CHAPTER 7

RULE SYSTEMS FOR CONJECTURAL REASONING

— in which various forms of conjectural reasoning will be axiomatised and semantically characterised —

IN THIS CHAPTER I will develop axiomatic and semantic accounts of various forms of conjectural reasoning. The main purpose of the resulting logical systems is to provide a descriptive taxonomy of conjectural reasoning. This taxonomy will contain two main families, corresponding to the two forms of conjectural reasoning considered in this thesis: explanatory and confirmatory reasoning. Within the family of confirmatory reasoning a further distinction is made between incremental and non-incremental forms, the former based on the semantic notion of regular models, the latter based on the notion of consistency. For each of these three forms of conjectural reasoning a characterisation is given in the form of a semantics accompanied by a sound and complete rule system. It should be noted that the representation results are obtained for a propositional language *L*.

§26. EXPLANATORY REASONING

This section provides a formalisation of the idea that explanatory reasoning preserves explanatory power. As defined in §19, given some explanation mechanism $\vdash$, the *explanatory power* of a formula α is defined as its *closure* $Cn_{\vdash}(\alpha) = \{\gamma \mid \alpha \vdash \gamma\}$. Using this definition, we may require of an explanatory argument $\alpha \mathrel{|\!\sim} \beta$ that $Cn_{\vdash}(\alpha) \subseteq Cn_{\vdash}(\beta)$, i.e. for every γ, if $\alpha \vdash \gamma$ then $\beta \vdash \gamma$. If $\vdash$ satisfies the rules of the system **M** from the KLM-framework this is equivalent with $\beta \vdash \alpha$ (Lemma 5.2).

In this section I will mostly restrict attention to explanation mechanisms that satisfy the rules of **M**. The resulting form of conjectural reasoning is referred to as *strong* explanatory reasoning; the adjective 'strong' will be often omitted if no confusion can arise. Weaker explanation mechanisms will be briefly considered at the end of the section. Strong explanatory reasoning is characterised in two steps. I will first define a system, $\mathbf{M_{rev}}$, which embodies a reversed version of the KLM system **M**. However, $\mathbf{M_{rev}}$ does not satisfy Consistency, and is therefore, strictly speaking, not a system for explanatory reasoning. Using rules discussed in the previous chapter, a more restrictive version of $\mathbf{M_{rev}}$, called **EM**, is defined and characterised.

7. *Rule systems for conjectural reasoning*

Reverse deductive consequence relations

This subsection presents a reversed version of the KLM system **M**, defined in such a way as to facilitate as much as possible the development of a system for strong explanatory reasoning in the next subsection. Since the latter is self-contained, the reader may want to skip the present subsection.

Given a set of models representing the implicit background theory, a reverse deductive consequence relation consists of those arguments of which the conclusion has at least the same explanatory power as the premisses in each of those models, where the explanatory power of a formula α in a model m is the set $\{\gamma \mid m \models \alpha \rightarrow \gamma\}$. This leads to the following definition.

> DEFINITION 7.1. A *reverse deductive structure* is a set $W \subseteq U$. The consequence relation it defines is denoted by $\mid\!\!\sim_W$ and is defined by: $\alpha \mid\!\!\sim_W \beta$ iff for every $m \in W$ and for every $\gamma \in L$: if $m \models \alpha \rightarrow \gamma$ then $m \models \beta \rightarrow \gamma$. A consequence relation is called reverse deductive iff it is defined by a reverse deductive structure.

For fixed m and γ the preservation condition boils down to $m \models \beta \rightarrow \alpha \vee \gamma$; quantifying over γ we obtain the equivalent condition $\alpha \mid\!\!\sim_W \beta$ iff for every $m \in W$: $m \models \beta \rightarrow \alpha$. This latter condition will be used in the proof of the representation theorem; however, as a definition the above formulation is preferred, because it expresses the idea of a preservation semantics more clearly.

The following system provides an axiomatisation of reverse deductive consequence relations; a formal proof of this statement follows the discussion of the main features of the system.

> DEFINITION 7.2. The system $\mathbf{M_{rev}}$ consists of the following axiom schema and inference rules:
>
> - **Reflexivity**: $\alpha \mid\!\!\sim \alpha$
> - **Predictive Incrementality**: $\dfrac{\models \alpha \wedge \gamma \rightarrow \beta \ , \ \alpha \mid\!\!\sim \gamma}{\beta \mid\!\!\sim \gamma}$
> - **Additivity**: $\dfrac{\alpha \mid\!\!\sim \gamma \ , \ \beta \mid\!\!\sim \gamma}{\alpha \wedge \beta \mid\!\!\sim \gamma}$
> - **Right Strengthening**: $\dfrac{\models \gamma \rightarrow \beta \ , \ \alpha \mid\!\!\sim \beta}{\alpha \mid\!\!\sim \gamma}$
> - **Conditionalisation**: $\dfrac{\alpha \mid\!\!\sim \beta \wedge \gamma}{\beta \rightarrow \alpha \mid\!\!\sim \gamma}$

$\mathbf{M_{rev}}$ can be readily obtained from the KLM system **M** (with some minor modifications) by applying the rewrite rule $\alpha \vdash \beta \Rightarrow \beta \mid\!\!\sim \alpha$. The first four rules have been discussed in the previous chapter; the intuitions behind Conditionalisation will be discussed in the next subsection.

LEMMA 7.3. *Every consequence relation satisfying the rules of* $\boldsymbol{M_{rev}}$ *is incremental, convex, disjunctively closed, and conjunctively closed.*
Proof. Predictive Incrementality implies Incrementality by Lemma 6.5.
By Lemma 6.9, Right Interval follows from Admissible Converse Entailment (which is an instance of Predictive Incrementality) and Left and Right Reflexivity (hence from Reflexivity).
In order to derive Right Or we will need the following rule:

- **Contraposition**: $\dfrac{\alpha \mid\!\sim \beta}{\neg\beta \mid\!\sim \neg\alpha}$

Contraposition can be derived as follows. Suppose $\alpha \mid\!\sim \beta$, then by Conditionalisation $\beta\rightarrow\alpha \mid\!\sim \mathtt{true}$, by Incrementality $\neg\alpha\rightarrow\neg\beta \mid\!\sim \mathtt{true}$, and by Right Strengthening $\neg\alpha\rightarrow\neg\beta \mid\!\sim \neg\alpha$. Furthermore, since by Reflexivity $\neg\alpha \mid\!\sim \neg\alpha$, we conclude by Additivity and Incrementality.
Now, to derive Right Or, suppose $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$, then by Contraposition $\neg\beta \mid\!\sim \neg\alpha$ and $\neg\gamma \mid\!\sim \neg\alpha$, by Additivity $\neg\beta\wedge\neg\gamma \mid\!\sim \neg\alpha$, and we conclude by Contraposition.
Right And follows from Right Strengthening. ■

The fact that reverse deductive consequence relations are conjunctively closed may seem surprising, but this is a consequence of the fact that an unsatisfiable formula entails everything. Indeed, we will see below that explanatory consequence relations are not conjunctively closed (they continue to be disjunctively closed, however). To see that $\mathbf{M_{rev}}$ does not satisfy Consistency, observe that for any unsatisfiable formula α we have $\models\alpha\rightarrow\neg\alpha$, but also $\alpha \mid\!\sim \alpha$ by Reflexivity.

In order to prove completeness of $\mathbf{M_{rev}}$ with respect to the set of reverse deductive structures we will need the following rule:

- **Converse Entailment**: $\dfrac{\models\beta\rightarrow\alpha}{\alpha \mid\!\sim \beta}$

To see that Converse Entailment is a derived rule of $\mathbf{M_{rev}}$, put $\alpha=\gamma$ in Incrementality and use Reflexivity.

The following theorem proves that reverse deductive structures characterise reverse deductive consequence relations.

THEOREM 7.4 (Representation theorem for reverse deductive consequence relations). *A consequence relation is reverse deductive iff it satisfies the rules of* $\boldsymbol{M_{rev}}$.
Proof. The only-if part involves demonstrating that, for a given reverse deductive structure W, the consequence relation $\mid\!\sim_W$ it defines satisfies the rules of $\mathbf{M_{rev}}$. This is quite trivial and will only be done for Predictive Incrementality. Suppose that $\models\alpha\wedge\gamma\rightarrow\beta$, i.e. for every $m\in U$: if $m \models \alpha$ and $m \models \gamma$ then $m \models \beta$. Furthermore, suppose that for every $m\in W$: $m \models \gamma\rightarrow\alpha$, i.e. if $m \models \gamma$ then $m \models \alpha$. Since $W\subseteq U$, this implies that for every $m\in W$, if $m \models \gamma$ then $m \models \beta$, i.e. $m \models \gamma\rightarrow\beta$.

