CHAPTER 7

RULE SYSTEMS FOR CONJECTURAL REASONING

— in which various forms of conjectural reasoning will be axiomatised and semantically characterised —

IN THIS CHAPTER I will develop axiomatic and semantic accounts of various forms of conjectural reasoning. The main purpose of the resulting logical systems is to provide a descriptive taxonomy of conjectural reasoning. This taxonomy will contain two main families, corresponding to the two forms of conjectural reasoning considered in this thesis: explanatory and confirmatory reasoning. Within the family of confirmatory reasoning a further distinction is made between incremental and non-incremental forms, the former based on the semantic notion of regular models, the latter based on the notion of consistency. For each of these three forms of conjectural reasoning a characterisation is given in the form of a semantics accompanied by a sound and complete rule system. It should be noted that the representation results are obtained for a propositional language L.

§26. EXPLANATORY REASONING

This section provides a formalisation of the idea that explanatory reasoning preserves explanatory power. As defined in §19, given some explanation mechanism $\mid\!\sim$, the *explanatory power* of a formula α is defined as its *closure* $Cn_{\mid\sim}(\alpha) = \{\gamma \mid \alpha \mid\!\sim \gamma\}$. Using this definition, we may require of an explanatory argument $\alpha \mid\!< \beta$ that $Cn_{\mid\sim}(\alpha) \subseteq Cn_{\mid\sim}(\beta)$, i.e. for every γ, if $\alpha \mid\!\sim \gamma$ then $\beta \mid\!\sim \gamma$. If $\mid\!\sim$ satisfies the rules of the system **M** from the KLM-framework this is equivalent with $\beta \mid\!\sim \alpha$ (Lemma 5.2).

In this section I will mostly restrict attention to explanation mechanisms that satisfy the rules of **M**. The resulting form of conjectural reasoning is referred to as *strong* explanatory reasoning; the adjective ‘strong’ will be often omitted if no confusion can arise. Weaker explanation mechanisms will be briefly considered at the end of the section. Strong explanatory reasoning is characterised in two steps. I will first define a system, $\mathbf{M_{rev}}$, which embodies a reversed version of the KLM system **M**. However, $\mathbf{M_{rev}}$ does not satisfy Consistency, and is therefore, strictly speaking, not a system for explanatory reasoning. Using rules discussed in the previous chapter, a more restrictive version of $\mathbf{M_{rev}}$, called **EM**, is defined and characterised.

Reverse deductive consequence relations

This subsection presents a reversed version of the KLM sys[redacted] as to facilitate as much as possible the development [redacted]a s[redacted]tem for s[redacted]ong explanatory reasoning in the next subsection. Since the latter is self[redacted]ned, the r[redacted]der may want to skip the present subsection.

Given a set of models representing the implicit back[redacted] theory, a[redacted] consequence relation consists of those arguments of w[redacted] same explanatory power as the premisses in each of th[redacted]dels[redacted]he[redacted] th[redacted] explanatory power of a formula α in a model m is the set $\{\gamma$ [redacted] $\alpha\rightarrow$[redacted]$\}$. T[redacted]is[redacted]eads to the following definition.

DEFINITION 7.1. A *reverse deductive stru[redacted]ure* is a[redacted]et [redacted] consequence relation it defines is denoted by $|\!\!<_W$ [redacted]nd is def[redacted]ed b[redacted] [redacted]$|\!\!<_W \beta$ iff for every $m\in W$ and for every $\gamma\in L$: if m [redacted]$\alpha\rightarrow\gamma$ th[redacted] consequence relation is called reverse deductive [redacted] it is def[redacted] deductive structure.

For fixed m and γ the preservation condition boils down to m [redacted]$\beta\rightarrow\alpha\vee\gamma$[redacted] quantifying over γ we obtain the equivalent condition $\alpha\ |\!\!<_W\ \beta$ iff for every [redacted] W: m [redacted]$\beta\rightarrow\alpha$. This latter condition will be used in the proof of the rep[redacted]sentation theorem; [redacted]wever, as a definition the above formulation is preferred, because [redacted] expresses the idea of a preservation semantics more clearly.

The following system provides an axiomatisatio[redacted] of reverse deductive consequence relations; a formal proof of this statement follows th[redacted]ures of the system.

DEFINITION 7.2. The system $\mathbf{M_{rev}}$ consi[redacted] schema and inference rules:

- **Reflexivity**: [redacted] $\alpha\ |\!\!<\ \alpha$
- **Predictive Incrementality**: [redacted]$\gamma\rightarrow\beta$, $\alpha\ |\!\!<\ \gamma$ / [redacted]
- **Additivity**: [redacted]$|\!\!<\ \gamma$, $\beta\ |\!\!<\ \gamma$ / $\alpha\wedge\beta\ |\!\!<\ \gamma$
- **Right Strengthening**: [redacted]$\rightarrow\beta$, $\alpha\ |\!\!<\ \beta$ / $\alpha\ |\!\!<\ \gamma$
- **Conditionalisation**: $\dfrac{\alpha\ |\!\!<\ \beta\wedge\gamma}{\beta\rightarrow\alpha\ |\!\!<\ \gamma}$

$\mathbf{M_{rev}}$ can be readily obtained from the KLM system **M** (with some minor modifications) by applying the rewrite rule $\alpha\ |\!\!\sim\ \beta \Rightarrow \beta\ |\!\!<\ \alpha$. The first four rules have been discussed in the previous chapter; the intuitions behind Conditionalisation will be discussed in the next subsection.

LEMMA 7.3. *Every consequence relation satisfying the rules of* $\mathbf{M_{rev}}$ *is incremental, convex, disjunctively closed, and conjunctively closed.*

Proof. Predictive Incrementality implies Incrementality by Lemma 6.5.

By Lemma 6.9, Right Interval follows from Admissible Converse Entailment (which is an instance of Predictive Incrementality) and Left and Right Reflexivity (hence from Reflexivity).

In order to derive Right Or we will need the following rule:

- **Contraposition**: $$\frac{\alpha \mathrel{|\!<} \beta}{\neg\beta \mathrel{|\!<} \neg\alpha}$$

Contraposition can be derived as follows. Suppose $\alpha \mathrel{|\!<} \beta$, then by Conditionalisation $\beta\to\alpha \mathrel{|\!<}$ **true**, by Incrementality $\neg\alpha\to\neg\beta \mathrel{|\!<}$ **true**, and by Right Strengthening $\neg\alpha\to\neg\beta \mathrel{|\!<} \neg\alpha$. Furthermore, since by Reflexivity $\neg\alpha \mathrel{|\!<} \neg\alpha$, we conclude by Additivity and Incrementality.

Now, to derive Right Or, suppose $\alpha \mathrel{|\!<} \beta$ and $\alpha \mathrel{|\!<} \gamma$, then by Contraposition $\neg\beta \mathrel{|\!<} \neg\alpha$ and $\neg\gamma \mathrel{|\!<} \neg\alpha$, by Additivity $\neg\beta\wedge\neg\gamma \mathrel{|\!<} \neg\alpha$, and we conclude by Contraposition.

[redacted] Right Strengthening.

[redacted]e fact that reverse deductive consequence relations are c[redacted]junctively closed may seem [redacted] but this is a [redacted]sequence of the fact that an [redacted]erything. Indeed, we will see below that explanatory c[redacted]sequence relations are not [redacted]njunctively closed (they continue to be disjunctively c[redacted]sed, however). To see that [redacted]$_{\mathbf{rev}}$ does not satisfy Consistency, observe that for any un[redacted]tisfiable formula α we have [redacted]$\to\neg\alpha$, but also $\alpha \mathrel{|\!<} \alpha$ by Reflexivity.

In order to prove completeness of $\mathbf{M_{rev}}$ with respect t[redacted]the set of reverse deductive structures we will need the following rule:

- **Converse Entailment**: $$\frac{\text{[redacted]}\to\alpha}{\text{[redacted]} \mathrel{|\!<} \beta}$$

To see that Converse Entailment is a [redacted]rived rule of $\mathbf{M_{rev}}$, put $\alpha=\gamma$ in Inc[redacted]entality and use Refle[redacted]ivity.

The f[redacted]owi[redacted] th[redacted]rem [redacted]oves th[redacted]reverse deductive structures chara[redacted]erise reverse deductive [redacted]s.

TH[redacted] reverse deductive cons[redacted] rel[redacted]*ductive iff it satis*[redacted]*es the* *ru*[redacted]

Pr[redacted] for a given [redacted]verse de[redacted] it defines satis[redacted]es the rul[redacted] for Pr[redacted]ictive In[redacted] i.e. for every $m\in U$: if m [redacted]α and m [redacted] m [redacted]hermo[redacted], suppose that for every $m\in W$: m [redacted]$\gamma\to\alpha$, i.e. [redacted]α. Si[redacted] $W\subseteq U$, this implies that for every[redacted]$\in W$, if m [redacted] then m [redacted]β, i.e. m [redacted]$\gamma\to\beta$.

[redacted]r the if part, [redacted]t $\mid\!<$ be an arbitrary consequence relation [redacted]atisfying the rules [redacted] $\mathbf{M_{rev}}$, and c[redacted]sider the following reverse deductive st[redacted]cture:

[redacted]t $\alpha \mid\!< \beta$: m [redacted]$\beta\to\alpha\}$

[redacted]e will [redacted]rove [redacted]at $\alpha \mid\!< \beta$ [redacted]f $\alpha \mid\!<_W \beta$. The only-if part fo[redacted]ows direc[redacted]y from [redacted] we will sho[redacted]that there[redacted]xists a [redacted]odel [redacted]
[redacted]efine [redacted] $= \{\neg$[redacted]$\} \cup \{\delta \mid$ [redacted] $\mid\!< \beta\}$; we will fi[redacted] show th[redacted] Γ_0 is sat[redacted]fiable. [redacted]ppose[redacted]ot, th[redacted] by com[redacted]actness there is a[redacted]nite $\Delta\subseteq\{$[redacted] more, s[redacted]ce $\delta \mid\!< \beta$ for [redacted] Δ, we have Δ[redacted]g Additiv[redacted] and Incremen[redacted]ity, we obtain α[redacted] β. Cont[redacted]diction [redacted] is satisfiab[redacted].
Let m_0 [redacted]Γ_0; clearly m_0 [redacted]α and, since by [redacted]e conclude that m_0 does no[redacted] satisfy $\beta\to\alpha$; it[redacted]mains to[redacted]ove that [redacted]W; i.e., that for all φ, ψ such that $\varphi \mid\!< \psi$ w[redacted]have m_0 [redacted]$\psi\to\varphi$. Suppose $\varphi \mid\!< \psi$, then by Right Strengthening $\varphi \mid\!< \psi$[redacted]β, and by Conditionalisation $\psi\to\varphi \mid\!< \beta$; thus $\psi\to\varphi\in\Gamma_0$ and therefore m_0 [redacted]$\psi\to\varphi$.

