(A[len-1] >= A[len-2])
        return len-1;

    for(int i=1; i < len-1; i=i+1) {
        if(A[i] >= A[i-1] && A[i] >= A[i+1])
            return i;
        }
    return -1;
}
```

C++ code

Peak Finding: Simple Algorithm

Problem PEAK FINDING: Write algorithm with properties:

1. **Input:** An integer array of length n
2. **Output:** A position $0 \leq i \leq n-1$ such that a_i is a peak

```
Require: Integer array A of length n
  if A[0] ≥ A[1] then
    return 0
  if A[n − 1] ≥ A[n − 2] then
    return n − 1
  for i = 1 ... n − 2 do
    if A[i] ≥ A[i − 1] and A[i] ≥ A[i + 1] then
      return i
  return −1
```

Pseudo code

Peak Finding: Problem well-defined?

Is Peak Finding well defined? Does every array have a peak?

Lemma

Every integer array has at least one peak.

Proof.

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Proof.
Let A be an integer array of length n. Suppose for the sake of a contradiction that A does not have a peak. Then $a_1 > a_0$ since otherwise a_0 is a peak. But then $a_2 > a_1$ since otherwise a_1 is a peak. Continuing, for the same reason, $a_i > a_{i-1}$ since otherwise a_{i-1} is a peak, for every $i \leq n-1$. But this implies $a_{n-1} > a_{n-2}$ and hence a_{n-1} is a peak. A contradiction. Hence, every array has a peak. □

0	1	2	3	4	5	6
a_0	a_1	a_2	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	a_2	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	$> a_1$	a_3	a_4	a_5	a_6

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0	1	2	3	4	5	6
a_0	$> a_0$	$> a_1$	$> a_2$	$> a_3$	$> a_4$	$> a_5$

Peak Finding: Problem well-defined?

Is Peak Finding well defined? Does every array have a peak?

Lemma

Every integer array has at least one peak.

Proof.

Every maximum is a peak. (Shorter and immediately convincing!)

□

Peak Finding: How fast is the Simple Algorithm?

How fast is our Algorithm?