For the if part, let $\mid\!\sim$ be an arbitrary consequence relation satisfying the rules of $\mathbf{M_{rev}}$, and consider the following reverse deductive structure:

$$W = \{m \in U \mid \text{for all } \alpha, \beta \text{ such that } \alpha \mid\!\sim \beta\text{: } m \models \beta \rightarrow \alpha\}$$

We will prove that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_W \beta$. The only-if part follows directly from the construction of W. Suppose that $\alpha \not\mid\!\sim \beta$, we will show that there exists a model $m_0 \in W$ that does not satisfy $\beta \rightarrow \alpha$.
Define $\Gamma_0 = \{\neg\alpha\} \cup \{\delta \mid \delta \mid\!\sim \beta\}$; we will first show that Γ_0 is satisfiable. Suppose not, then by compactness there is a finite $\Delta \subseteq \{\delta \mid \delta \mid\!\sim \beta\}$ such that $\models \Delta \rightarrow \alpha$[72], i.e. $\models \beta \rightarrow (\Delta \rightarrow \alpha)$; by Converse Entailment $\Delta \rightarrow \alpha \mid\!\sim \beta$. Furthermore, since $\delta \mid\!\sim \beta$ for $\delta \in \Delta$, we have $\Delta \mid\!\sim \beta$ by Additivity[73]. Using Additivity and Incrementality, we obtain $\alpha \mid\!\sim \beta$. Contradiction, so Γ_0 is satisfiable.
Let $m_0 \models \Gamma_0$; clearly $m_0 \not\models \alpha$ and, since by Reflexivity $\beta \in \Gamma_0$, $m_0 \models \beta$. We conclude that m_0 does not satisfy $\beta \rightarrow \alpha$; it remains to prove that $m_0 \in W$; i.e., that for all φ, ψ such that $\varphi \mid\!\sim \psi$ we have $m_0 \models \psi \rightarrow \varphi$. Suppose $\varphi \mid\!\sim \psi$, then by Right Strengthening $\varphi \mid\!\sim \psi \wedge \beta$, and by Conditionalisation $\psi \rightarrow \varphi \mid\!\sim \beta$; thus $\psi \rightarrow \varphi \in \Gamma_0$ and therefore $m_0 \models \psi \rightarrow \varphi$. ■

As indicated above, reverse deductive consequence relations are not consistent, since unsatisfiable formulas do count as explanations. Overcoming this defect leads us to the notion of strong explanatory reasoning.

Explanatory consequence relations

In this subsection I will define a system **EM** (explanatory reasoning wrt. a monotonic explanation mechanism) that satisfies Consistency but contains weaker versions of Reflexivity and Right Strengthening. The semantic characterisation of explanatory reasoning demonstrates that **EM** is strictly more restrictive than $\mathbf{M_{rev}}$.

I will start again with a semantic definition of explanatory reasoning. Like a reverse deductive structure, an explanatory structure is a set of models. However, an explanatory structure defines a different consequence relation, since not only must the conclusion of every argument have at least the same explanatory power as the premisses, but in addition the conclusion should be satisfiable.

DEFINITION 7.5. A *strong explanatory structure* is a set $W \subseteq U$. The consequence relation it defines is denoted by $\mid\!\sim_W$ and is defined by: $\alpha \mid\!\sim_W \beta$ iff (*i*) there is an $m \in W$ such that $m \models \beta$, and (*ii*) for every $m \in W$ and for every $\gamma \in L$: if $m \models \alpha \rightarrow \gamma$ then $m \models \beta \rightarrow \gamma$. A consequence relation is called strong explanatory iff it is defined by an explanatory structure.

[72] By a slight abuse of notation, Δ denotes both a finite set of formulas, and their conjunction.
[73] If $\Delta = \emptyset$, we have $\mathbf{true} \mid\!\sim \beta$ by Reflexivity and Convergence.

As said before, the adjective 'strong' will be often omitted if no confusion can arise. Furthermore, in the proofs of the lemmas below condition (*ii*) will be read as the equivalent condition (*ii'*) for every $m \in W$: $m \models \beta \rightarrow \alpha$.

As for an axiomatisation of explanatory consequence relations, we should note that the conclusion of an explanatory argument is required to be admissible. As has been argued in §23, this means that Reflexivity should be weakened in various ways. It turns out that only Explanatory Reflexivity will be among the rules defining the system **EM**: both Left and Right Reflexivity are derived rules. The other rule of $\mathbf{M_{rev}}$ that is not valid in **EM** is Right Strengthening. It is easy to see why this is so: if γ is inadmissible, then $\models \gamma \rightarrow \beta$, yet $\alpha \not\mid\sim \gamma$ for any α. We therefore add a condition stating that γ must be admissible, leading to the rule of Admissible Right Strengthening. Finally, a rule should be added to ensure the validity of Consistency. In the light of Lemma 6.8 it is sufficient to add Left Consistency. We thus obtain the following system for explanatory reasoning.

DEFINITION 7.6. The system **EM** consists of the following inference rules:

- **Explanatory Reflexivity**: $\dfrac{\alpha \mid\sim \alpha \ , \ \neg\beta \not\mid\sim \alpha}{\beta \mid\sim \beta}$
- **Predictive Incrementality**: $\dfrac{\models \alpha \wedge \gamma \rightarrow \beta \ , \ \alpha \mid\sim \gamma}{\beta \mid\sim \gamma}$
- **Additivity**: $\dfrac{\alpha \mid\sim \gamma \ , \ \beta \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$
- **Admissible Right Strengthening**: $\dfrac{\models \gamma \rightarrow \beta \ , \ \alpha \mid\sim \beta \ , \ \gamma \mid\sim \gamma}{\alpha \mid\sim \gamma}$
- **Conditionalisation**: $\dfrac{\alpha \mid\sim \beta \wedge \gamma}{\beta \rightarrow \alpha \mid\sim \gamma}$
- **Left Consistency**: $\dfrac{\alpha \mid\sim \beta}{\neg\alpha \not\mid\sim \beta}$

The significance of the rules of **EM** as properties of explanatory reasoning has been discussed in the previous chapter, with the exception of the rule of Conditionalisation. This rule can be best understood if one recalls a discussion from §9, where a distinction was drawn between two different representations of examples: the 'examples as implications' approach, and the 'examples as classifications' approach. In the former case, examples are ground implications with the description of an instance as antecedent, and a classification as consequent. In the latter case, the description of the instance belongs to the background theory, while the example comprises only the classification of the instance. Conditionalisation expresses that the former approach is as powerful as the latter: anything that can be induced by means of the 'examples as classifications' approach can also be induced by means of the 'examples as implications' approach[74]. It should be added that, in the case of strong explanatory reasoning, both approaches are actually

[74]Since the background theory is left implicit in our framework, the description of the instance (β) is added to the hypothesis rather than the background theory.

equivalent, since the following rule is a derived rule of **EM**:

- **Reverse of Conditionalisation**: $\dfrac{\beta\rightarrow\alpha \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \beta\wedge\gamma}$

The derivation of this rule is left to the reader.

The following lemma lists a few properties of strong explanatory reasoning.

LEMMA 7.7. *Every consequence relation satisfying the rules of **EM** is consistent, incremental, convex, and disjunctively closed, but not conjunctively closed.*

Proof. By Lemma 6.8, Left Consistency implies Consistency in the presence of Right Reflexivity and Admissible Converse Entailment (both instances of Predictive Incrementality).

Predictive Incrementality implies Incrementality.

By Lemma 6.9, Right Interval follows from Admissible Right Strengthening, Admissible Converse Entailment, and Left and Right Reflexivity. In order to show that Left Reflexivity is a derived rule of **EM**, suppose $\alpha \mathrel{|\!\sim} \beta$, then by Predictive Incrementality $\beta \mathrel{|\!\sim} \beta$. Furthermore, by Left Consistency $\neg\alpha \not\mathrel{|\!\sim} \beta$, and we conclude by Explanatory Reflexivity.

Right Or can be derived by means of the following rule (note that Contraposition is not valid in **EM**):

- **Admissible Contraposition**: $\dfrac{\alpha \mathrel{|\!\sim} \beta \ , \neg\alpha \mathrel{|\!\sim} \neg\alpha}{\neg\beta \mathrel{|\!\sim} \neg\alpha}$

To derive Admissible Contraposition, suppose $\alpha \mathrel{|\!\sim} \beta$, then by Conditionalisation $\beta\rightarrow\alpha \mathrel{|\!\sim} \mathbf{true}$, by Incrementality $\neg\alpha\rightarrow\neg\beta \mathrel{|\!\sim} \mathbf{true}$, and, since by assumption $\neg\alpha \mathrel{|\!\sim} \neg\alpha$, by Admissible Right Strengthening $\neg\alpha\rightarrow\neg\beta \mathrel{|\!\sim} \neg\alpha$. We conclude by Additivity and Incrementality.

In order to derive Right Or, first note that $\alpha \mathrel{|\!\sim} \beta$ implies $\beta \mathrel{|\!\sim} \beta$ by Right Reflexivity (an instance of Predictive Incrementality), hence $\beta\vee\gamma \mathrel{|\!\sim} \beta$ by Incrementality and $\beta\vee\gamma \mathrel{|\!\sim} \beta\vee\gamma$ by Left Reflexivity. Suppose $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$. Now, either $\neg\alpha \mathrel{|\!\sim} \neg\alpha$ or $\neg\alpha \not\mathrel{|\!\sim} \neg\alpha$; in the former case we can apply Admissible Contraposition to obtain $\neg\beta \mathrel{|\!\sim} \neg\alpha$ and $\neg\gamma \mathrel{|\!\sim} \neg\alpha$, hence $\neg\beta\wedge\neg\gamma \mathrel{|\!\sim} \neg\alpha$ by Additivity, and we conclude by Admissible Contraposition. On the other hand, if $\neg\alpha \not\mathrel{|\!\sim} \neg\alpha$ then $\alpha \mathrel{|\!\sim} \beta\vee\gamma$ by Explanatory Reflexivity. ■

The following derived rules of **EM** are used in the proof of the representation theorem: Incrementality, Right Reflexivity and Admissible Converse Entailment (instances of Predictive Incrementality), Consistent Right Strengthening (Lemma 6.11), and Consistency (Lemma 6.8).

To see that **EM** is not conjunctively closed, let p and q such that $\neg p \not\mathrel{|\!\sim} q$ and $q \mathrel{|\!\sim} q$, then by Consistent Right Strengthening $q \mathrel{|\!\sim} q\wedge p$. However, since $q \not\mathrel{|\!\sim} (q\wedge p)\wedge(q\wedge\neg p)$ by Consistency, Right And would give us $q \not\mathrel{|\!\sim} q\wedge\neg p$, which, together with $q \mathrel{|\!\sim} q$, results in $p \mathrel{|\!\sim} q$ by Consistent Right Strengthening. That is, adding Right And to **EM** would lead

to a situation in which every admissible hypothesis would either explain an arbitrary observation or its negation, which is clearly counterintuitive.