As indicated above, reverse deductive consequence relations are not consistent, since unsatisfiable formulas do count as explanations. Overcoming this defect leads us to the notion of strong explanatory reasoning.

Explanatory consequence relations

In this subsection I will define a system **EM** (explanatory reasoning wrt. a monotonic explanation mechanism) that satisfies Co[redacted]istency but contains weaker versions of Reflexivity and Right [redacted]trengthening. [redacted]he[redacted]semantic characterisation of explanatory reasoning demonstrates [redacted]at **EM** is stric[redacted] m[redacted]re restrictive than $\mathbf{M_{rev}}$.

I will start again wit[redacted]a semantic de[redacted]nit[redacted]n of explanatory reasoning. Like a reverse deductive structure, an e[redacted]planatory stru[redacted]ure[redacted] an explanatory structure defines a diffe[redacted]st the conclusion of every argument have at [redacted]st the same ex[redacted]la[redacted]ory power as the premisses, but in addition the conclusion should be[redacted]atisfiable.

DEFINITION 7.[redacted] is a set $W\subseteq U$. The consequence relat[redacted]n it defines is [redacted]d by $\mid\!<_W$ and is defined by: $\alpha \mid\!<_W \beta$ iff (*i*) there is an[redacted]$\in W$ such tha[redacted]m [redacted]β, and (*ii*) for every $m\in W$ and for every $\gamma\in L$: if m [redacted]$\alpha\to\gamma$ then m [redacted]$\beta\to\gamma$. A consequence relation is called strong explanator[redacted] iff it is defined by an explanatory structure.

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[72] By a slight abuse of notation, Δ denotes both a finite set of formulas, and their conjunction.

[73] If $\Delta=\varnothing$, we have **true** $\mid\!< \beta$ by Reflexivity and Convergence.

As said before, the adjective ‘strong’ will [redacted] often omitted if no confusion ca[redacted]arise. Furthermore, in the proofs of the lemmas [redacted]elow condition (*ii*) will be read [redacted] equivalent condition (*ii′*) for every $m\in W$: m [redacted]$\beta\to\alpha$.

As for an axiomatisation of explanatory co[redacted]sequence relations, we should note [redacted]t the conclusion of an explanatory argument is required to be admissible. As has been ar[redacted] in §23, this means that Reflexivity should be weakened in various ways. It turns [redacted] that only Explanatory Reflexivity will be among the rules defining the system **EM**: bo[redacted] Left and Right Reflexivity are derived rules. The other ru[redacted] of $\mathbf{M_{rev}}$ that is not valid in [redacted]**M** is Right Strengthening. It is easy to see why this is s[redacted] if γ is inadmissible, then [redacted]$\to\beta$, yet $\alpha \not\mathrel{|\!\!<} \gamma$ for any α. We therefore add a condition [redacted]tating that γ must be adm[redacted]ssible, leading to the rule of Admissible Right Strengthenin[redacted] Finally, a rule should be added to ensure the validity of Consistency. In the light of L[redacted]dd Left Consistency. We thus obtain the following system fo[redacted]planatory reasoning.

DEFINITION 7.6. The system **EM** consists of t[redacted]

- **Explanatory Reflexivity**: [redacted] $\alpha\ ,\ \neg\beta \not\mathrel{|\!\!<} \alpha$ / $\beta \mathrel{|\!\!<} \beta$
- **Predictive Incrementality**: [redacted]$\gamma\to\beta\ ,\ \alpha \mathrel{|\!\!<} \gamma$ / [redacted]
- **Additivity**: $\dfrac{\text{[redacted]}\ \gamma\ ,\ \beta \mathrel{|\!\!<} \gamma}{\alpha\wedge\beta \mathrel{|\!\!<} \gamma}$
- **Admissible Right Strengthening**: [redacted]$\to\beta\ ,\ \alpha \mathrel{|\!\!<} \beta\ ,\ \gamma \mathrel{|\!\!<} \gamma$ / $\alpha \mathrel{|\!\!<} \gamma$
- **Conditionalisation**: $\dfrac{\alpha \mathrel{|\!\!<} \beta\wedge\gamma}{\beta\to\alpha \mathrel{|\!\!<} \gamma}$
- **Left Consistency**: $\dfrac{\alpha \mathrel{|\!\!<} \beta}{\neg\alpha \not\mathrel{|\!\!<} \beta}$

The significance of the rules of **EM** as properties of explanatory reasoning has been discussed in the previous chapter, with the exception of the rule of Conditionalisation. This rule can be best understood if one recalls a discussion from §9, where a distinction was drawn between two different representations of examples: the ‘examples as implications’ approach, and the ‘examples as classifications’ approach. In the former case, examples are ground implications with the description of an instance as antecedent, and a classification as consequent. In the latter case, the description of the instance belongs to the background theory, while the example comprises only the classification of the instance. Conditionalisation expresses that the former approach is as powerful as the latter: anything that can be induced by means of the ‘examples as classifications’ approach can also be induced by means of the ‘examples as implications’ approach[74]. It should be added that, in the case of strong explanatory reasoning, both approaches are actually

[74] Since the background theory is left implicit in our framework, the description of the instance (β) is added to the hypothesis rather than the background theory.

equivalent, since the following rule is a derived rule of **EM**:

- **Reverse of Conditionalisation**: $\dfrac{\beta\rightarrow\alpha \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \beta\wedge\gamma}$

The derivation of this rule is left to the reader.

The following lemma lists a few properties of strong explanatory reasoning.

LEMMA 7.7. *Every consequence relation satisfying the rules of* ***EM*** *is consistent, incremental, convex, and disjunctively closed, but not conjunctively closed.*

Proof. By Lemma 6.8, Left Consistency implies Consistency in the presence of Right Reflexivity and Admissible Converse Entailment (both instances of Predictive Incrementality).

Predictive Incrementality implies Incrementality.

By Lemma 6.9, Right Interval follows from Admissible Right Strengthening, Admissible Converse Entailment, and Left and Right Reflexivity. In order to show that Left Reflexivity is a derived rule of **EM**, suppose $\alpha \mathrel{|\!\sim} \beta$, then by Predictive Incrementality $\beta \mathrel{|\!\sim} \beta$. Furthermore, by Left Consistency $\neg\alpha \mathrel{|\!\not\sim} \beta$, and we conclude by Explanatory Reflexivity.

Right Or can be derived by means of the following rule (note that Contraposition is not valid in **EM**):

- **Admissible Contraposition**: $\dfrac{\alpha \mathrel{|\!\sim} \beta \,,\, \neg\alpha \mathrel{|\!\sim} \neg\alpha}{\neg\beta \mathrel{|\!\sim} \neg\alpha}$

To derive Admissible Contraposition, suppose $\alpha \mathrel{|\!\sim} \beta$, then by Conditionalisation $\beta\rightarrow\alpha \mathrel{|\!\sim} \mathbf{true}$, by Incrementality $\neg\alpha\rightarrow\neg\beta \mathrel{|\!\sim} \mathbf{true}$, and, since by assumption $\neg\alpha \mathrel{|\!\sim} \neg\alpha$, by Admissible Right Strengthening $\neg\alpha\rightarrow\neg\beta \mathrel{|\!\sim} \neg\alpha$. We conclude by Additivity and Incrementality.

In order to derive Right Or, first note that $\alpha \mathrel{|\!\sim} \beta$ implies $\beta \mathrel{|\!\sim} \beta$ by Right Reflexivity (an instance of Predictive Incrementality), hence $\beta\vee\gamma \mathrel{|\!\sim} \beta$ by Incrementality and $\beta\vee\gamma \mathrel{|\!\sim} \beta\vee\gamma$ by Left Reflexivity. Suppose $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$. Now, either $\neg\alpha \mathrel{|\!\sim} \neg\alpha$ or $\neg\alpha \mathrel{|\!\not\sim} \neg\alpha$; in the former case we can apply Admissible Contraposition to obtain $\neg\beta \mathrel{|\!\sim} \neg\alpha$ and $\neg\gamma \mathrel{|\!\sim} \neg\alpha$, hence $\neg\beta\wedge\neg\gamma \mathrel{|\!\sim} \neg\alpha$ by Additivity, and we conclude by Admissible Contraposition. On the other hand, if $\neg\alpha \mathrel{|\!\not\sim} \neg\alpha$ then $\alpha \mathrel{|\!\sim} \beta\vee\gamma$ by Explanatory Reflexivity.

The following derived rules of **EM** are used in the proof of the representation theorem: Incrementality, Right Reflexivity and Admissible Converse Entailment (instances of Predictive Incrementality), Consistent Right Strengthening (Lemma 6.11), and Consistency (Lemma 6.8).

To see that **EM** is not conjunctively closed, let p and q such that $\neg p \mathrel{|\!\not\sim} q$ and $q \mathrel{|\!\sim} q$, then by Consistent Right Strengthening $q \mathrel{|\!\sim} q\wedge p$. However, since $q \mathrel{|\!\not\sim} (q\wedge p)\wedge(q\wedge\neg p)$ by Consistency, Right And would give us $q \mathrel{|\!\not\sim} q\wedge\neg p$, which, together with $q \mathrel{|\!\sim} q$, results in $p \mathrel{|\!\sim} q$ by Consistent Right Strengthening. That is, adding Right And to **EM** would lead to a situation in which every admissible hypothesis would either explain an arbitrary observation or its negation, which is clearly counterintuitive.