```
Require: Integer array A of length n
  if A[0] ≥ A[1] then
    return 0
  if A[n − 1] ≥ A[n − 2] then
    return n − 1
  for i = 1 ... n − 2 do
    if A[i] ≥ A[i − 1] and A[i] ≥ A[i + 1] then
      return i
  return −1
```

How often do we look at the array elements? (worst case!)

- $A[0]$ and $A[n-1]$: twice
- $A[1] \ldots A[n-2]$: 4 times
- Overall: $2 + 2 + (n-2) \cdot 4 = 4(n-1)$

Can we do better?!

Peak Finding: An even faster Algorithm

Finding Peaks even Faster: FAST-PEAK-FINDING

1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return FAST-PEAK-FINDING($A[0, \lfloor n/2 \rfloor - 1]$)
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$
   FAST-PEAK-FINDING($A[\lfloor n/2 \rfloor + 1, n - 1]$)

Comments:

- FAST-PEAK-FINDING is *recursive* (it calls itself)
- $\lfloor x \rfloor$ is the floor function ($\lceil x \rceil$: ceiling)

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 16/2 \rfloor] = A[8]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[7] \geq A[8]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,7])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 8

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 8/2 \rfloor] = A[4]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[3] \geq A[4]$ then **return** FAST-PEAK-FINDING($A[0, 3]$)

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 4

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 4/2 \rfloor] = A[2]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[1] \geq A[2]$ then **return** $\textsc{Fast-Peak-Finding}(A[0, 1])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Else **return** FAST-PEAK-FINDING($A[3]$), which returns 3

Peak Finding: How fast is the Improved Algorithm?

How often does the Algorithm look at the array elements?

- Without the recursive calls, the algorithm looks at the array elements at most 5 times
- Let $R(n)$ be the number of calls to FAST-PEAK-FINDING when the input array is of length n. Then:

$$\begin{aligned} R(1) &= R(2) = 1 \\ R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \text{ , for } n \geq 3 \text{ .} \end{aligned}$$

- Solving the recurrence (see lecture on recurrences):

$$\begin{aligned} R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \leq R(n/2) + 1 = R(\lfloor n/4 \rfloor) + 2 \\ &\leq R(n/4) + 2 = \cdots \leq \lceil \log n \rceil \text{ .} \end{aligned}$$

- Hence, we look at most at $5\lceil \log n \rceil$ array elements!

Peak Finding: Correctness

Why is the Algorithm correct?!

Steps 1,2,3 are clearly correct

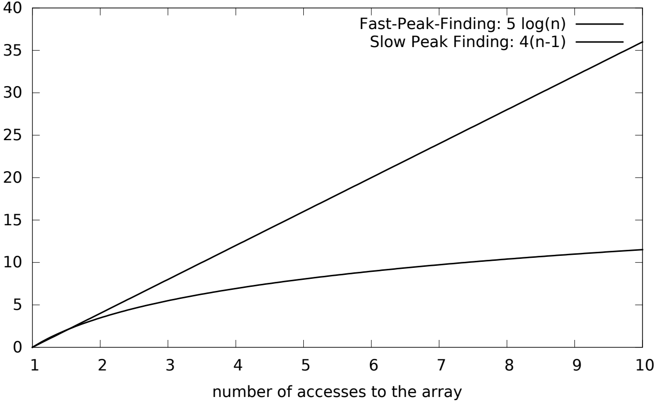
1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return $\textsc{Fast-Peak-Finding}(A[0, \lfloor n/2 \rfloor - 1])$
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$
   $\textsc{Fast-Peak-Finding}(A[\lfloor n/2 \rfloor + 1, n - 1])$

Why is step 4 correct? (step 5 is similar)

- Need to prove: peak in $A[0, \lfloor n/2 \rfloor - 1]$ is a peak in A
- Critical case: $\lfloor n/2 \rfloor - 1$ is a peak in $A[0, \lfloor n/2 \rfloor - 1]$
- Condition in step 4 guarantees $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ and hence $\lfloor n/2 \rfloor - 1$ is a peak in A as well (very important!) □

Peak Finding: Runtime Comparison

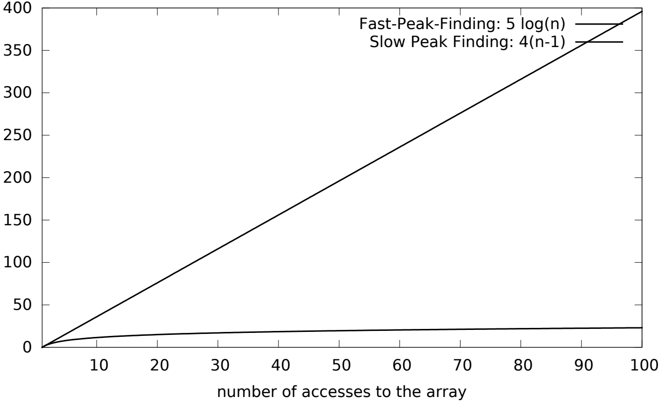
$4(n-1)$ **versus** $5 \log n$



Conclusion: $5 \log n$ is so much better than $4(n-1)$!

Peak Finding: Runtime Comparison

$4(n-1)$ **versus** $5\log n$



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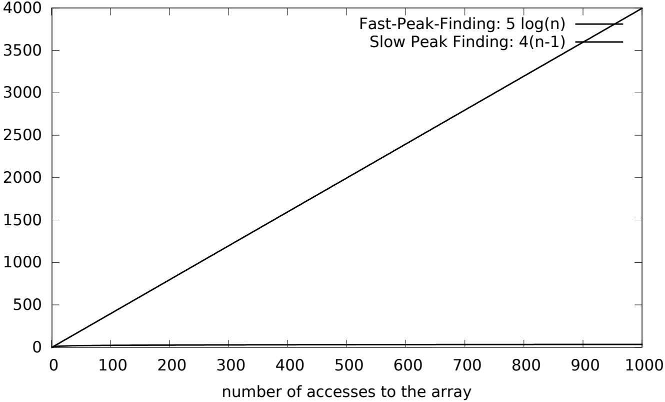

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