The characterisation of explanatory consequence relations requires a few steps more than in the case of reverse deductive reasoning. I start by proving that the rules of **EM** are sound with respect to the semantics defined by explanatory structures.

LEMMA 7.8 (Soundness of **EM**). *Any explanatory consequence relation satisfies the rules of **EM**.*
Proof. Let $W \subseteq U$ be an explanatory structure; we need to demonstrate that $\mid\!\sim_W$, as defined in Definition 7.5, satisfies the rules of **EM**. Since all the rules of **EM**, with the exception of Left Consistency, are valid rules of reverse deductive reasoning, we only need to check condition (*i*) for those rules. This is trivial for Predictive Incrementality, Additivity, Admissible Right Strengthening, and Conditionalisation.
For Explanatory Reflexivity, since $\alpha \mid\!\sim_W \alpha$ means that some model in W satisfies α, $\neg\beta \not\mid\!\sim_W \alpha$ implies that not all models in W satisfy $\alpha \rightarrow \neg\beta$, i.e. there is a model in W satisfying $\alpha \wedge \beta$ and hence β.
For Left Consistency, suppose that $m_0 \in W$ satisfies β, while all models in W satisfy $\beta \rightarrow \alpha$. It follows that m_0 satisfies α, hence not all models in W satisfy $\beta \rightarrow \neg\alpha$. ■

In order to prove completeness, we need to build an explanatory structure W from a given consequence relation $\mid\!\sim$ satisfying the rules of **EM**, such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_W \beta$. For non-empty consequence relations the construction of W is the same as for reverse deductive reasoning:

$$W = \{m \in U \mid \text{for all } \alpha, \beta \text{ such that } \alpha \mid\!\sim \beta\text{: } m \models \beta \rightarrow \alpha\}$$

The chief difference with reverse deductive reasoning is that every explanatory hypothesis is satisfiable in W.

LEMMA 7.9. *Let $\mid\!\sim$ be a consequence relation satisfying the rules of **EM**, and let W be defined as above. If $\alpha \mid\!\sim \beta$ then there is a model $m \in W$ such that $m \models \beta$.*
Proof. Let $\alpha \mid\!\sim \beta$; we will prove that $\{\beta\} \cup \{\delta \rightarrow \gamma \mid \gamma \mid\!\sim \delta\}$ is satisfiable. Suppose not, then by compactness there is a finite $\Delta \subseteq \{\delta \rightarrow \gamma \mid \gamma \mid\!\sim \delta\}$ such that $\models \beta \rightarrow \neg\Delta$. Furthermore, since $\varphi \mid\!\sim \psi$ for any $\psi \rightarrow \varphi \in \Delta$, we have $\psi \rightarrow \varphi \mid\!\sim \mathtt{true}$ for any $\psi \rightarrow \varphi \in \Delta$ by Conditionalisation, $\Delta \mid\!\sim \mathtt{true}$ by Additivity, and $\Delta \mid\!\sim \beta$ by Right Reflexivity and Admissible Right Strengthening. But then by Consistency $\not\models \beta \rightarrow \neg\Delta$, a contradiction. ■

Furthermore, we have that every inadmissible formula is unsatisfiable in W.

LEMMA 7.10. *Let $\mid\!\sim$ be a non-empty consequence relation satisfying the rules of **EM**, and let W be defined as above. If $\gamma \not\mid\!\sim \gamma$ then γ is unsatisfiable in W.*

Proof. Let $\alpha \nvdash \beta$, then $\mathbf{true} \nvdash \mathbf{true}$ by Incrementality and Left Reflexivity. Furthermore, if $\gamma \not\nvdash \gamma$ then $\neg\gamma \nvdash \mathbf{true}$ by Explanatory Reflexivity, hence $m \models \mathbf{true} \rightarrow \neg\gamma$ for every $m \in W$. ■

I will now show that W defines a consequence relation that is included in $\nvdash$.

LEMMA 7.11. *Let $\nvdash$ be a non-empty consequence relation satisfying the rules of **EM**, and let W be defined as above. If $\alpha \nvdash_W \beta$ then $\alpha \nvdash \beta$.*

Proof. Suppose that $\alpha \not\nvdash \beta$, we will show that either no model in W satisfies β, or there exists a model $m_0 \in W$ that does not satisfy $\beta \rightarrow \alpha$.

First of all, if $\beta \not\nvdash \beta$ then β is unsatisfiable in W according to Lemma 7.10. In the remainder of the proof we will assume that $\beta \nvdash \beta$. Define $\Gamma_0 = \{\neg\alpha\} \cup \{\delta \mid \delta \nvdash \beta\}$; we will first show that Γ_0 is satisfiable. Suppose not, then by compactness there is a finite $\Delta \subseteq \{\delta \mid \delta \nvdash \beta\}$ such that $\models \Delta \rightarrow \alpha$, i.e. $\models \beta \rightarrow (\Delta \rightarrow \alpha)$; by Admissible Converse Entailment $\Delta \rightarrow \alpha \nvdash \beta$ (recall that $\beta \nvdash \beta$). But by Additivity $\Delta \nvdash \beta$; using Additivity and Incrementality, we obtain $\alpha \nvdash \beta$. Contradiction, so Γ_0 is satisfiable.

Let $m_0 \models \Gamma_0$; clearly $m_0 \not\models \alpha$ and, since $\beta \in \Gamma_0$, $m_0 \models \beta$. It remains to prove that m_0 is in W; i.e., that for all φ, ψ such that $\varphi \nvdash \psi$ we have $m_0 \models \psi \rightarrow \varphi$. Let $\varphi \nvdash \psi$; if $\neg\beta \not\nvdash \psi$, then by Consistent Right Strengthening $\varphi \nvdash \psi \wedge \beta$, and by Conditionalisation $\psi \rightarrow \varphi \nvdash \beta$; thus $\psi \rightarrow \varphi \in \Gamma_0$ and therefore $m_0 \models \psi \rightarrow \varphi$. On the other hand, if $\neg\beta \nvdash \psi$ then by Conditionalisation and Incrementality $\beta \rightarrow \neg\psi \nvdash \mathbf{true}$, by Admissible Right Strengthening $\beta \rightarrow \neg\psi \nvdash \beta$, and by Additivity and Incrementality $\neg\psi \nvdash \beta$; thus $\neg\psi \in \Gamma_0$ and therefore $m_0 \not\models \psi$, hence $m_0 \models \psi \rightarrow \varphi$. ■

Armed with the previous three lemmas we can prove the completeness of **EM**.

THEOREM 7.12 (Representation theorem for explanatory consequence relations). *A consequence relation is explanatory iff it satisfies the rules of **EM**.*

Proof. The only-if part is Lemma 7.8. For the if part, let $\nvdash$ be an arbitrary non-empty consequence relation satisfying the rules of **EM**, and let

$$W = \{m \in U \mid \text{for all } \alpha, \beta \text{ such that } \alpha \nvdash \beta\text{: } m \models \beta \rightarrow \alpha\}$$

Suppose $\alpha \nvdash \beta$, then by the construction of W, $m \models \beta \rightarrow \alpha$ for all $m \in W$. Furthermore, by Lemma 7.9 there is a model in W satisfying β. We may conclude that $\alpha \nvdash_W \beta$. Conversely, if $\alpha \nvdash_W \beta$ then Lemma 7.11 proves that $\alpha \nvdash \beta$. We conclude that W defines a consequence relation that is exactly $\nvdash$. For an empty consequence relation put $W = \varnothing$. ■

We may note, to round off our discussion of strong explanatory reasoning, that this semantic characterisation of **EM** clearly demonstrates that strong explanatory reasoning is strictly more restrictive than reverse deductive reasoning, in the following sense (recall also the discussion of comparison criteria for different forms of reasoning in §16). A reverse deductive reasoner and a strong explanatory reasoner build their respective

consequence relation, if they share the same background knowledge, from the same set of models $W \subseteq U$. The resulting consequence relations differ in the following respect: for every formula β that is unsatisfiable in W, the reverse deductive reasoner will include arguments $\alpha \mid\!\sim \beta$ for arbitrary $\alpha \in L$, while the explanatory reasoner will include none of these (notice that such an α always exists, *viz.* **`false`**). The two reasoners will agree on all other arguments. So the strong explanatory restriction of a reverse deductive consequence relation is always a proper subset of the latter.[75]

This concludes the investigations into the formal properties of strong explanatory reasoning. Before developing rule systems for confirmatory reasoning in the next sections, I will now spend a few words on weaker forms of explanatory reasoning.

Weaker notions of explanation

If $\vdash$ is not transitive, then there exist α, β and γ such that $\alpha \vdash \beta$ and $\beta \vdash \gamma$, yet $\alpha \nvdash \gamma$, i.e. $Cn_{\vdash}(\beta) \not\subseteq Cn_{\vdash}(\alpha)$. As has been proved by Kraus *et. al*, transitivity and monotonicity are equivalent in the presence of the rules of **C**, the weakest rule system for plausible reasoning. The upshot is that for plausible consequence relations, the nice equivalence between $Cn_{\vdash}(\alpha) \subseteq Cn_{\vdash}(\beta)$ and $\beta \vdash \alpha$, as expressed by Lemma 5.2, breaks down. However, sometimes we need a plausible explanation mechanism, for instance if we want to induce default rules with exceptions (*cf.* Bain & Muggleton, 1991).