The characterisation of explanatory consequence relations requires a few steps more than in the case of reverse deductive reasoning. I start by proving that the rules of **EM** are sound with respect to the semantics defined by explanatory structures.

LEMMA 7.8 (Soundness of **EM**). *Any explanatory consequence relation satisfies the rules of **EM**.*

Proof. Let $W \subseteq U$ be an explanatory structure; we need to demonstrate that $\mid\!\sim_W$, as defined in Definition 7.5, satisfies the rules of **EM**. Since all the rules of **EM**, with the exception of Left Consistency, are valid rules of reverse deductive reasoning, we only need to check condition (*i*) for those rules. This is trivial for Predictive Incrementality, Additivity, Admissible Right Strengthening, and Conditionalisation.

For Explanatory Reflexivity, since $\alpha \mid\!\sim_W \alpha$ means that some model in W satisfies α, $\neg\beta \not\mid\!\sim_W \alpha$ implies that not all models in W [redacted]tisfy $\alpha\rightarrow\neg\beta$, i.e. there is a model in W satisfying $\alpha\wedge\beta$ and hence β.

For Left Consistency, suppose that $m_0 \in W$ satisfies β, [redacted]hile all models in W satisfy $\beta\rightarrow\alpha$. It follows that m_0 satisfies α, hence [redacted] all models in W satisfy $\beta\rightarrow\neg\alpha$.

In order to prove [redacted]mpleteness, we need to build an expla[redacted]tory structure W from a given consequence re[redacted]ion $\mid\!\sim$ satisfying the rules of **EM**, such [redacted] non-empty consequ[redacted]ce relations the construction of W i[redacted]the same as for reverse deductive re[redacted]oning:

$$W [redacted] \text{that } \alpha \mid\!\sim \beta\text{: } m [redacted] \beta\rightarrow\alpha\}$$

The chief di[redacted]ren[redacted] with reverse deductive reas[redacted]ing is that every explanatory hyp[redacted] is satisfiabl[redacted]

LEM[redacted]9. *Let $\mid\!\sim$ be a consequence re[redacted]tion satisfying the rules of **EM**, and* [redacted] *$\alpha \mid\!\sim \beta$* [redacted] *such that* [redacted].

Proo[redacted]t $\alpha \mid\!\sim \beta$; we will prove that $\{\beta$[redacted]$\cup \{\delta\rightarrow\gamma \mid \gamma \mid\!\sim \delta\}$ is satisfiable. Supp[redacted]e not, then by compactness there[redacted] such that [redacted]$\beta\rightarrow\neg\Delta$. Furthermore, since φ [redacted] we have $\psi\rightarrow\varphi$[redacted] **true** for any $\psi\rightarrow\varphi\in\Delta$ by C[redacted]ditionalisation, $\Delta \mid\!\sim$ **true** by Additivity, and $\Delta \mid\!\sim \beta$ by Right Ref[redacted]xivity and Admissible Right Strengthening. But then by Consistency [redacted]$\rightarrow\neg\Delta$, a contradiction.

Furthermore, we have that every inadmissible formula is unsatisfiable in W.

LEMMA 7.10. *Let $\mid\!\sim$ be a non-empty consequence relation satisfying the rules of **EM**, and let W be defined as above. If $\gamma \not\mid\!\sim \gamma$ then γ is unsatisfiable in W.*

Proof. Let $\alpha \mathrel{|\!\sim} \beta$, t[redacted]n **true** $\mathrel{|\!\sim}$ **true** by Incrementality and Left Reflexivity. Furtherm[redacted]e, if $\gamma \mathrel{|\!\not\sim} \gamma$ then $\neg\gamma \mathrel{|\!\sim}$ **true** by [redacted]xplanatory [redacted]eflexivity, hence m [redacted]rue$\rightarrow\neg\gamma$ for every $m\in W$.

I will [redacted]w show that W defines a consequence relation that is included[redacted]$\mathrel{|\!\sim}$.

[redacted]EMMA[redacted].11. *Let $\mathrel{|\!\sim}$ be* [redacted]*non-empty consequence*[redacted]*elation s*[redacted]*ined as above. If $\alpha \mathrel{|\!\sim}$*[redacted]*β then α* [redacted]*β.*
[redacted]*oo*[redacted]ppose that α [redacted]β, we will show that [redacted]ther no [redacted]del in W [redacted]is[redacted]es[redacted], or there exists[redacted]model $m_0\in W$ that does[redacted]ot satisfy [redacted]ainder of the pr[redacted]ve will assume that β [redacted]β. Define[redacted]$_0 = \{\neg\alpha\}$ [redacted]$\{$[redacted] that Γ_0 is satisfi[redacted]le. Suppo[redacted] not, then [redacted]α); by Admiss[redacted]β (recall that β [redacted]β)[redacted]β[redacted] by Additivity[redacted]β; using [redacted]dditivity [redacted]d [redacted]crementality, we obta[redacted]β. Contradicti[redacted]o Γ_0 is sat[redacted]fiable.
Let [redacted]Γ_0; clearly m_0 [redacted]nd since[redacted] that [redacted]$_0$ is in W; i.e., th[redacted] for all [redacted]ψ such tha[redacted]$\varphi \mathrel{|\!\sim} \psi$ we have m_0 [redacted]$\psi\rightarrow\varphi$. Let $\varphi \mathrel{|\!\sim} \psi$[redacted]if $\neg\beta \mathrel{|\!\not\sim}$[redacted], then by [redacted]onsistent Right Stre[redacted]gthening $\varphi \mathrel{|\!\sim} \psi\wedge\beta$, [redacted]d by Co[redacted]tionalisation[redacted] thus $\psi\rightarrow\varphi\in\Gamma_0$ and therefore m_0 [redacted]$\psi\rightarrow\varphi$. On[redacted]e other hand,[redacted] then by Conditionalisation and Incr[redacted]mentality [redacted]$\rightarrow\neg\psi \mathrel{|\!\sim}$ **tru**[redacted] by Admissible Right Strengthening $\beta\rightarrow\neg\psi \mathrel{|\!\sim} \beta$, and [redacted]y Additivit[redacted]d Incrementality $\neg\psi \mathrel{|\!\sim} \beta$; thus $\neg\psi\in\Gamma_0$ and therefore m_0 [redacted]ψ, hence m[redacted]$\psi\rightarrow\varphi$.

Armed with the previous three lemmas we can prove the com[redacted]

THEOREM 7.12 (Representation theorem for expl[redacted] relations). *A consequence relation is explanatory iff* [redacted]*isfies the rules of* ***EM***.
Proof. The only-if part is Lemma 7.8. For the if part[redacted]$\mathrel{|\!\sim}$ be an arbitrary non-empty consequence relation satisfying the rules o[redacted], and let

$$W = \{m\in U \mid \text{for all } \alpha, \beta \text{ such that } \alpha \mathrel{|\!\sim} \beta\text{:}\ \text{[redacted]}\ \beta\rightarrow\alpha\}$$

Suppose $\alpha \mathrel{|\!\sim} \beta$, then by the construction of W, m [redacted]$\beta\rightarrow\alpha$ for all $m\in W$. Furthermore, by Lemma 7.9 there is a model in W [redacted]atisfying β. We may conclude that $\alpha \mathrel{|\!\sim}_W \beta$. Conversely, if $\alpha \mathrel{|\!\sim}_W \beta$ then Lemma 7.11 proves that $\alpha \mathrel{|\!\sim} \beta$. We conclude that W defines a consequence relation that is exactly $\mathrel{|\!\sim}$. For an empty consequence relation put $W=\emptyset$.

We may note, to round off our discussion of strong explanatory reasoning, that this semantic characterisation of **EM** clearly demonstrates that strong explanatory reasoning is strictly more restrictive than reverse deductive reasoning, in the following sense (recall also the discussion of comparison criteria for different forms of reasoning in §16). A reverse deductive reasoner and a strong explanatory reasoner build their respective consequence relation, if they share the same background knowledge, from the same set of models $W \subseteq U$. The resulting consequence relations differ in the following respect: for every formula β that is unsatisfiable in W, the reverse deductive reasoner will include arguments $\alpha \mathrel{|\!\!<} \beta$ for arbitrary $\alpha \in L$, while the explanatory reasoner will include none of these (notice that such an α always exists, *viz.* **`false`**). The two reasoners will agree on all other arguments. So the strong explanatory restriction of a reverse deductive consequence relation is always a proper subset of the latter.[75]

This concludes the investigations into the formal properties of strong explanatory reasoning. Before developing rule systems for confirmatory reasoning in the next sections, I will now spend a few words on weaker forms of explanatory reasoning.

Weaker notions of explanation

If $\mathrel{|\!\!\sim}$ is not transitive, then there exist α, β and γ such that $\alpha \mathrel{|\!\!\sim} \beta$ and $\beta \mathrel{|\!\!\sim} \gamma$, yet $\alpha \not\mathrel{|\!\!\sim} \gamma$, i.e. $Cn_{|\!\sim}(\beta) \not\subseteq Cn_{|\!\sim}(\alpha)$. As has been proved by Kraus *et. al*, transitivity and monotonicity are equivalent in the presence of the rules of **C**, the weakest rule system for plausible reasoning. The upshot is that for plausible consequence relations, the nice equivalence between $Cn_{|\!\sim}(\alpha) \subseteq Cn_{|\!\sim}(\beta)$ and $\beta \mathrel{|\!\!\sim} \alpha$, as expressed by Lemma 5.2, breaks down. However, sometimes we need a plausible explanation mechanism, for instance if we want to induce default rules with exceptions (*cf.* Bain & Muggleton, 1991).

It seems that we have two options for formalising induction of such weak explanations. One option, which is left for future research, is to define $\alpha \mathrel{|\!\!<} \beta$ iff $Cn_{|\!\sim}(\alpha) \subseteq Cn_{|\!\sim}(\beta)$, and to investigate how, if at all, properties of $\mathrel{|\!\!\sim}$ carry over to $\mathrel{|\!\!<}$. The other option, that has been investigated to some extent in (Flach, 1991), is to put $\alpha \mathrel{|\!\!<} \beta$ iff $\beta \mathrel{|\!\!\sim} \alpha$ (and β consistent). Again applying the rewrite rule $\alpha \mathrel{|\!\!\sim} \beta \Rightarrow \beta \mathrel{|\!\!<} \alpha$ to rules of KLM, it is easily shown that Incrementality and Additivity of $\mathrel{|\!\!<}$ correspond to Right Weakening and Right And of $\mathrel{|\!\!\sim}$, respectively. Since both latter rules are satisfied in the weakest KLM system **C**, it seems safe to assume that even with plausible explanation mechanisms we have Incrementality and Additivity. However, it can also be shown that Right Strengthening of $\mathrel{|\!\!<}$ corresponds to Monotonicity of $\mathrel{|\!\!\sim}$. In other words, when strengthening a given plausible explanation one may reach a ‘hole’ in the Version Space: a hypothesis that is in between the S and G sets, yet does not explain the examples. Further research is needed to characterise the implications of this observation.

§27. REGULARITY-BASED CONFIRMATORY REASONING

In this section and the next one I will consider various formalisations of confirmatory reasoning. The idea underlying the formalisation in this section has essentially been stated in §19: to consider as possible hypotheses those formulas that are satisfied by every

[75]As a consequence, no strong explanatory consequence relation coincides with a reverse deductive consequence relation. This contrasts with e.g. the relation between monotonic and preferential reasoning, since every monotonic consequence relation is preferential. This phenomenon can be traced back to the inclusion of Left Consistency in **EM** (all other rules of **EM** are valid in $\mathbf{M_{rev}}$): this rule, like its relatives Consistency and Right Consistency, expresses that certain arguments should be excluded from the consequence relation.

regular interpretation, where a regular interpretation is one in which the objects behave in a way similar to the observed objects. The resulting form of confirmatory reasoning is called *regularity-based* confirmatory reasoning. Various possible notions of regularity come to mind. For instance, it could be established by a partition on the set of interpretations, such that an interpretation is regular if it falls in the same equivalence class as a model of the observations. As has been indicated in §19, such a notion of regularity would invalidate Hempel's conditions (C1) and (C2) — since Hempel's views formed the starting point for my investigations into confirmatory induction, I have chosen to develop a notion of regularity that remains more faithful to his ideas.

In proposing this notion I do not make any claim to originality: it can be directly traced back to Helft and De Raedt on the one hand, and Kraus *et al.* on the other. The idea is to use a partial ordering on interpretations, and to consider as regular interpretations those models of the observations that are minimal with respect to this ordering. Such a preference ordering on interpretations seems very natural for plausible reasoning — but is it also natural for conjectural reasoning? One could raise the following objections:

(*i*) regularity is a property of interpretations, not an ordering relation between interpretations;

(*ii*) even if regularity is an ordering between interpretations, this ordering must depend on the observations.

One possible defence against these objections is to point at the truth-ordering employed by Helft and De Raedt. This ordering selects as regular interpretations those in which no objects other than the observed ones have the attributed properties, which seems very reasonable. Nevertheless, each of the two points above makes some sense and deserves further investigation.

Below I give a characterisation of regularity-based confirmatory reasoning on the basis of such a preference ordering by means of so-called preferential confirmatory consequence relations. This system is a variation of the KLM system **P**, the main difference being the added requirement of consistency of the observations. After that I will demonstrate how closed-world reasoning on the basis of the truth-ordering of interpretations fits into the framework of preferential confirmatory reasoning.

Preferential co[redacted]rmatory consequence relations

I will start w[redacted] a semantic definition of preferential confirmatory reasoning. This definition esta[redacted]ishes a close variant of KLM's notion of a preferential structure (see Definition 4.[redacted]t only satisfiable formulas are allowed in confirmatory a[redacted]uments.

DEFIN[redacted]*confirmatory structure* is a triple $W = \langle S,l,<\rangle$, [redacted] $S \rightarrow U$ is a function that labels every state w[redacted] a model, and $<$ is a strict partial order[76] on S, called the *prefere[redacted]e ordering*, that is smooth[77]. A state $s \in S$ satisfies a formula $\alpha \in L$ iff $l(s)$ [redacted]α; the set of states satisfying α is denoted by $[\alpha]$, and a minimal

[76] I.e., $<$ is irreflexive and transitive.

[77] I.e. for any $S' \subseteq S$ and for any $s \in S'$, either s is minimal in S', or there is a $t \in S'$ such that $t<s$ and t is minimal in S'. This condition is satisfied if $<$ does not allow infinite descending chains.

element of $[\alpha]$ (wrt. $<$) will be called a *regular* state for α. The consequence relation defined by W is denoted by $\mid\!\sim_W$ and is defined by: $\alpha \mid\!\sim_W \beta$ iff (*i*) there is a state $s \in S$ satisfying α, and (*ii*) every regular state for α satisfies β. A consequence relation is called preferential confirmatory iff it is defined by a preferential confirmatory structure.

States are labelled with models, so the preference ordering between states can be used to define a relation between models — however, since [redacted] e same model can label several states, this relation will not, in general, be a partial o[redacted]r. According to Kraus *et al.*, the additional freedom provided by states 'is vital for [redacted] representation theorem to hold' (p.181), and I will follow them in this respect.

The following set of rules will be proved to a[redacted]atory consequence relations.

DEFINITION 7.14. The system **CP** consists of [redacted]

- **Confirmatory Reflexivity**: [redacted] $\alpha \;,\; \alpha \not\mid\!\sim \neg\beta$ [redacted]
- **Left Logical Equivalence**: [redacted] $\leftrightarrow\beta \;,\; \alpha \mid\!\sim \gamma$ / $\beta \mid\!\sim \gamma$
- **Predictive Right Weakening**: [redacted] $\wedge\beta\rightarrow\gamma \;,\; \alpha \mid\!\sim \beta$ / $\alpha \mid\!\sim \gamma$
- **Cautious Monotonicity**: $\dfrac{\alpha \mid\!\sim \beta \;,\; \alpha \mid\!\sim \gamma}{\alpha\wedge\beta \mid\!\sim \gamma}$
- **Right And**: $\dfrac{\alpha \mid\!\sim \beta \;,\; \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta\wedge\gamma}$
- **Left Or**: $\dfrac{\alpha \mid\!\sim \gamma \;,\; \beta \mid\!\sim \gamma}{\alpha\vee\beta \mid\!\sim \gamma}$
- **Right Consistency**: $\dfrac{\alpha \mid\!\sim \beta}{\alpha \not\mid\!\sim \neg\beta}$

In comparison with Hempelian consequence relations (Definition 6.12) two rules are added: Left Or and Cautious Monotonicity, both of which are valid principles of preferential reasoning. As argued in §24, Cautious Monotonicity can be seen as a strengthening of Verification, which states that if α confirms γ, then any predicted observation β provides further confirming evidence — Cautious Monotonicity extends this to any β that is also confirmed by α. The rule of Left Or states that if both α and β provide confirming evidence for γ, the knowledge that at least one of them is true should not refute γ. As will be demonstrated below (Lemma 7.16), both of these rules make use of the fact that regular states are minimal elements of the preference ordering — in other words, choosing another mechanism to select regular states would most probably violate both Cautious Monotonicity and Left Or. Note also that without these two rule the system would be rather weak: the other rules (with the exception of Left Logical

Equivalence) only say something about the right-hand side of confirmatory argumen[redacted]

The following lemma gathers some further properties of **CP**.

LEMMA 7.15. *Every consequence relation satisfying the rule*[redacted]*f **CP** is consistent, convex, disjunctively and conjunctively closed.*
Proof. By Lemma 6.7, Right Consistency implies Consiste[redacted] presence of Admssible Entailment and Left Reflexivity (whic[redacted]are both instances of Predictive Right Weakening).
Right Interval and Right Or are implied by Right Weakening[redacted] Predictive Right Weakening.
Right And is a rule of **CP**.

To see that **CP** is not incremental, let p and q be such that $p \mathrel{|\!\sim} q$ and [redacted]$\rightarrow q$. Cautious Monotonicity yields $p \wedge q \mathrel{|\!\sim} q$, and Predictive Right Weakening [redacted]es $p \wedge q \mathrel{|\!\sim} p$. Incrementality would give $q \mathrel{|\!\sim} p$, which means that p and q have the same 'confirmatory power', which defies our intuitions. Alternatively, it is easy to show that Incrementality is invalid by constructing an appropriate preferential confirmatory structure. For instance, let S h[redacted]e two states s and t with $s<t$, and let $[p]=\{t\}$ and $[q]=\{s,t\}$, then $p \wedge q \mathrel{|\!\sim} p$ but $q \not\mathrel{|\!\sim} p$[redacted]

I w[redacted]now prove the validity of the rules of **CP**.