It seems that we have two options for formalising induction of such weak explanations. One option, which is left for future research, is to define $\alpha \mid\!\sim \beta$ iff $Cn_{\vdash}(\alpha) \subseteq Cn_{\vdash}(\beta)$, and to investigate how, if at all, properties of $\vdash$ carry over to $\mid\!\sim$. The other option, that has been investigated to some extent in (Flach, 1991), is to put $\alpha \mid\!\sim \beta$ iff $\beta \vdash \alpha$ (and β consistent). Again applying the rewrite rule $\alpha \vdash \beta \Rightarrow \beta \mid\!\sim \alpha$ to rules of KLM, it is easily shown that Incrementality and Additivity of $\mid\!\sim$ correspond to Right Weakening and Right And of $\vdash$, respectively. Since both latter rules are satisfied in the weakest KLM system **C**, it seems safe to assume that even with plausible explanation mechanisms we have Incrementality and Additivity. However, it can also be shown that Right Strengthening of $\mid\!\sim$ corresponds to Monotonicity of $\vdash$. In other words, when strengthening a given plausible explanation one may reach a 'hole' in the Version Space: a hypothesis that is in between the *S* and *G* sets, yet does not explain the examples. Further research is needed to characterise the implications of this observation.

§27. REGULARITY-BASED CONFIRMATORY REASONING

In this section and the next one I will consider various formalisations of confirmatory reasoning. The idea underlying the formalisation in this section has essentially been stated in §19: to consider as possible hypotheses those formulas that are satisfied by every

[75] As a consequence, no strong explanatory consequence relation coincides with a reverse deductive consequence relation. This contrasts with e.g. the relation between monotonic and preferential reasoning, since every monotonic consequence relation is preferential. This phenomenon can be traced back to the inclusion of Left Consistency in **EM** (all other rules of **EM** are valid in $\mathbf{M}_{\mathbf{rev}}$): this rule, like its relatives Consistency and Right Consistency, expresses that certain arguments should be excluded from the consequence relation.

regular interpretation, where a regular interpretation is one in which the objects behave in a way similar to the observed objects. The resulting form of confirmatory reasoning is called *regularity-based* confirmatory reasoning. Various possible notions of regularity come to mind. For instance, it could be established by a partition on the set of interpretations, such that an interpretation is regular if it falls in the same equivalence class as a model of the observations. As has been indicated in §19, such a notion of regularity would invalidate Hempel's conditions (C1) and (C2) — since Hempel's views formed the starting point for my investigations into confirmatory induction, I have chosen to develop a notion of regularity that remains more faithful to his ideas.

In proposing this notion I do not make any claim to originality: it can be directly traced back to Helft and De Raedt on the one hand, and Kraus *et al*. on the other. The idea is to use a partial ordering on interpretations, and to consider as regular interpretations those models of the observations that are minimal with respect to this ordering. Such a preference ordering on interpretations seems very natural for plausible reasoning — but is it also natural for conjectural reasoning? One could raise the following objections:

(*i*) regularity is a property of interpretations, not an ordering relation between interpretations;

(*ii*) even if regularity is an ordering between interpretations, this ordering must depend on the observations.

One possible defence against these objections is to point at the truth-ordering employed by Helft and De Raedt. This ordering selects as regular interpretations those in which no objects other than the observed ones have the attributed properties, which seems very reasonable. Nevertheless, each of the two points above makes some sense and deserves further investigation.

Below I give a characterisation of regularity-based confirmatory reasoning on the basis of such a preference ordering by means of so-called preferential confirmatory consequence relations. This system is a variation of the KLM system **P**, the main difference being the added requirement of consistency of the observations. After that I will demonstrate how closed-world reasoning on the basis of the truth-ordering of interpretations fits into the framework of preferential confirmatory reasoning.

Preferential confirmatory consequence relations

I will start with a semantic definition of preferential confirmatory reasoning. This definition establishes a close variant of KLM's notion of a preferential structure (see Definition 4.8), the difference being that only satisfiable formulas are allowed in confirmatory arguments.

> DEFINITION 7.13. A *preferential confirmatory structure* is a triple $W = \langle S,l,<\rangle$, where S is a set of *states*, l: $S\rightarrow U$ is a function that labels every state with a model, and $<$ is a strict partial order[76] on S, called the *preference ordering*, that is smooth[77]. A state $s\in S$ satisfies a formula $\alpha\in L$ iff $l(s) \models \alpha$; the set of states satisfying α is denoted by $[\alpha]$, and a minimal

[76] I.e., $<$ is irreflexive and transitive.

[77] I.e. for any $S'\subseteq S$ and for any $s\in S'$, either s is minimal in S', or there is a $t\in S'$ such that $t<s$ and t is minimal in S'. This condition is satisfied if $<$ does not allow infinite descending chains.

element of $[\alpha]$ (wrt. $\prec$) will be called a *regular* state for α. The consequence relation defined by W is denoted by $\mid\!\sim_W$ and is defined by: $\alpha \mid\!\sim_W \beta$ iff (*i*) there is a state $s \in S$ satisfying α, and (*ii*) every regular state for α satisfies β. A consequence relation is called preferential confirmatory iff it is defined by a preferential confirmatory structure.

States are labelled with models, so the preference ordering between states can be used to define a relation between models — however, since the same model can label several states, this relation will not, in general, be a partial order. According to Kraus *et al.*, the additional freedom provided by states 'is vital for the representation theorem to hold' (p.181), and I will follow them in this respect.

The following set of rules will be proved to axiomatise preferential confirmatory consequence relations.

DEFINITION 7.14. The system **CP** consists of the following inference rules:

- **Confirmatory Reflexivity**: $\dfrac{\alpha \mid\!\sim \alpha \ , \ \alpha \not\mid\!\sim \neg\beta}{\beta \mid\!\sim \beta}$
- **Left Logical Equivalence**: $\dfrac{\models \alpha \leftrightarrow \beta \ , \ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$
- **Predictive Right Weakening**: $\dfrac{\models \alpha \wedge \beta \rightarrow \gamma \ , \ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$
- **Cautious Monotonicity**: $\dfrac{\alpha \mid\!\sim \beta \ , \ \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$
- **Right And**: $\dfrac{\alpha \mid\!\sim \beta \ , \ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$
- **Left Or**: $\dfrac{\alpha \mid\!\sim \gamma \ , \ \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$
- **Right Consistency**: $\dfrac{\alpha \mid\!\sim \beta}{\alpha \not\mid\!\sim \neg\beta}$

In comparison with Hempelian consequence relations (Definition 6.12) two rules are added: Left Or and Cautious Monotonicity, both of which are valid principles of preferential reasoning. As argued in §24, Cautious Monotonicity can be seen as a strengthening of Verification, which states that if α confirms γ, then any predicted observation β provides further confirming evidence — Cautious Monotonicity extends this to any β that is also confirmed by α. The rule of Left Or states that if both α and β provide confirming evidence for γ, the knowledge that at least one of them is true should not refute γ. As will be demonstrated below (Lemma 7.16), both of these rules make use of the fact that regular states are minimal elements of the preference ordering — in other words, choosing another mechanism to select regular states would most probably violate both Cautious Monotonicity and Left Or. Note also that without these two rule the system would be rather weak: the other rules (with the exception of Left Logical

Equivalence) only say something about the right-hand side of confirmatory arguments.

The following lemma gathers some further properties of **CP**.

LEMMA 7.15. *Every consequence relation satisfying the rules of **CP** is consistent, convex, disjunctively and conjunctively closed.*

Proof. By Lemma 6.7, Right Consistency implies Consistency in the presence of Admssible Entailment and Left Reflexivity (which are both instances of Predictive Right Weakening).

Right Interval and Right Or are implied by Right Weakening, hence by Predictive Right Weakening.

Right And is a rule of **CP**. ■

To see that **CP** is not incremental, let p and q be such that $p \nvdash q$ and $\models p \rightarrow q$. Cautious Monotonicity yields $p \wedge q \nvdash q$, and Predictive Right Weakening gives $p \wedge q \nvdash p$. Incrementality would give $q \nvdash p$, which means that p and q have the same 'confirmatory power', which defies our intuitions. Alternatively, it is easy to show that Incrementality is invalid by constructing an appropriate preferential confirmatory structure. For instance, let S have two states s and t with $s<t$, and let $[p]=\{t\}$ and $[q]=\{s,t\}$, then $p \wedge q \nvdash p$ but $q \not\nvdash p$.

I will now prove the validity of the rules of **CP**.

LEMMA 7.16 (Soundness of **CP**). *Any preferential confirmatory consequence relation satisfies the rules of **CP**.*

Proof. For Confirmatory Reflexivity, suppose $[\alpha]$ is non-empty, and not all regular states for α satisfy $\neg\beta$; it follows that some state in S satisfies β.

For Left Logical Equivalence, notice that logically equivalent formulas are satisfied by the same states.

For Predictive Right Weakening, if all regular states for α satisfy β and $\models \alpha \wedge \beta \rightarrow \gamma$, then (since all regular states for α satisfy α) all regular states for α satisfy γ.

For Cautious Monotonicity, we need the fact that $<$ is a smooth partial order. Suppose that $[\alpha]$ is non-empty, and all regular states for α satisfy β and γ — clearly, $[\alpha \wedge \beta]$ is non-empty. Now, let s be regular for $\alpha \wedge \beta$, then $s \in [\alpha]$; I will prove that s is regular for α. Suppose not, then there is a $t \in [\alpha]$ such that $t<s$ and t is regular for α. Now, every state regular for α satisfies β, hence $t \in [\alpha \wedge \beta]$. But this contradicts the minimality of s in $[\alpha \wedge \beta]$, hence s is regular for α and thus satisfies γ.

For Right And, if all regular states for α satisfy β and γ, then clearly they satisfy $\beta \wedge \gamma$.

For Left Or, note that $[\alpha \vee \beta] = [\alpha] \cup [\beta]$; thus, if $[\alpha]$ and $[\beta]$ are non-empty then so is $[\alpha \vee \beta]$. Furthermore, the set of regular states for $\alpha \vee \beta$ is a subset of the union of the regular states for α and β, since a state cannot be minimal in $[\alpha \vee \beta]$ without being minimal in at least one of $[\alpha]$ and $[\beta]$.

For Right Consistency, suppose $[\alpha]$ is non-empty, and all regular states for α satisfy β; it follows that no regular state for α satisfies $\neg\beta$. ■

In order to prove completeness, we need to build a preferential confirmatory structure W from a given consequence relation $\mid\!\sim$ satisfying the rules of **CP**, such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_W \beta$. As in the case of explanatory structures, such a confirmatory structure is built from a specific set of models. These models are selected relative to a given formula, as follows.

> DEFINITION 7.17. Let $\mid\!\sim$ be a conjectural consequence relation. The model $m \in U$ is said to be *normal for* α iff for all β in L such that $\alpha \mid\!\sim \beta$, $m \models \beta$.

So, a model is normal for a formula if it satisfies every confirmed hypothesis. Thus, given certain evidence the set of normal models decreases when the set of confirmed hypotheses increases. Notice that every model in U is normal for an inadmissible formula, which is therefore not satisfied by some of its normal models. An admissible formula is satisfied by every normal model, however. Notice also that if α is admissible and γ is inadmissible, then by Confirmatory Reflexivity $\alpha \mid\!\sim \neg\gamma$, hence no normal model for α satisfies γ.

The set of models normal for admissible formulas will be used to build a preferential confirmatory structure. The following lemma states the key result about normal models: they can characterise arbitrary Hempelian consequence relations.

> LEMMA 7.18. *Suppose a consequence relation $\mid\!\sim$ satisfies Right Weakening and Right And, and let α be an admissible formula. All normal models for α satisfy β iff $\alpha \mid\!\sim \beta$.*
> *Proof.* The if part follows from Definition 7.17.
> For the only-if part, suppose $\alpha \mid\!\sim \alpha$ and $\alpha \not\mid\!\sim \beta$; I will show that there is a normal model for α that does not satisfy β. Let $\Gamma_0 = \{\neg\beta\} \cup \{\delta \mid \alpha \mid\!\sim \delta\}$; it suffices to show that Γ_0 is satisfiable. Suppose not, then by compactness there is a finite $\Delta \subseteq \{\delta \mid \alpha \mid\!\sim \delta\}$ such that $\models \Delta \rightarrow \beta$, i.e. $\models \alpha \rightarrow (\Delta \rightarrow \beta)$; by Right Weakening $\alpha \mid\!\sim \Delta \rightarrow \beta$. But by Right And $\alpha \mid\!\sim \Delta$; using Right And and Right Weakening we obtain $\alpha \mid\!\sim \beta$, a contradiction. ■

Notice from the proof of Lemma 7.18 that normal models exist for any admissible α.

Given an arbitrary preferential confirmatory consequence relation $\mid\!\sim$, the completeness proof is based on a preferential confirmatory structure $W = \langle S, l, < \rangle$ constructed as follows:

(1) $S = \{\langle m, \alpha \rangle \mid \alpha$ is an admissible formula, and m is a normal model for $\alpha\}$;
(2) $l(\langle m, \alpha \rangle) = m$;
(3) $\langle m, \alpha \rangle < \langle n, \beta \rangle$ iff $\alpha \vee \beta \mid\!\sim \alpha$ and $m \not\models \beta$.

Thus, states are pairs of admissible formulas and normal models. The labelling function simply maps a state to the model it contains. Condition (3) defines the preference ordering between states: note that $\beta \mid\!\sim \alpha$ is a special case of $\alpha \vee \beta \mid\!\sim \alpha$ by means of Left Or, and the fact that α is admissible. The condition $m \not\models \beta$ is added to make the ordering irreflexive; note that as a consequence any $\langle m, \alpha \rangle \in S$ is minimal in $[\alpha]$.

The main difference between the preferential consequence relations of Kraus *et al.* and my preferential confirmatory consequence relations is the way unsatisfiable formulas are treated. In the KLM framework unsatisfiable formulas are characterised by the fact that they have every formula in L as a plausible consequence, which means that they don't

have normal models. In my framework, unsatisfiable formulas confirm no hypotheses, and have all models in U as normal models. In both cases, the structure W that is used to prove completeness contains only satisfiable formulas in its states. This means that we can replicate most of KLM's results about the structure W.

PROPOSITION 7.19. *(1) (KLM 5.13) The relation $\prec$ is a strict partial order.*
(2) (KLM 5.15) The relation $\prec$ is smooth: for any $s \in [\alpha]$, either s is minimal in $[\alpha]$ or there exists a state $t \prec s$ minimal in $[\alpha]$.
(3) (KLM 5.11) If $\alpha \vee \beta \mathrel{|\!\sim} \alpha$ and m is a normal model for α that satisfies β, then m is a normal model for β.
(4) (KLM 5.14) $\langle m, \alpha \rangle$ is minimal in $[\beta]$ iff $m \models \beta$ and $\alpha \vee \beta \mathrel{|\!\sim} \alpha$.

The first two statements express that W is a preferential confirmatory structure. The remaining two are used in the proof of the following lemma.

LEMMA 7.20. *Let $\mathrel{|\!\sim}$ be a consequence relation satisfying the rules of **CP**, and let W be defined as above. If $\alpha \mathrel{|\!\sim} \beta$ then $\alpha \mathrel{|\!\sim}_W \beta$.*
Proof. Suppose that $\alpha \mathrel{|\!\sim} \beta$; we will show that (*i*) there is a state in S satisfying α, and (*ii*) every minimal state in $[\alpha]$ satisfies β.
(*i*) By Left Reflexivity α is admissible; furthermore, by Right Consistency $\alpha \not\mathrel{|\!\sim} \neg\beta$, so by Lemma 7.18 there exists a model m normal for α. We conclude that $\langle m, \alpha \rangle \in [\alpha]$.
(*ii*) Suppose $\langle n, \gamma \rangle$ is minimal in $[\alpha]$, then γ is an admissible formula, n is a normal model for γ that satisfies α, and $\gamma \vee \alpha \mathrel{|\!\sim} \gamma$ by Proposition 7.19 (4). By Proposition 7.19 (3) n is a normal model for α, hence $n \models \beta$. ■

The following lemma proves the converse of Lemma 7.20, and completes the proof of the representation theorem.

LEMMA 7.21. *Let $\mathrel{|\!\sim}$ be a consequence relation satisfying the rules of **CP**, and let W be defined as above. If $\alpha \mathrel{|\!\sim}_W \beta$ then $\alpha \mathrel{|\!\sim} \beta$.*
Proof. Suppose $\alpha \mathrel{|\!\sim}_W \beta$, then α must be admissible (since no state in S satisfies an inadmissible formula). Furthermore, given any model m normal for α, $\langle m, \alpha \rangle$ is minimal in $[\alpha]$, hence m satisfies β, and the conclusion follows by Lemma 7.18. ■

We may now summarise.

THEOREM 7.22 (Representation theorem for preferential confirmatory consequence relations). *A consequence relation is preferential confirmatory iff it satisfies the rules of **CP**.*
Proof. The only-if part is Lemma 7.16. For the if part, let $\mathrel{|\!\sim}$ be a consequence relation satisfying the rules of **CP** and let W be defined as above. Lemmas 7.20 and 7.21 prove that $\alpha \mathrel{|\!\sim} \beta$ iff $\alpha \mathrel{|\!\sim}_W \beta$, i.e. $\mathrel{|\!\sim}$ is preferential confirmatory. ■

The rule system **CP** demonstrates that Hempel's adequacy conditions for confirmation

can be extended to a complete axiomatisation; the semantics of preferential confirmatory structures provide an operationalisation of the relation of confirmation. I will now show that closed-world reasoning *à la* Helft and De Raedt fits nicely into this framework.

Closed-world reasoning as preferential confirmatory reasoning

The concept of closed-world reasoning is borrowed from logic programming, where a logic program consisting of definite clauses can only have positive literals among its ground atomic consequences: it can't say that some ground atom is false. Consequently, the Herbrand base of ground atoms is divided into two subsets: those that are logical consequences of the program (these are true), and those that are not (the truthvalues of these are unknown). The well-known Closed-World Assumption (CWA) now proposes to interpret the latter as being actually false. In the lattice of Herbrand models of the program, this amounts to taking the bottom element of this lattice as the intended model.

The ordering in this lattice is the ordering of truth-content: one Herbrand model is smaller than another if the set of ground atoms considered true in the first model is a subset of those considered true in the second model[78]. This truth-ordering provides a link with the preferential confirmatory structures discussed above.

LEMMA 7.23. *The consequence relation established by the truth-minimal model semantics is preferential confirmatory.*
Proof. Such a consequence relation is defined by the following preferential confirmatory structure: take the set of Herbrand interpretations for S, the identity function for l, and the proper subset ordering for $<$[79]. ■

This means that the truth-minimal model semantics inherits all the properties of preferential confirmatory consequence relations. Note that in the case of an indefinite program this semantics would require truth in **all** minimal models of the program: there is no sophisticated treatment of negation in the body of clauses.

When restricted to definite programs, the truth-minimal model semantics also enjoys some properties not shared with every form of preferential confirmatory reasoning.

LEMMA 7.24. *The consequence relation established by the truth-minimal model semantics for definite clauses satisfies the following property:*

- **Admissible Completeness**: $\dfrac{\alpha \not\mid\sim \neg\beta \ , \ \alpha \mid\sim \alpha}{\alpha \mid\sim \beta}$

Proof. If α is admissible it is satisfied by some state, hence it has a unique minimal state labelled by a single Herbrand model, in which every formula is either true or false. ■

Notice that Admissible Completeness is satisfied by every preferential confirmatory structure in which the set of states satisfying a formula forms a downward semilattice under the preference ordering.