[redacted] of **CP**). *Any preferential confirmatory* [redacted]*nsequence relation satisfies the rules of **CP**.*
[redacted]*oof*. For Confirmatory Reflexivity, suppose $[\alpha]$ is non-empty, and not all [redacted]; it follows that some state in S satisfies β.
[redacted], notice that logically equivalent formulas are [redacted]isfied by the same states.
[redacted]r Predictive Right Weakening, if all regular states for α satisfy β and [redacted]$\wedge\beta\rightarrow\gamma$, then (since all regular states for α satisfy α) all regular states [redacted] α satisfy γ.
For Cautious Monotonicity, we need the fact that $<$ is a smooth partial order. Suppose that $[\alpha]$ is non-empty, and all regular states for α satisfy β and γ — clearly, $[\alpha\wedge\beta]$ is non-empty. Now, let s be regular for $\alpha\wedge\beta$, then $s\in[\alpha]$; I will prove that s is regular for α. Suppose not, then there is a $t\in[\alpha]$ such that $t<s$ and t is regular for α. Now, every state regular for α satisfies β, hence $t\in[\alpha\wedge\beta]$. But this contradicts the minimality of s in $[\alpha\wedge\beta]$, hence s is regular for α and thus satisfies γ.
For Right And, if all regular states for α satisfy β and γ, then clearly they satisfy $\beta\wedge\gamma$.
For Left Or, note that $[\alpha\vee\beta] = [\alpha] \cup [\beta]$; thus, if $[\alpha]$ and $[\beta]$ are non-empty then so is $[\alpha\vee\beta]$. Furthermore, the set of regular states for $\alpha\vee\beta$ is a subset of the union of the regular states for α and β, since a state cannot be minimal in $[\alpha\vee\beta]$ without being minimal in at least one of $[\alpha]$ and $[\beta]$.
For Right Consistency, suppose $[\alpha]$ is non-empty, and all regular states for α satisfy β; it follows that no regular state for α satisfies $\neg\beta$.

In order to prove completeness, we need to build a preferential confirma[redacted] W from a given consequence relation $\mid\!\sim$ satisfying the rules of **CP**, such [redacted]t $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_W \beta$. As in the case of explanatory structures, such a confirmatory str[redacted]ure is built from a specific set of models. These models are selected relative to a give[redacted] follows.

> DEFINITION 7.17. Let $\mid\!\sim$ be a conjectural consequence relation. The [redacted]odel $m \in U$ is said to be *normal for* α iff for all β in L such that $\alpha \mid\!\sim \beta$, m [redacted]β.

So, a model is normal for a formula if it satisfies every confirmed hypothesis. Thus, given certain evidence the set of normal models decreases when the set of confirmed hypotheses increases. Notice that every model in U is normal for an inadmissible formula, which is therefore not satisfied by some of its normal models. An admissible formula is satisfied by every normal model, however. Notice also that if α is admissible and γ is inadmissible, then by Confirmatory Reflexivity $\alpha \mid\!\sim \neg\gamma$, hence no normal model for α satisfies γ.

The set of models normal for admissible formu[redacted]s will be us[redacted]d to build a prefe[redacted] confirmatory structure. The following lemma state[redacted]he key res[redacted] about normal m[redacted] they can characterise arbitrary Hempelian consequen[redacted] relations.

> LEMMA 7.18. *Suppose a consequence relatio*[redacted] *and Right And, and let α be an admissible* [redacted]*mula. All* [redacted]*rmal models for* [redacted] *α satisfy β iff $\alpha \mid\!\sim \beta$.*
>
> *Proof.* The if part follows from Definition 7[redacted]
> For the only-if part, suppose $\alpha \mid\!\sim \alpha$ and α [redacted]; I will sh[redacted]w that there is a [redacted] normal model for α that does not satisf[redacted]β. [redacted] $\Gamma_0 = \{\neg\beta$[redacted]$\cup \{\delta \mid \alpha \mid\!\sim \delta\}$; it suffices to show that Γ_0 is satisfiable. [redacted]upp[redacted]se not, the[redacted]by compactness there is a finite $\Delta \subseteq \{\delta \mid \alpha \mid\!\sim \delta\}$ such [redacted]at [redacted]$\Delta \rightarrow \beta$, i.e. [redacted]$\alpha \rightarrow (\Delta \rightarrow \beta)$; by Right Weakening $\alpha \mid\!\sim \Delta \rightarrow \beta$. But by R[redacted]ght And $\alpha \mid\!\sim \Delta$; using Right And and Right Weakening we obtain $\alpha \mid\!\sim$ [redacted]

Notice from the proof of Lemma 7.18 that n[redacted]l models exist for any admissible α.

Given an arbitrary preferential confirmat[redacted] the completeness proof is based on a preferential confirmatory[redacted]cted as follows:

(1) $S = \{\langle m,\alpha\rangle \mid \alpha$ is an admissible [redacted]ula, and m is a normal model for $\alpha\}$;
(2) $l(\langle m,\alpha\rangle) = m$;
(3) $\langle m,\alpha\rangle < \langle n,\beta\rangle$ iff $\alpha\vee\beta \mid\!\sim \alpha$ and [redacted]

Thus, states are pairs of admissible formul[redacted]belling function simply maps a state to the model it contains. [redacted]ondition (3) defines the preference ordering between states: note that $\beta \mid\!\sim \alpha$ is a special c[redacted]e of $\alpha\vee\beta \mid\!\sim \alpha$ by means of Left Or, and the fact that α is admissible. The condition m [redacted]β is added to make the ordering irreflexive; note that as a consequence any $\langle m,\alpha\rangle \in S$ is [redacted]nimal in $[\alpha]$.

The main difference between the preferential consequence relations of Kraus *et al.* and my preferential confirmatory consequence relations is the way unsatisfiable formulas are treated. In the KLM framework unsatisfiable formulas are characterised by the fact that they have every formula in L as a plausible consequence, which means that they don't have normal models. In my framework, unsatisfiable formul[redacted] confirm no hypotheses, and have all models in U as normal models. In both cases, t[redacted] structure W that is used to prove completeness contains only satisfiable formulas in[redacted]s states. This means that we can replicate most of KLM's results about the structure W[redacted]

PROPOSITION 7.19. *(1) (KLM 5.13) The relation* [redacted]*s a strict partial order.*
(2) (KLM 5.15) The relation < is smooth: for any [redacted] *either s is*
minimal in [α] or there exists a state t<s minim[redacted]*n [α].*
(3) (KLM 5.11) If $\alpha\vee\beta \mathrel{|\!\sim} \alpha$ *and m is a normal m*[redacted]*del for α that satisfies β,*
then m is a normal model for β.
(4) (KLM 5.14) $\langle m,\alpha\rangle$ *is minimal in [β] iff m* [redacted]*β and* $\alpha\vee\beta \mathrel{|\!\sim} \alpha$.

The first two statements express that W is a preferential confirmatory structur[redacted] remaining two are used in the proof of the following lemma.

LEMMA 7.20. *Let* $\mathrel{|\!\sim}$ *be a consequence relation satisfying the ru*[redacted]*s of* ***CP****,*
and let W be defined as above. If $\alpha \mathrel{|\!\sim} \beta$ *then* $\alpha \mathrel{|\!\sim}_W \beta$.
Proof. Suppose that $\alpha \mathrel{|\!\sim} \beta$; we will show that (*i*) there is a [redacted]
satisfying α, and (*ii*) every minimal state in [α] satisfies β.
(*i*) By Left Reflexivity α is admissible; furthermore, by Right C[redacted]sistency
$\alpha \mathrel{|\!\not\sim} \neg\beta$, so by Lemma 7.18 there exists a model *m* normal [redacted]
conclude that $\langle m,\alpha\rangle \in [\alpha]$.
(*ii*) Suppose $\langle n,\gamma\rangle$ is minimal in [α], then γ is an admissible fo[redacted]ula, *n* is
a normal model for γ that satisfies α, and $\gamma\vee\alpha \mathrel{|\!\sim} \gamma$ by Propos[redacted]on 7.19
(4). By Proposition 7.19 (3) *n* is a normal model for α, hence *n* [redacted]β.

The following lemma proves the converse of Lemma 7.20, and completes the proof[redacted] representation theorem.

LEMMA 7.21. *Let* $\mathrel{|\!\sim}$ *be a consequence relation satisfying the rules of* ***CP****,*
and let W be defined as above. If $\alpha \mathrel{|\!\sim}_W \beta$ *then* $\alpha \mathrel{|\!\sim} \beta$.
Proof. Suppose $\alpha \mathrel{|\!\sim}_W \beta$, then α must be admissible (since no state in *S* satisfies an inadmissible formula). Furthermore, given any model *m* normal for α, $\langle m,\alpha\rangle$ is minimal in [α], hence *m* satisfies β, and the conclusion follows by Lemma 7.18.

We may now summarise.

THEOREM 7.22 (Representation theorem for preferential confirmatory consequence relations). *A consequence relation is preferential confirmatory iff it satisfies the rules of* ***CP****.*
Proof. The only-if part is Lemma 7.16. For the if part, let $\mathrel{|\!\sim}$ be a consequence relation satisfying the rules of **CP** and let W be defined as above. Lemmas 7.20 and 7.21 prove that $\alpha \mathrel{|\!\sim} \beta$ iff $\alpha \mathrel{|\!\sim}_W \beta$, i.e. $\mathrel{|\!\sim}$ is preferential confirmatory.

The rule system **CP** demonstrates that Hempel's adequacy conditions for confirmation can be extended to a complete axiomatisation; the semantics of preferential confirmatory structures provide an operationalisation of the relation of confirmation. I will now show that closed-world reasoning *à la* Helft and De Raedt fits nicely into this framework.

Closed-world reasoning as preferential confirmatory reasoning

The concept of closed-world reasoning is borrowed from logic programming, where a logic program consisting of definite clauses can only have positive literals among its ground atomic consequences: it can't say that some ground atom is false. Consequently, the Herbrand base of ground atoms is divided into two subsets: those that are logical consequences of the program (these are true), and those that are not (the truthval [redacted] these are unknown). The well-known Closed-World Assumption (CWA) now propo [redacted] interpret the latter as being actually false. In the lattice of Herbrand models [redacted] program, this amounts to taking the bottom element of this lattice as the intended m [redacted]

The ordering in this lattice is the ordering of truth-content: one Herbrand mo [redacted] smaller than another if the set of ground atoms considered true in the first mod [redacted] subset of those considered true in the second model[78]. This truth-ordering provides [redacted] with the preferential confirmatory structures discussed above.

> LEMMA 7.23. *The consequence relation established by the truth-minimal model semantics is preferential confirmatory.*
> *Proof*. Such a consequence relation is defined by the following preferential confirmatory structure: take the set of Herbrand interpretations for *S*, the identity function for *l*, and the proper subset ordering for <[79].