[78] If we identify a (two-valued) Herbrand interpretation with the set of ground atoms it assigns the truthvalue **`true`**, then this ordering coincides with the subset ordering.

[79] Thus, the distinction between states and models is not needed for modelling minimal Herbrand model semantics as a preferential structure.

7. Rule systems for conjectural reasoning

In order to characterise the main feature of the truth-minimal model semantics, namely that it minimises the assignment of truth to ground atoms, we could try something like

- **Negation as Failure**: $$\frac{\alpha \nvdash \mathtt{p}}{\alpha \mid\!\sim \neg \mathtt{p}}$$

in which α is a definite program and $\mathtt{p}$ is a positive literal. However, note that also $\alpha \nvdash \neg \mathtt{p}$, since definite programs only have **positive** ground atomic consequences, while $\alpha \not\mid\!\sim \mathtt{p}$ by Right Consistency — in other words, the rule of Negation as Failure is invalid if $\mathtt{p}$ is a negative literal. A complete axiomatisation of the truth-minimal model semantics thus calls for a more fine-grained tool than consequence relations, that operate on the complete language. The interested reader is referred to (Dix, 1994ab) for a possible approach.

§28. CONSISTENCY-BASED CONFIRMATORY REASONING

The preference ordering in preferential confirmatory structures picks out certain models of the premisses, and draws conclusions justified by those models. When the evidence is incomplete, as is usually the case, the intended model may be not among those deemed most regular by the preference ordering. If this becomes evident by further observations, previously refuted hypotheses will have to be reconsidered. In this section I will demonstrate that it is possible to avoid such non-incremental behaviour. The main idea is to keep track of all models of the premisses α, and to consider a hypothesis β to be refuted ($\alpha \not\mid\!\sim \beta$) only if β is satisfied by none of the models of α. Thus, we switch from entailment over preferred models of the observations to consistency relative to all models of α. The resulting form of confirmatory reasoning is therefore termed *consistency-based*.

The form of reasoning just described is characterised below by the system **CW**, for *weak* confirmatory reasoning. It defines 'α confirms β' as 'β is compatible with α', which is clearly the weakest possible definition of confirmation. Moreover, the system is also related to the rule systems **EM** and **CP** considered previously, since each of these is strictly more restrictive than **CW**: every explanatory or preferential confirmatory argument is also weak confirmatory, but not *vice versa* (with fixed background knowledge). Thus, the system **CW** represents the root of our hierarchy of rule systems for conjectural reasoning.

An alternative characterisation of consistency-based confirmatory reasoning is also (partly) worked out in this section. Here the idea is to represent the indeterminacy of the observations not by a large set of models, but instead by a few well-chosen **partial** or three-valued models, namely those partial models that are minimal with respect to the information they contain. We thus see that the concept of a minimal model again plays a role, but here its connotation is quite different from the preferential setting, where the minimal models represent educated guesses for the intended model. In contrast, the set of information-minimal models implies that the intended model is at least as informed as one of them. This approach has been inspired by the Version Space model of concept learning (§9), where the *S*-set of most specific concepts plays a similar role in delineating the set of all concepts consistent with the examples.

§28. Consistency-based confirmatory reasoning

Weak confirmatory reasoning

As mentioned above, in weak confirmatory reasoning a hypothesis is confirmed if it is compatible with the observations.

> DEFINITION 7.25. A *weak confirmatory structure* is a set $W \subseteq U$. The consequence relation it defines is denoted by $\mid\!\sim_W$ and is defined by: $\alpha \mid\!\sim_W \beta$ iff there is an $m \in W$ such that $m \models \alpha \wedge \beta$. A consequence relation is called weak confirmatory iff it is defined by a weak confirmatory structure.

From this definition it is clear that weak confirmatory consequence relations satisfy both Right Weakening and Left Weakening (i.e. Incrementality), as well as Consistency. One additional rule is needed.

> DEFINITION 7.26. The system **CW** consists of the following rules:
>
> - **Predictive Incrementality**: $\dfrac{\models \alpha \wedge \gamma \rightarrow \beta \ , \ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$
> - **Predictive Right Weakening**: $\dfrac{\models \alpha \wedge \beta \rightarrow \gamma \ , \ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$
> - **Disjunctive Rationality**: $\dfrac{\alpha \vee \beta \mid\!\sim \gamma \ , \ \beta \not\mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$
> - **Consistency**: $\dfrac{\alpha \mid\!\sim \beta}{\not\models \beta \rightarrow \neg\alpha}$

Disjunctive Rationality has not been considered before. The name has been borrowed from Kraus *et al*., who identify it as a valid principle of plausible reasoning (although it is not a derived rule of their system **P**, nor of my system **CP**). In the context of confirmatory reasoning, Disjunctive Rationality is a rather strong rule, which states that if a hypothesis is confirmed by disjunctive observations it is confirmed by at least one of the disjuncts.

Every consequence relation satisfying the rules of **CW** is confirmatory, but not Hempelian.

> LEMMA 7.27. *Every consequence relation satisfying the rules of **CW** is consistent, incremental, convex, and disjunctively closed.*
> *Proof*. Consistency is a rule of **CW**.
> Predictive Incrementality implies Incrementality by Lemma 6.5.
> Right Interval and Right Or are implied by Right Weakening, hence by Predictive Right Weakening. ■

To see that **CW** is not conjunctively closed, suppose $\alpha \mid\!\sim \beta$, then (since $\alpha \not\mid\!\sim \beta \wedge \neg\beta$ by Consistency) Right And would imply $\alpha \not\mid\!\sim \neg\beta$. However, it is easy to find formulas α and β such that both $\alpha \wedge \beta$ and $\alpha \wedge \neg\beta$ are consistent.

The following theorem proves the equivalence of weak confirmatory structures and the system **CW**.

THEOREM 7.28 (Representation theorem for weak confirmatory consequence relations). *A consequence relation* $\mid\!\sim$ *is weak confirmatory iff it satisfies the rules of* ***CW***.

Proof. The only-if part involves demonstrating that $\mid\!\sim_W$, as defined in Definition 7.25, satisfies the rules of **CW**, which is trivial.

For the if part, let $\mid\!\sim$ be an arbitrary consequence relation satisfying the rules of **CW**, and consider the following weak confirmatory structure:

$$W = \{m \in U \mid \text{for all } \alpha,\beta \in L \text{ such that } m \models \alpha \wedge \beta\colon\ \alpha \mid\!\sim \beta\}$$

We will prove that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_W \beta$. The if part follows directly from the construction of W. Suppose that $\alpha \mid\!\sim \beta$, we will show that there exists a model $m_0 \in W$ that satisfies $\alpha \wedge \beta$.

Define $\Gamma_0 = \{\alpha\} \cup \{\delta \mid \neg\delta \not\mid\!\sim \beta\}$; we will first show that Γ_0 is satisfiable. Suppose not, then by compactness there is a finite $\Delta \subseteq \{\delta \mid \neg\delta \not\mid\!\sim \beta\}$ such that $\models \neg(\Delta \wedge \alpha)$, i.e. $\models \beta \rightarrow \neg(\Delta \wedge \alpha)$; by Consistency $\Delta \wedge \alpha \not\mid\!\sim \beta$. Furthermore, since $\neg\delta \not\mid\!\sim \beta$ for $\delta \in \Delta$, we have $\neg\Delta \not\mid\!\sim \beta$ by Disjunctive Rationality and $\neg\Delta \wedge \alpha \not\mid\!\sim \beta$ by Incrementality. Combining this with $\Delta \wedge \alpha \not\mid\!\sim \beta$ we obtain $(\neg\Delta \wedge \alpha) \vee (\Delta \wedge \alpha) \not\mid\!\sim \beta$ by Disjunctive Rationality and $\alpha \not\mid\!\sim \beta$ by Incrementality. Contradiction, so Γ_0 is satisfiable.

Let $m_0 \models \Gamma_0$; clearly $m_0 \models \alpha$ and, since by Consistency $\beta \in \Gamma_0$, $m_0 \models \beta$. It remains to prove that $m_0 \in W$; i.e., that for all φ, ψ such that $m_0 \models \varphi \wedge \psi$ we have $\varphi \mid\!\sim \psi$. Let $m_0 \models \varphi \wedge \psi$, then $\neg(\varphi \wedge \psi) \notin \Gamma_0$, hence $\varphi \wedge \psi \mid\!\sim \beta$; by Predictive Right Weakening $\varphi \wedge \psi \mid\!\sim \psi \wedge \beta$, by Incrementality $\varphi \mid\!\sim \psi \wedge \beta$, and by Right Weakening $\varphi \mid\!\sim \psi$. ■

One may remark that we could define a preferential variant of weak confirmatory reasoning by including a preference ordering on models and defining a hypothesis to be confirmed if it is satisfied by at least one of the minimal models of the observations. This will be worked out below for the special case of the information ordering on partial models.

Consistency-based confirmatory reasoning with partial models

I will start with a brief review of partial logic. A partial interpretation is a function from L to {**true**,**unknown**,**false**}[80]. The symbol $\models$ will be used for satisfaction by partial interpretations; thus, $m \models \alpha$ stands for $m(\alpha)$=**true** (m *verifies* α), $m \models \neg\beta$ stands for $m(\beta)$=**false** (m *falsifies* β), $m \not\models \neg\gamma$ stands for $m(\gamma) \in$ {**unknown**,**true**}, and so on. U is now a set of partial models, representing the background knowledge. Furthermore, $\alpha \models \beta$ stands for $\forall m \in U$: if $m \models \alpha$ then $m \models \beta$.

Since there are now three truth-values, the truth-tables for the logical connectives need to be extended. The enlarged truth-tables can be derived from the intended interpretation of the third truth-value **unknown**, which represents lack of information as to whether a formula is true or false — we might say that the 'information-content' of **unknown** is less than either **true** or **false**. A compound formula is assigned the truth-value

[80]Or equivalently, a *partial* function from L to {**true**,**false**}.

unknown if the two ways in which the truth-value of an undetermined component could be increased result in different truth-values of the formula, which prompts us to put $m(\alpha\rightarrow\beta)$=**unknown** if for instance $m(\alpha)$=**unknown** and $m(\beta)$=**false**. On the other hand, we put $m(\alpha\wedge\beta)$=**false** in this case, since this is the truth-value obtained regardless whether $m(\alpha)$ is increased to **true** or **false**. That is, we adopt Kleene's strong three-valued connectives (see Turner, 1984, Ch.3 for two alternative interpretations of the third truth-value and their associated truth-tables).

An important observation is that there are no tautologies under this regime: for instance, $p \not\models q\vee\neg q$, since if $m(q)$=**unknown** then $m(q\vee\neg q)$=**unknown**. This also means that the Deduction Theorem is invalid: for instance, we have $p \models p$, but $\not\models p\rightarrow p$. One half of the Deduction Theorem is valid, however: from $\alpha \models \beta\rightarrow\gamma$ we may conclude $\alpha\wedge\beta \models \gamma$ and $\alpha\wedge\neg\gamma \models \neg\beta$. The latter two statements are not equivalent, since contraposition is invalid: for instance, we have $q\wedge\neg q \models \neg p$, but we saw earlier that $p \not\models q\vee\neg q$. Similarly, there are different notions of compatibility: $\alpha\wedge\beta \not\models$ **false** means that there is a model in which both α and β are true, while the weaker statements $\alpha \not\models \neg\beta$ and $\beta \not\models \neg\alpha$ (which are not equivalent) mean that there is a model in which one statement is true while the other is not false. The reader is referred to (Thijsse, 1992) for a thorough exposition of these and related points.

Partial models may be ordered, just as truth-values, according to their information-content.

> DEFINITION 7.29. Let U be a set of partial models. The *information ordering* on U is defined by $m_1\leq m_2$ if for every propositional atom $p\in L$, either $m_1(p)$=**unknown** or $m_1(p)=m_2(p)$.

That is, we have $m_1 \models \alpha$ implies $m_2 \models \alpha$ for both $\alpha=p$ and $\alpha=\neg p$. This can be extended to the complete language L, assuming that Kleene's strong three-valued connectives are employed (see Thijsse, 1992, p.67, Proposition 3.4):

> PROPOSITION 7.30 (Persistence). *$m_1\leq m_2$ iff for every formula $\alpha\in L$, if $m_1 \models \alpha$ then $m_2 \models \alpha$.* ■

That is, once formulas α and β have received truth-values **true** ($m_1 \models \alpha$) and **false** ($m_2 \models \neg\beta$) respectively, these truth-values persist everywhere higher in the information ordering. The only changes in truth-value occurring when climbing the information ordering are from **unknown** to **true**, and from **unknown** to **false**. Furthermore, we may note that the information ordering is smooth, at least in the case of propositional logic.

> LEMMA 7.31. *For any formula $\alpha\in L$ and any $m\in U$, if $m \models \alpha$ then there exists an $n\in U$ such that $n \models \alpha$, $n\leq m$ and n is minimal in $[\alpha]$, where $[\alpha]$ denotes the set of models verifying α.*[81]
> *Proof*. Let m be a model of α, and let m' be the interpretation obtained from m by changing the truthvalue of any propositional atom not occurring in α to **unknown**. It is easy to prove, by induction on the structure of α,

[81]This is a translation of Kraus *et al.*'s definition of smoothness to a non-strict partial order.

that m' is a model of α if m is; furthermore, by the definition of $\leq$, $m' \leq m$. If m' is minimal in $[\alpha]$, we're done; if it is not, we can change the truthvalue of some propositional atom from **true** or **false** into **unknown** without changing the truthvalue of α. We can repeat this process until a minimal model has been found; since m' assigns **true** or **false** to only finitely many proposition symbols (i.e. a subset of those occurring in α) the process only takes a finite number of steps. ■

It is clear, then, that an information-minimal model will assign **unknown** to as many propositional atoms as possible. For instance, the formula $p \vee \neg p$ has two information-minimal models, one assigning **true** to p, the other assigning **false** (and **unknown** to every other atom). Furthermore, the intended model (which is *total*, i.e. assigning truth or falsity to every formula) will be at least as informed as one of the information-minimal models of the observations — in other words, a hypothesis that is falsified by every information-minimal model of the observations is necessarily false in the intended model. This justifies the use of the information ordering as a preference ordering for consistency-based confirmatory reasoning, and to define $\alpha \mathrel{|\!\sim} \beta$ if β is not falsified in at least one information-minimal model of α[82]. However, due to the intricacies of partial logic this condition is too weak in one respect: suppose that for every minimal model m verifying α we have $m(\beta)$=**unknown**; it follows that not only both $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \neg\beta$, but also $\alpha \mathrel{|\!\sim} \beta \wedge \neg\beta$. This can be handled by adding a second condition for $\alpha \mathrel{|\!\sim} \beta$ to hold, to the effect that β should be verified by at least one model of α. Notice that this is not necessarily a **minimal** model: we may still have $m(\beta)$=**unknown** for every minimal model m verifying α, and thus both $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \neg\beta$. Such a situation simply means that α does not contain enough information to discriminate between the hypotheses β and $\neg\beta$.

I will now proceed as follows: I will first define partial preferential consistency-based confirmatory consequence relations (ppcc consequence relations for short) in the general case, where the preference ordering can be any partial order on U. After listing some of the properties of ppcc consequence relations in the general case, I will study the special case where the preference ordering is the information ordering.

DEFINITION 7.32. A *partial preferential consistency-based confirmatory structure* (*ppcc structure* for short) is a pair $W = \langle V, \leq \rangle$, where $V \subseteq U$ is a set of partial models, and $\leq$ is a smooth partial order on V. The consequence relation defined by W is denoted by $\mathrel{|\!\sim}_W$ and is defined by: $\alpha \mathrel{|\!\sim}_W \beta$ iff (*i*) there is a model $m \in V$ verifying α and β, and (*ii*) there is a minimal (wrt. $\leq$) model $m_0 \in [\alpha]$ not falsifying β, where $[\alpha]$ denotes the set of models verifying α. A consequence relation is called ppcc iff it is defined by a ppcc structure.

It has been noted above that Right And and Right Consistency are invalid — the latter is replaced by a partial variant of Consistency:

- **Partial Consistency**: $\dfrac{\alpha \mathrel{|\!\sim} \beta}{\alpha \wedge \beta \not\equiv \mathbf{false}}$

[82] A more appropriate intuitive reading would be 'α does not disconfirm β'.

Other rules that are valid when reformulated in partial terms are Left Logical Equivalence and Predictive Right Weakening:

- **Partial Left Logical Equivalence**: $\dfrac{\alpha \models \beta \ , \ \beta \models \alpha \ , \ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$
- **Partial Predictive Right Weakening**: $\dfrac{\alpha \models \beta \rightarrow \gamma \ , \ \alpha \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \gamma}$

As instances of Partial Predictive Right Weakening we obtain Left Reflexivity and (Partial) Admissible Entailment, as usual.

One may note that these rules are partial variants of rules from **CP**. However, the remaining rules of **CP** are invalid for ppcc consequence relations, most notably Confirmatory Reflexivity and Cautious Monotonicity. Confirmatory Reflexivity (from $\alpha \mathrel{|\!\sim} \alpha$ and $\alpha \not\mathrel{|\!\sim} \neg\beta$ conclude $\beta \mathrel{|\!\sim} \beta$) is invalid because the condition $\alpha \not\mathrel{|\!\sim} \neg\beta$ is not necessarily caused by every minimal model of α falsifying $\neg\beta$, the other possible reason being that no model verifying α also verifies $\neg\beta$. Only in the first case, but not in the second, can we conclude that there exists a model verifying β, as required by the conclusion of the rule. We may add that Right Reflexivity is valid.

The failure of Cautious Monotonicity (from $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$ conclude $\alpha\wedge\beta \mathrel{|\!\sim} \gamma$) can be traced back to the following observation: $\alpha \mathrel{|\!\sim} \beta$ means that the minimal models of α do not falsify β (i.e. $m \not\models \neg\beta$); but this does **not** imply, as it does in the non-partial case, that these models also verify β (i.e. $m \models \beta$), hence the minimal models of $\alpha\wedge\beta$ may be completely different from those of α. For instance, we may have $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \neg\beta$, but clearly we don't have $\alpha\wedge\beta \mathrel{|\!\sim} \neg\beta$. A partial variant of Verification, which represents a weaker form of Cautious Monotonicity, is however valid:

- **Partial Verification**: $\dfrac{\alpha \models \beta \rightarrow \gamma \ , \ \alpha \mathrel{|\!\sim} \beta}{\alpha\wedge\gamma \mathrel{|\!\sim} \beta}$

As indicated in §24, this rule weakens Cautious Monotonicity by demanding that the hypotheses β and γ are based on the same assumptions.

The rules found valid so far are collected in the following lemma.