This means that the truth-minimal model semantics inherits all the propert [redacted] preferential confirmatory consequence relations. Note that in the case of an ind [redacted] program this semantics would require truth in **all** minimal models of the program [redacted] is no sophisticated treatment of negation in the body of clauses.

When restricted to definite programs, the truth-minimal model semantics also [redacted] some properties not shared with every form of preferential confirmatory reasoning.

> LEMMA 7.24. *The consequence relation established by the truth-minimal model semantics for definite clauses satisfies the following property:*
>
> - **Admissible Completeness**: $\dfrac{\alpha \not\mathrel{|\!<} \neg\beta \;,\; \alpha \mathrel{|\!<} \alpha}{\alpha \mathrel{|\!<} \beta}$
>
> *Proof*. If α is admissible it is satisfied by some state, hence it has a unique minimal state labelled by a single Herbrand model, in which every formula is either true or false.

Notice that Admissible Completeness is satisfied by every preferential confirmatory structure in which the set of states satisfying a formula forms a downward semilattice under the preference ordering.

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[78]If we identify a (two-valued) Herbrand interpretation with the set of ground atoms it assigns the truthvalue **`true`**, then this ordering coincides with the subset ordering.

[79]Thus, the distinction between states and models is not needed for modelling minimal Herbrand model semantics as a preferential structure.

[redacted] order to characterise the main feature of the truth-minima[redacted]model semantics, namely tha[redacted]t minimises the assignment of truth to ground atoms, we [redacted]uld try something like

- [redacted]ilure: $$\frac{\alpha \; [redacted] \; p}{\alpha \not\mathrel{|\!\sim} \neg p}$$

in [redacted]ich α is a definite program and p is a positive literal. However, note that also α [redacted]$\neg p$, since definite programs only have **positive** ground atomic consequences, wh[redacted]e $\alpha \not\mathrel{|\!\sim} p$ by Right Consistency — in other words, the rule of Negation as Failure is invalid if p is a negative literal. A complete axiomatisation of the truth-minimal model semantics thus calls for a more fine-grained tool than consequence relations, that operate on the complete language. The interested reader is referred to (Dix, 1994ab) for a possible approach.

§28. Consistency-based confirmatory reasoning

The preference ordering in preferential confirmatory structures picks out certain models of the premisses, and draws conclusions justified by those models. When the evidence is incomplete, as is usually the case, the intended model may be not among those deemed most regular by the preference ordering. If this becomes evident by further observations, previously refuted hypotheses will have to be reconsidered. In this section I will demonstrate that it is possible to avoid such non-incremental behaviour. The main idea is to keep track of all models of the premisses α, and to consider a hypothesis β to be refuted ($\alpha \not\mathrel{|\!\sim} \beta$) only if β is satisfied by none of the models of α. Thus, we switch from entailment over preferred models of the observations to consistency relative to all models of α. The resulting form of confirmatory reasoning is therefore termed *consistency-based*.

The form of reasoning just described is characterised below by the system **CW**, for *weak* confirmatory reasoning. It defines 'α confirms β' as 'β is compatible with α', which is clearly the weakest possible definition of confirmation. Moreover, the system is also related to the rule systems **EM** and **CP** considered previously, since each of these is strictly more restrictive than **CW**: every explanatory or preferential confirmatory argument is also weak confirmatory, but not *vice versa* (with fixed background knowledge). Thus, the system **CW** represents the root of our hierarchy of rule systems for conjectural reasoning.

An alternative characterisation of consistency-based confirmatory reasoning is also (partly) worked out in this section. Here the idea is to represent the indeterminacy of the observations not by a large set of models, but instead by a few well-chosen **partial** or three-valued models, namely those partial models that are minimal with respect to the information they contain. We thus see that the concept of a minimal model again plays a role, but here its connotation is quite different from the preferential setting, where the minimal models represent educated guesses for the intended model. In contrast, the set of information-minimal models implies that the intended model is at least as informed as one of them. This approach has been inspired by the Version Space model of concept learning (§9), where the *S*-set of most specific concepts plays a similar role in delineating the set of all concepts consistent with the examples.

Weak confirmatory reasoning

As mentioned above, in weak confirma[redacted]y reasoning a hypothesis is confirmed if it is compatible with the observations.

DEFINITION 7.25. A *weak con*[redacted]*rmatory s*[redacted]*ucture* is a set $W \subseteq U$. The consequence relation it defines is d[redacted]noted by $\mid\!\sim$[redacted] and is defined by: $\alpha \mid\!\sim_W \beta$ iff there is an $m \in W$ such that m [redacted] $\alpha \wedge \beta$. A c[redacted]sequence relation is called weak confirmatory iff it is defined [redacted]

From this definition it is clear that weak confirmator[redacted] consequence relations satisfy both Right Weakening and Left Weakening (i.e. Increme[redacted]y. One additional rule is needed.

DEFINITION 7.26. The system **CW** consists o[redacted]

- **Predictive Incrementality**: $\dfrac{\text{[redacted]} \wedge \gamma \text{ [redacted] } \beta \ ,\ \alpha \mid\!\sim \gamma}{\text{[redacted]} \mid\!\sim \gamma}$
- **Predictive Right Weakening**: $\dfrac{\text{[redacted]} \wedge \text{[redacted]} \rightarrow \gamma \ ,\ \alpha \mid\!\sim \beta}{\text{[redacted]} \mid\!\sim \gamma}$
- **Disjunctive Rationality**: $\dfrac{\alpha \vee \beta \text{ [redacted]}}{\text{[redacted]} \mid\!\sim \gamma}$
- **Consistency**: $\dfrac{\text{[redacted]} \mid\!\sim \beta}{\text{[redacted]} \beta \rightarrow \neg\alpha}$

Disjunctive Rationality has not been considered before. The name has been borrowe[redacted] Kraus *et al.*, who identify it as a valid principle of plausible reasoning (although it[redacted] a derived rule of their system **P**, nor of my system **CP**). In the context of confir[redacted] reasoning, Disjunctive Rationality is a rather strong rule, which states that if a hyp[redacted] is confirmed by disjunctive observations it is confirmed by at least one of the disjun[redacted]

Every consequence relation satisfying the rules of **CW** is confirmatory, b[redacted] Hempelian.

LEMMA 7.27. *Every consequence relation satisfying the rules of* ***CW*** *is* [redacted] *consistent, incremental, convex, and disjunctively closed.*
Proof. Consistency is a rule of **CW**.
Predictive Incrementality implies Incrementality by Lemma 6.5.
Right Interval and Right Or are implied by Right Weakening, hence by [redacted] Predictive Right Weakening.

To see that **CW** is not conjunctively closed, suppose $\alpha \mid\!\sim \beta$, then (since $\alpha \not\mid\!\sim \beta \wedge \neg\beta$ by Consistency) Right And would imply $\alpha \not\mid\!\sim \neg\beta$. However, it is easy to find formulas α and β such that both $\alpha \wedge \beta$ and $\alpha \wedge \neg\beta$ are consistent.

The following theorem proves the equivalence of weak confirmatory structures and the system **CW**.

THEOREM 7.28 (Representation theorem for weak c[redacted]firmatory consequence relations). *A consequence relation* $\mathrel{|\!\sim}$ *is weak confir*[redacted] *rules of* ***CW***.

Pro[redacted] The only-i[redacted]art involves demonstrating [redacted]at $\mathrel{|\!\sim}_W$, as defined in Defi[redacted]ion 7.25, sat[redacted]ies the rules of **CW**, which i[redacted]

For t[redacted] if part, let $\mathrel{|\!\sim}$[redacted]e an arbitrary consequence re[redacted]ion satisfying the rules of **C**[redacted], and conside[redacted]he following weak confirmat[redacted] structure:

[redacted] $\alpha\wedge\beta$: $\alpha \mathrel{|\!\sim} \beta\}$

We [redacted]rove that [redacted]f $\alpha \mathrel{|\!\sim}_W \beta$. The if part follows directl[redacted] the cons[redacted]sts a mod[redacted]$\alpha\wedge\beta$.

Defi[redacted]hat Γ_0 is s[redacted] Supp[redacted]not, then [redacted]pactness there is a finite $\Delta\subseteq\{\delta \mid \neg\delta$ [redacted] that [redacted]$\wedge\alpha)$, i.e. [redacted]$\Delta\wedge\alpha \not\mathrel{|\!\sim} \beta$. Fu[redacted]ore, since [redacted] and $\neg\Delta\wedge\alpha$ [redacted] $(\neg\Delta\wedge\alpha)$[redacted]$(\Delta\wedge\alpha) \not\mathrel{|\!\sim} \beta$ b[redacted] and $\alpha \not\mathrel{|\!\sim} \beta$ by [redacted]men-tality. C[redacted]ntradiction, s[redacted]

Let m_0 [redacted]Γ_0; clearly [redacted]α and, since by Consistency $\beta\in\Gamma_0$, [redacted]β. It remains [redacted] prove that [redacted]W; i.e., that for all φ, ψ such that m_0 [redacted] we have $\varphi \mathrel{|\!\sim} \psi$. Let m_0 [redacted]$\varphi\wedge\psi$, then $\neg(\varphi\wedge\psi)\notin\Gamma_0$, hence $\varphi\wedge\psi$[redacted]β; by Predictive Right Weake[redacted]ing $\varphi\wedge\psi \mathrel{|\!\sim} \psi\wedge\beta$, by I[redacted]rementality $\varphi \mathrel{|\!\sim} \psi\wedge\beta$, and by Right Weakening $\varphi \mathrel{|\!\sim} \psi$[redacted]

One may remark that we cou[redacted]define[redacted]prefere[redacted]ial variant of weak confirm[redacted]ory reasoning by inc[redacted]ding a preferen[redacted]orderi[redacted]on mo[redacted]be confirmed if it is[redacted]tisfied by at le[redacted]s of the observations[redacted] will be worked [redacted]t below for th[redacted]special[redacted]p[redacted]tial models.

Consistency-base[redacted]*confirmatory re*[redacted]*oning w*[redacted]*partia*[redacted]

I will start with [redacted]om L to {**true**,**u**[redacted]**nown**,**false**[redacted][80]. The[redacted]mbol [redacted]l be used for satisfacti[redacted] by partial interpre[redacted]$\neg\beta$ stands for $m(\beta)$=[redacted]**alse** (m *falsifies* β), m [redacted]γ stands [redacted]r $m(\gamma)\in\{$**unknown**,**tr**[redacted]$\}$, and so on. U is[redacted]ow a set of partial mode[redacted]represent[redacted]g the background knowledge. Furthermore, α [redacted]β stands for $\forall m\in U$: if m [redacted]α then m [redacted]β.

Since there a[redacted]now three truth-values, th[redacted]truth-tables for the logical connectives need to be extended. The enlarged truth-tables can be derived from the intended interpretation of the third truth-value **unknown**, which represents lack of information as to whether a formula is true or false — we might say that the ‘information-content’ of **unknown** is less than either **true** or **false**. A compound formula is assigned the truth-value

[80] Or equivalently, a *partial* function from L to {**true**,**false**}.

unkn[redacted]**n** i[redacted]uth-value of an undete[redacted]mined [redacted]mpone[redacted] could be inc[redacted]ased[redacted]result in[redacted]ifferent truth-values of the form[redacted]a, w[redacted]ich p[redacted] m([redacted]¬[redacted])=u[redacted]**known** i[redacted]for instance m(α)=**unknown** an[redacted]m(β[redacted] ha[redacted]ce th[redacted]is [redacted] truth[redacted]valu[redacted]tained reg[redacted]ss w[redacted]er m([redacted]) is increased to **true** or **fals**[redacted] str[redacted]er, 1984, Ch.3 fo[redacted]wo [redacted] of [redacted]s).

[redacted] are no tauto[redacted] ins[redacted]hen m([redacted]**unk**[redacted] me[redacted]s[redacted]hat the[redacted]educt[redacted]n Theorem is invalid: for instanc[redacted]p. Or[redacted], however: fr[redacted]α[redacted]β→γ [redacted] α∧[redacted]γ and [redacted]∧¬γ[redacted]¬β. The latter two statemen[redacted] are not equiva[redacted]nt, since co[redacted] have q∧¬q [redacted]¬p, but we saw [redacted]rlier that p[redacted]q∨¬q. S[redacted]milarly there are di[redacted]erent notions of compatibility: α∧β [redacted]**false** me[redacted]s that the[redacted]is a[redacted]odel in which[redacted]oth α and β are true, while the weaker statements α [redacted]¬β and β [redacted]¬α [redacted]hich are not [redacted]uivalent) mean that there is a model in which one sta[redacted]ent is tru[redacted]whil[redacted]the other is no[redacted]alse. The reader is referred to (Thijsse, 1992[redacted] thorough exposition [redacted]

Partial mo[redacted]ls m[redacted] be [redacted]dered, j[redacted]t as truth-values, according to their inform[redacted] content.

DEF[redacted]TION [redacted]29[redacted]Let *U* b[redacted] set of partial models. The *inf*[redacted]*mation* [redacted] *orde*[redacted]onal a[redacted]m p∈*L*, [redacted] eith[redacted]m_1(p)=[redacted]**own** or m[redacted]p)=m_2(p).

That[redacted], w[redacted] implies m_2 [redacted]α for both α=p and α=¬p. This[redacted]an be ex[redacted] to th[redacted]co[redacted]plete langu[redacted] *L*, assumin[redacted] that Kleene's strong three-value[redacted] emp[redacted]:

P[redacted]OPOSITION[redacted].30 (Persistence).[redacted]$m_1 \leq m_2$ *iff for every formula* [redacted]

That[redacted]s, once formulas α and β have re[redacted]h_1[redacted]α) and **false** (m_2 [redacted]¬β) respectively, these truth-val[redacted]s persist everywhere higher [redacted]he information ordering. The only changes in truth-val[redacted] occurring when climbing the[redacted]formation ordering are from **unknown** to **true**, a[redacted] from [redacted]Furthermore, we may note that the information order[redacted]g is smooth, at least in the c[redacted] logic.

LEMMA 7.31. *For any for*[redacted]*la α∈L and any m∈U, if m* [redacted]*α then there exists an n∈U such that n* [redacted]*α, n≤m and n is minimal in* [redacted]*], where [α] denotes the set of models ve*[redacted]*fying α.*[81]

Proof. Let m be a model of α, and let m' be the interpretation obtained from m by changing the truthvalue of any propositional atom not occurring in α to **unknown**. It is easy to prove, by induction on the structure of α,

[81] This is a translation of Kraus *et al.*'s definition of smoothness to a non-strict partial order.

that m' is a model of α if m is; furthermore, by the definition of ≤, $m' \le m$. If m' is minimal in [α], we're done; if it is not, we can change the truthvalue of some propositional atom from **`true`** or **`false`** into **`unknown`** without changing the truthvalue of α. We can repeat this process until a minimal model has been found; since m' assigns **`true`** or **`false`** to only finitely many proposition symbols (i.e. a subset of those occurring in α) the process only takes a finite number of steps.

It is clear, then, that an information-minimal model will assign **`unknown`** to as many propositional atoms as possible. For instance, the formula `p`∨¬`p` has two information-minimal models, one assigning **`true`** to `p`, the other assigning **`false`** (and **`unknown`** to every other atom). Furthermore, the intended model (which is *total*, i.e. assigning truth or falsity to every formula) will be at least as informed as one of the information-minimal models of the observations — in other words, a hypothesis that is falsified by every information-minimal model of the observations is necessarily false in the intended model. This justifies the use of the information ordering as a preference ordering for consistency-based confirmatory reasoning, and to define $\alpha \mathrel{|\!\!\!<} \beta$ if β is not falsified in at least one information-minimal model of α[82]. However, due to the intricacies of partial logic this condition is too weak in one respect: suppose that for every minimal model m verifying α we have $m(\beta)$=**`unknown`**; it follows that not only both $\alpha \mathrel{|\!\!\!<} \beta$ and $\alpha \mathrel{|\!\!\!<} \neg\beta$, but also $\alpha \mathrel{|\!\!\!<} \beta\wedge\neg\beta$. This can be handled by adding a second condition for $\alpha \mathrel{|\!\!\!<} \beta$ to hold, to the effect that β should be verified by at least one model of α. Notice that this is not necessarily a **minimal** model: we may still have $m(\beta)$=**`unknown`** for every minimal model m verifying α, and thus both $\alpha \mathrel{|\!\!\!<} \beta$ and $\alpha \mathrel{|\!\!\!<} \neg\beta$. Such a situation simply means that α does not contain enough information to discriminate between the hypotheses β and ¬β.

I will now proceed as follows: I will first define partial preferential consistency-based confirmatory consequence relations (ppcc consequence relations for short) in the general case, where the preference ordering can be any partial order on U. After listing some of the properties of ppcc consequence relations in the general case, I will study the special case where the preference ordering is the information ordering.

DEFINITION 7.32. A *partial preferential consistenc*[redacted]*based confirmatory structure* (*ppcc structure* for short) is a pair $W = \langle V, \le$ [redacted] where $V \subseteq U$ is a set of partial models, and ≤ is a smooth partial order on [redacted]he consequence relation defined by W is denoted by $\mathrel{|\!\!\!<}_W$ and is defined b[redacted] is a model $m \in V$ verifying α and β, and (*ii*) there is a[redacted]inimal (wrt. ≤) model $m_0 \in [\alpha]$ not falsifying β, where [α] denotes the s[redacted]of models verifying α. A consequence relation is called ppcc iff it is define[redacted]

It has been noted above that Right And and Right Consis[redacted]ncy are invalid — the latter is replaced by a partial variant of Consistency:

- **Partial Consistency**: $\dfrac{\alpha\,[\text{redacted}] < \beta}{\alpha\wedge\beta\,[\text{redacted}]\,\mathbf{false}}$

[82] A more appropriate intuitive reading would be 'α does not disconfirm β'.

Other rules that are valid when reformulated in partial term[redacted] and Predictive Right Weakening:

- **Partial Left Logical Equivalence**: α [redacted] / β |< γ
- **Partial Predictive Right Weakening**: $\dfrac{\alpha\ [\text{redacted}]\ \beta\to\gamma\ ,\ \alpha \mathrel{|\!\!<} \beta}{\alpha \mathrel{|\!\!<} \gamma}$

As instances of Partial Predictive Right Weakening we obtain Left Reflexivity and (Partial) Admissible En[redacted]llment, as usual.

One may note that [redacted]ese rules are [redacted]rtial variants of rules from **CP**. However, the remaining rules of **C**[redacted] are invalid f[redacted] ppcc consequence relations, most notably Confirmatory Reflexivi[redacted]. Confirmatory Reflexivity (from $\alpha \mathrel{|\!\!<} \alpha$ and $\alpha \not\mathrel{|\!\!<} \neg\beta$ con[redacted]ude $\beta \mathrel{|\!\!<} \beta$) is [redacted]on $\alpha \not\mathrel{|\!\!<} \neg\beta$ is not necessarily caused by e[redacted]ry minimal m[redacted]el of α falsif[redacted]g ¬β, the other possible reason being that no model v[redacted]On[redacted] in the first case, but not in the second, can we concl[redacted]e that there e[redacted] as required by the conclusion of the rule. [redacted]iv[redacted] is valid.