LEMMA 7.33. *Any ppcc consequence relation satisfies the following rules:*

- **Right Reflexivity**: $\dfrac{\alpha \mathrel{|\!\sim} \beta}{\beta \mathrel{|\!\sim} \beta}$
- **Partial Left Logical Equivalence**: $\dfrac{\alpha \models \beta \ , \ \beta \models \alpha \ , \ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$
- **Partial Predictive Right Weakening**: $\dfrac{\alpha \models \beta \rightarrow \gamma \ , \ \alpha \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \gamma}$
- **Partial Verification**: $\dfrac{\alpha \models \beta \rightarrow \gamma \ , \ \alpha \mathrel{|\!\sim} \beta}{\alpha\wedge\gamma \mathrel{|\!\sim} \beta}$
- **Partial Consistency**: $\dfrac{\alpha \mathrel{|\!\sim} \beta}{\alpha\wedge\beta \not\models \mathtt{false}}$

Proof. For Right Reflexivity, if $\alpha \mid\!\sim \beta$ then some model $m \in [\alpha]$ verifies β, i.e. $[\beta]$ is non-empty, and every model minimal in $[\beta]$ verifies β.
For Partial Left Logical Equivalence, if $\alpha \models \beta$ and $\beta \models \alpha$ then α and β are verified by the same models.
For Partial Predictive Right Weakening, let m_0 be the model minimal in $[\alpha]$ that does not falsify β, i.e. $m_0 \models \alpha$ and $m_0 \not\models \neg\beta$. If additionally $\alpha \models \beta \rightarrow \gamma$, we obtain $m_0 \models \gamma$.
For Partial Verification, let m_0 be the model minimal in $[\alpha]$ that does not falsify β, i.e. $m_0 \models \alpha$ and $m_0 \not\models \neg\beta$. Now, since $\alpha \models \beta \rightarrow \gamma$, we have $m_0 \models \gamma$ and therefore $m_0 \in [\alpha \wedge \gamma]$; since $[\alpha \wedge \gamma] \subseteq [\alpha]$, m_0 is also minimal in $[\alpha \wedge \gamma]$. Furthermore, let m be the model in $[\alpha]$ such that $m_0 \leq m$ and $m \models \beta$; since $\alpha \models \beta \rightarrow \gamma$, we have $m \models \gamma$ and therefore $m \in [\alpha \wedge \gamma]$.
For Partial Consistency, if some model in $[\alpha]$ verifies β, then clearly $\alpha \wedge \beta \not\models \mathbf{false}$. ■

The completeness of the set of rules in Lemma 7.33 is questionable. For instance, from the definition of $\mid\!\sim_W$ we see that if $\alpha \mid\!\sim_W \beta$ then $\alpha \mid\!\sim_W \alpha \wedge \beta$; yet, substituting $\alpha \wedge \beta$ for γ in Partial Predictive Right Weakening will not work, since $\alpha \models \beta \rightarrow (\alpha \wedge \beta)$ is invalid if $m(\beta) = \mathbf{unknown}$ for some model m. One possible solution[83] to this problem is to replace Partial Predictive Right Weakening with the sound but rather tedious

$$\frac{\alpha \wedge \beta \models \gamma \ , \ \alpha \wedge \neg\gamma \models \neg\beta \ , \ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

The reader may want to check that the first two conditions vanish when $\alpha \wedge \beta$ is substituted for γ. However, my main concern at this stage is conceptual analysis rather than logical rigour, and I leave the issue of axiomatising ppcc consequence relations as an open problem.

I will now demonstrate that the information ordering can be used to build a ppcc structure defining consequence relations that are closely related to weak confirmatory consequence relations.

DEFINITION 7.34. A *partial weak confirmatory structure* is a ppcc structure $W = \langle V, \leq \rangle$, where $\leq$ is the information ordering on U, restricted to V. A consequence relation is called partial weak confirmatory iff it is defined by a partial weak confirmatory structure.

The following lemma demonstrates the close relation with weak confirmatory consequence relations.

LEMMA 7.35. *Let W be a partial weak confirmatory structure.* $\alpha \mid\!\sim_W \beta$ *iff there exists a model in* $[\alpha]$ *verifying* β.

[83] Another option is to assign to $\rightarrow$ the Lukasiewicz interpretation, which differs from Kleene's strong interpretation by putting $m(\alpha \rightarrow \beta) = \mathbf{true}$ if $m(\alpha) = m(\beta) = \mathbf{unknown}$. However, as a result the language is no longer persistent: increasing the truth-value of α or β may decrease the truth-value of $\alpha \rightarrow \beta$ wrt. the information ordering.

Proof. The only-if part follows from Definition 7.32. For the if part, let m be a model in $[\alpha]$ verifying β, then by Lemma 7.31 there exists a model $n \leq m$ such that n is minimal in $[\alpha]$. Furthermore, $m \not\models \neg\beta$, and therefore $n \not\models \neg\beta$ by Proposition 7.30. We conclude that $\alpha \mathrel{|\!\sim}_W \beta$. ■

That is, for any partial weak confirmatory structure $W = \langle V, \leq \rangle$ we have $\alpha \mathrel{|\!\sim}_W \beta$ iff there is an $m \in V$ such that $m \models \alpha \wedge \beta$ — in other words, partial weak confirmatory reasoning corresponds to verifiability of premisses and conclusion with respect to a set of partial models, and the information ordering does not affect the set of arguments but only serves as a computational tool. We may further note that if $\alpha \wedge \beta$ has a partial model, it has a total model (construct a total model m' from a partial model m by putting $m'(\mathtt{p}) = m(\mathtt{p})$ if $m(\mathtt{p}) \neq$ **unknown**, and by arbitrarily putting $m'(\mathtt{p})$ to **true** or **false** otherwise). This means that for any partial weak confirmatory consequence relation there exists an equivalent weak confirmatory consequence relation. Since the converse is trivially true (any set of total models is also a set of partial models), this proves the equivalence of weak and partial weak confirmatory reasoning.

The rules satisfied by partial weak confirmatory consequence relations are thus partial versions of the rules in **CW**. The validity of Partial Predictive Right Weakening and Partial Consistency has been proved above, in the general case of ppcc consequence relations — for completeness' sake I prove the validity of the remaining two rules below, which has become very easy in the light of Lemma 7.35.

COROLLARY 7.36. *Any partial weak confirmatory consequence relation satisfies the following rules:*

- **Partial Predictive Incrementality**: $\dfrac{\alpha \models \gamma \rightarrow \beta \ , \ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$
- **Disjunctive Rationality**: $\dfrac{\alpha \vee \beta \mathrel{|\!\sim} \gamma \ , \ \beta \not\mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \gamma}$

Proof. For Partial Predictive Incrementality, suppose that there exists a model in $[\alpha]$ verifying γ, and $\alpha \models \gamma \rightarrow \beta$; it follows that this model verifies β, hence there exists a model in $[\beta]$ verifying γ.
For Disjunctive Rationality, first note that $[\alpha \vee \beta] = [\alpha] \cup [\beta]$. Furthermore, if $\beta \not\mathrel{|\!\sim} \gamma$ then no model in $[\beta]$ verifies γ, and thus the model in $[\alpha \vee \beta]$ verifying γ must be in $[\alpha]$. ■

In this section I have defined weak confirmatory reasoning, which is also the weakest form of conjectural reasoning since it only requires consistency between evidence and hypothesis. This form of reasoning is axiomatised by the system **CW**. I have further defined a partial, preferential variant of weak confirmatory reasoning (the axiomatisation of which remains, as yet, incomplete), and proved the equivalence with weak confirmatory reasoning if the information ordering on partial models is taken as the preference ordering. This form of confirmatory reasoning will be put to work in the next chapter, because it has a distinct advantage over preferential confirmatory reasoning: it is incremental.

I should add that consistency-based confirmatory reasoning, as defined in this section, does not, in the general case, establish a preservation semantics (§18). If we define $\alpha \mid\!\sim \beta$ if there is a model satisfying both α and β, this cannot be reduced to a preservation function f constructing, from the models of α, a set of interpretations satisfying β, since this preservation function should operate independently from β. Even if we use the information ordering to select minimal models of α this does not establish a preservation function, since α may have several information-minimal models, and if both $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$ the minimal model α has in common with β may be different from the minimal model it has in common with γ. Only in the case that premisses always have a single information-minimal model (i.e. they are definite) does consistency-based confirmatory reasoning correspond to a preservation semantics. This indicates that the concept of a preservation semantics needs to be extended, or complemented by an alternative concept.

§29. SUMMARY AND CONCLUSIONS

In this chapter I presented the main formal results of this thesis, in the form of axiomatic characterisations of three different kinds of conjectural reasoning. The three rule systems **EM** (explanatory reasoning with a monotonic explanation mechanism), **CP** (preferential confirmatory reasoning), and **CW** (weak confirmatory reasoning) light parts of the map of conjectural reasoning, and thus provide a starting point for a descriptive theory of conjectural reasoning. In addition I have provided an alternative characterisation of weak confirmatory reasoning in terms of information-minimal partial models. Open problems include: characterising explanatory reasoning based on non-monotonic explanation mechanisms, characterising preferential consistency-based confirmatory reasoning, and extending the concept of a preservation semantics to cover consistency-based reasoning.

Each of the semantic structures characterising these rule systems has been designed to reflect current practice in the field of machine learning (chapter 3). Explanatory semantics models preservation of explanatory power, where an explanation is identified with a deductive proof, as in classification-oriented machine learning approaches. Preferential confirmatory structures generalise closed-world reasoning, as applied in Helft's and De Raedt's approaches to induction of integrity constraints. Weak confirmatory structures are based on compatibility between evidence and hypothesis, an idea that has been applied to incremental induction of integrity constraints in databases (see the next chapter).

However, neither of these semantics is claimed to fully capture the essence of inductive reasoning as performed by humans. For instance, identifying an explanation with a deductive proof seems to be quite crude, even if it is not uncommon in philosophy of science, since explanations often indicate a causal relation between observations and explanans. Also, formalising regular interpretations as minima with respect to a fixed ordering does not seem to be appropriate in all cases, since the ordering may depend on the observations. Even if further work is needed on these and related points, I believe that such future refinements can be incorporated in the formal framework set up in this thesis.

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