The failure of Cauti[redacted]s Monotonicit[redacted] can be traced back to the fo[redacted]owing observa[redacted]n: $\alpha \mathrel{|\!\!<} \beta$ mea[redacted] that the minimal models of α do not falsify β (i.e. *m* [redacted]¬β); but this d[redacted]s **not** imply, [redacted]case, that these models also [redacted]erify β (i.e. *m* [redacted]β), hence the[redacted]inimal models of $\alpha\wedge\beta$ may be completely different from those of α. Fo[redacted] instance, we [redacted], but clearly we don't have $\alpha\wedge\beta \mathrel{|\!\!<} \neg\beta$. A partial variant of[redacted]ts a weaker form of Cautious Monotonicity, is however vali[redacted]

- **Partial Verification**: α [redacted] β[redacted]γ , α [redacted] β / α[redacted]γ |< β

As indicated in §24, this rule weakens Cautious Monoto[redacted] hypotheses β and γ are based on the same assumptions.

The rules found valid so far are collected in the following[redacted]ma.

LEMMA 7.33. *Any ppcc consequence relation satisfie[redacted] follo[redacted]ing rules:*

- **Right Reflexivity**: [redacted]β
- **Partial Left Logical Equivalence**: α [redacted] / β[redacted]
- **Partial Predictive Right Weakening**: α [redacted]
- **Partial Verification**: α [redacted]β[redacted]γ , α |< β / α[redacted]
- **Partial Consistency**: [redacted]< β / α∧β [redacted]**false**

Pr[redacted]. For Righ[redacted] i.e[redacted]es β.
Fo[redacted]tial Left[redacted]o[redacted]al E[redacted]ivale[redacted] ve[redacted]
Fo[redacted] let m_0 be [redacted]odel minimal in [α[redacted] α [redacted]
Fo[redacted]tial Ver[redacted]a[redacted]on, le[redacted] model minimal i[redacted]α] that does not fal[redacted] we have m_0[redacted]and ther[redacted]ore m_0∈[redacted]; s[redacted] m_0 [redacted] [α[redacted]Furthe[redacted] le[redacted] be the[redacted]el in [α] such[redacted]hat $m_0 \leq m$ and *m* [redacted]β[redacted]since α [redacted]β→γ, w[redacted]have *m* [redacted]γ [redacted]d therefore *m*∈[redacted]α∧γ].
Fo[redacted]Consistency, [redacted] some [redacted] in [α] verifie[redacted] α∧β [redacted]**false**.

The completeness of the set of r[redacted]stance, from the definition of $\mid\!\sim_W$ we see that i[redacted]α $\mid\!\sim_W$ β then[redacted] $\mid\!\sim_W$ α∧β; yet[redacted]substituting α∧β for γ in Partial Predictive Right Weak[redacted]ning will no[redacted]work, since α [redacted]β→(α∧β) is invalid if *m*(β)=**unknown** for some mo[redacted]to this problem is to replace Partial Predictive Right W[redacted]akening with[redacted]e sound but rather tedious

α∧β [redacted]γ , α∧¬γ [redacted]¬β , α $\mid\!\sim$ β
α $\mid\!\sim$ γ

The reader may want to check that the first two conditions vanish when α∧β is substituted for γ. However, my main concern at this stage is conceptual analysis rather than logical rigour, and I leave the issue of axiomatising ppcc consequence relations as an open problem.

I will now demonstrate that the information ordering can be used to build a ppcc structure defining consequence relations that are closely related to weak confirmatory consequence relations.

DEFINITION 7.34. A *partial weak confirmatory structure* is a ppcc structure $W = \langle V, \leq \rangle$, where ≤ is the information ordering on *U*, restricted to *V*. A consequence relation is called partial weak confirmatory iff it is defined by a partial weak confirmatory structure.

The following lemma demonstrates the close relation with weak confirmatory consequence relations.

LEMMA 7.35. *Let W be a partial weak confirmatory structure.* $\alpha \mid\!\sim_W \beta$ *iff there exists a model in [α] verifying β.*

[83] Another option is to assign to → the Lukasiewicz interpretation, which differs from Kleene's strong interpretation by putting *m*(α→β)=**true** if *m*(α)=*m*(β)=**unknown**. However, as a result the language is no longer persistent: increasing the truth-value of α or β may decrease the truth-value of α→β wrt. the information ordering.

> *Pr*[redacted] from Definition 7.32. [redacted]r the if part, let *m* be[redacted]model in [redacted]ma 7.31 [redacted]re exists a model $n\leq$[redacted] such that [redacted]is minimal in $[\alpha]$. Furthermore, *m* [redacted]$\neg\beta$, and therefore *n* [redacted]$\neg\beta$ by Pro[redacted]at $\alpha \mid\!\!<_W \beta$.

That is, for any partial[redacted]eak confirmatory structure $W = \langle V,\leq\rangle$ we have $\alpha \mid\!\!<_W \beta$ iff there is an $m\in V$ such that *m* [redacted]$\alpha\wedge\beta$ — in other words, partial weak confirmatory reasoning corresponds to verifia[redacted]ity of premisses and conclusion with respect to a set of partial models, and the information ordering does not affect the set of arguments but only serves as a computational tool. We may further note that if $\alpha\wedge\beta$ has a partial model, it has a total model (construct a total model m' from a partial model *m* by putting $m'(\mathrm{p})=m(\mathrm{p})$ if $m(\mathrm{p})\neq$**`unknown`**, and by arbitrarily putting $m'(\mathrm{p})$ to **`true`** or **`false`** otherwise). This means that for any partial weak confirmatory consequence relation there exists an equivalent weak confirmatory consequence relation. Since the converse is trivially true (any set of total models is also a set of partial models), this[redacted]roves the equivalence of weak and partial weak confirmatory reasoning.

The rules satisfied by partial weak confirmatory consequen[redacted] relations are thus partial versions of the rules in **CW**. The validity of Partial Predic[redacted] Partial Consistency has been proved above, in the general [redacted]se of ppcc consequence relations — for completeness' sake I prove the validity of the [redacted] which has become very easy in the light[redacted]f Lemma 7.35.

> COROLLARY 7.36. *Any partial*[redacted]*eak confirmatory c*[redacted] *satisfies the following rules:*
>
> - **Partial Predictive In**[redacted] $\dfrac{\alpha \;[\text{redacted}]\; \gamma\rightarrow\beta \;,\; \alpha \mid\!\!< \gamma}{[\text{redacted}] \mid\!\!< \gamma}$
>
> - **Disjunctive Rationali**[redacted] $\dfrac{\alpha\vee\beta \mid\!\!< \gamma \;,\; \beta \not\mid\!\!< \gamma}{[\text{redacted}] \mid\!\!< \gamma}$
>
> *Proof*. For Partial Predictive In[redacted]ementality, suppose that there exists a model in $[\alpha]$ verifying γ, and α [redacted]$\gamma\rightarrow\beta$; it follows that this model verifies β, hence there exists a model in [redacted] verifying γ.
> For Disjunctive Rationality, first note that $[\alpha\vee\beta] = [\alpha] \cup [\beta]$. Furthermore, if $\beta \not\mid\!\!< \gamma$ then no model in $[\beta]$ verifies γ, and thus the model in $[\alpha\vee\beta]$ verifying γ must be in $[\alpha]$.

In this section I have defined weak confirmatory reasoning, which is also the weakest form of conjectural reasoning since it only requires consistency between evidence and hypothesis. This form of reasoning is axiomatised by the system **CW**. I have further defined a partial, preferential variant of weak confirmatory reasoning (the axiomatisation of which remains, as yet, incomplete), and proved the equivalence with weak confirmatory reasoning if the information ordering on partial models is taken as the preference ordering. This form of confirmatory reasoning will be put to work in the next chapter, because it has a distinct advantage over preferential confirmatory reasoning: it is incremental.

I should add that consistency-based confirmatory reasoning, as defined in this section, does not, in the general case, establish a preservation semantics (§18). If we define $\alpha \mathrel{|\!\!<} \beta$ if there is a model satisfying both α and β, this cannot be reduced to a preservation function f constructing, from the models of α, a set of interpretations satisfying β, since this preservation function should operate independently from β. Even if we use the information ordering to select minimal models of α this does not establish a preservation function, since α may have several information-minimal models, and if both $\alpha \mathrel{|\!\!<} \beta$ and $\alpha \mathrel{|\!\!<} \gamma$ the minimal model α has in common with β may be different from the minimal model it has in common with γ. Only in the case that premisses always have a single information-minimal model (i.e. they are definite) does consistency-based confirmatory reasoning correspond to a preservation semantics. This indicates that the concept of a preservation semantics needs to be extended, or complemented by an alternative concept.

§29. SUMMARY AND CONCLUSIONS

In this chapter I presented the main formal results of this thesis, in the form of axiomatic characterisations of three different kinds of conjectural reasoning. The three rule systems **EM** (explanatory reasoning with a monotonic explanation mechanism), **CP** (preferential confirmatory reasoning), and **CW** (weak confirmatory reasoning) light parts of the map of conjectural reasoning, and thus provide a starting point for a descriptive theory of conjectural reasoning. In addition I have provided an alternative characterisation of weak confirmatory reasoning in terms of information-minimal partial models. Open problems include: characterising explanatory reasoning based on non-monotonic explanation mechanisms, characterising preferential consistency-based confirmatory reasoning, and extending the concept of a preservation semantics to cover consistency-based reasoning.

Each of the semantic structures characterising these rule systems has been designed to reflect current practice in the field of machine learning (chapter 3). Explanatory semantics models preservation of explanatory power, where an explanation is identified with a deductive proof, as in classification-oriented machine learning approaches. Preferential confirmatory structures generalise closed-world reasoning, as applied in Helft's and De Raedt's approaches to induction of integrity constraints. Weak confirmatory structures are based on compatibility between evidence and hypothesis, an idea that has been applied to incremental induction of integrity constraints in databases (see the next chapter).

However, neither of these semantics is claimed to fully capture the essence of inductive reasoning as performed by humans. For instance, identifying an explanation with a deductive proof seems to be quite crude, even if it is not uncommon in philosophy of science, since explanations often indicate a causal relation between observations and explanans. Also, formalising regular interpretations as minima with respect to a fixed ordering does not seem to be appropriate in all cases, since the ordering may depend on the observations. Even if further work is needed on these and related points, I believe that such future refinements can be incorporated in the formal framework set up in this thesis.

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