CHAPTER 6

PROPERTIES OF CONJECTURAL CONSEQUENCE RELATIONS

— in which the study of general patterns of inductive reasoning is commenced, leading to a catalogue of rules for explanatory and confirmatory reasoning —

THIS CHAPTER IS intended to provide an initial, conceptual analysis of conjectural consequence relations. It provides a detailed analysis of the adequacy conditions for confirmatory and explanatory reasoning, formulated as metalevel inference rules for conjectural consequence relations. I also discuss some additional properties not considered by Hempel, most notably: verification, falsification, and incrementality. Furthermore, I indicate the interrelationships that exist between a number of these rules.

A main difference with Hempel's approach is that in my framework observations and hypothesis are required to be compatible. This has led to a slight reformulation of some of Hempel's adequacy conditions. In the context of semantically expressed background knowledge in the form of a restricted set of models U, this requires a proof-theoretic counterpart: the concept of an admissible formula, with which I will start my investigations.

§21. THE CONCEPT OF AN ADMISSIBLE FORMULA

Recall from §6 that, according to Hempel's adequacy conditions for confirmation, contradictory evidence confirms any hypothesis. This choice can be justified by an analogy with deductive reasoning, where an inconsistent formula entails any formula, and allows a statement of the entailment condition (H1) in the form originally proposed by Hempel: any sentence which is entailed by an observation report is confirmed by it.

However, this choice does not carry over to the explanatory case: contradictory evidence is not explained by every hypothesis but only by contradictory ones, since the explanatory power of contradictory evidence encompasses every formula in the language, hence the explanatory power of non-contradictory formulas will always be less comprehensive. The only choice that can be made consistently in both the confirmatory and the explanatory case is to require that evidence and hypothesis are compatible, hence contradictory evidence does not confirm, nor is explained by, any hypothesis. The price to pay is that some adequacy conditions become slightly more complicated.

Since we will employ an implicit background theory by restricting the set of models U, a formula is contradictory iff it is unsatisfiable with respect to U. Clearly, this is a semantic notion — if we want an independent proof-theoretic development of conjectural reasoning, we will need a counterpart that is formulated in terms of a consequence relation. Such a counterpart is provided by the following definition.

> DEFINITION 6.1. Given a conjectural consequence relation $\mathrel{|\!\!<}$, a formula $\alpha\in L$ will be called *admissible* iff $\alpha \mathrel{|\!\!<} \alpha$, and *inadmissible* otherwise.

Of course, the full proof that a formula is admissible if and only if it is satisfiable by some model in *U* requires a representation theorem. However, if the consequence relation satisfies some simple properties, part of the relation between admissibility and satisfiability wrt. *U* can already be formulated. For instance, if the consequence relation is such that premisses and conclusion are always compatible, it follows that any contradictory α is inadmissible.

When translating adequacy conditions for conjectural reasoning to rules for conjectural consequence relations, the concept of admissibility is used as follows. Whenever a condition requires an observation report or a hypothesis to be consistent, it is translated to a requirement that the formula in question be admissible. In order to indicate that the resulting rule has an antecedent to this effect, we add the qualification 'admissible' to the name of the rule. Note that in the context of learning from examples, the intuitive reading of a condition $\alpha \mathrel{|\!\!<} \alpha$ would be 'α does not cover any negative example' if α occurs as a hypothesis elsewhere in the rule, and 'α does not conflict with the negative examples' if it occurs as evidence (as explained in §12, negative examples are assumed to be part of the background theory).

§22. ADEQUACY CONDITIONS FOR CONFIRMATORY REASONING

I will now translate Hempel's set of adequacy conditions f[redacted] confirmation (or rather, the slightly reformulated conditions (C1–4) listed in §8) [redacted] to rules for confirmatory consequence relations. Throughout this section, the intend[redacted] interpretation of $\alpha \mathrel{|\!\!<} \beta$ is 'observations α confirm hypothesis β'.[64]

We start with the entailment condition:

(C1) *Entailment condition*: any sentence which is [redacted]ailed by a consistent observation report is confirmed by it.

(C1.1) Any consistent observation report is [redacted]

As a first approximation of (C1), consider the following rul[redacted]

- **Entailment**: [redacted] $\to\beta$ / [redacted] $\mathrel{|\!\!<} \beta$

This rule is too generous, since it also applies if α is contradictory. As discussed in the previous section, we should add an antecedent to this rule requiring that α be admissible:

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[64] Here, the notion of confirmation should be taken liberally, including the possibility that α is indifferent regarding β. In Hempel's terminology, $\alpha \mathrel{|\!\!<} \beta$ means 'α does not disconfirm β'.

- **Admissible Entailment**: $$\frac{\alpha \rightarrow \beta \;,\; \alpha \mathrel{|\!<} \alpha}{\alpha \mathrel{|\!<} \beta}$$

In words: an admissible observation report confirms any of its consequences.

As for condition (C1.1), notice that by putting $\beta=\alpha$ in Entailment we obtain the axiom schema $\alpha \mathrel{|\!<} \alpha$ (Reflexivity), expressing that any observation report confirms itself. Clearly, this axiom schema is too strong, since it implies that any formula would be admissible. However, applying the same substitution to Admissible Entailment, or translating (C1.1) by reading ‘consistent’ as ‘admissible’, would yield a tautology. We should therefore explicitly add sufficiently weakened forms of Reflexivity. As it turns out, the following three rules are sufficient, each of them expressing some aspect of admissibility:

- **Left Reflexivity**: $$\frac{\alpha \mathrel{|\!<} \beta}{\alpha \mathrel{|\!<} \alpha}$$
- **Right Reflexivity**: $$\frac{\alpha \mathrel{|\!<} \beta}{\beta \mathrel{|\!<} \beta}$$
- **Confirmatory Reflexivity**: $$\frac{\alpha \mathrel{|\!<} \alpha \;,\; \alpha \mathrel{|\!\not<} \neg\beta}{\beta \mathrel{|\!<} \beta}$$

Left Reflexivity states that any formula that occurs as evidence in a conjectural argument is admissible; Right Reflexivity expresses the same for hypotheses occurring in some conjectural argument. These rules imply that a formula is admissible iff it occurs in some conjectural argument.

The third weakening of Reflexivity is much less intuitive, which is remarkable since Reflexivity itself seems such a simple rule. Confirmatory Reflexivity can perhaps best be understood when considering its contrapositive:

$$\frac{\alpha \mathrel{|\!<} \alpha \;,\; \beta \mathrel{|\!\not<} \beta}{\alpha \mathrel{|\!<} \neg\beta}$$

This rule states that if β is inadmissible, i.e. too strong a statement with regard to the background knowledge, its negation $\neg\beta$ is so weak that it is confirmed by arbitrary admissible formulas α.

Next, we arrive at the group of consequence conditions.

(C2) *Consequence condition*: if an observation report confirms every one of a set *K* of sentences, then it also confirms any sentence which is a logical consequence of *K*.

(C2.1) *Special consequence condition*: if an observation report confirms a hypothesis *H*, then it also confirms every consequence of *H*.

(C2.2) *Equivalence condition*: if an observation report confirms a hypothesis *H*, then it also confirms every hypothesis which is logically equivalent with *H*.

(C2.3) *Conjunction condition*: if an observation report confirms each of two hypotheses, then it also confirms their conjunction.

Putting general consequence condition (C2) aside for t[redacted]of the other three rules has an immediate translation into a [redacted]e for confirmatory consequence relations. Special consequence condition (C2.1) transla[redacted] to the rule of Right Weakening:

- **Right Weakening**: [redacted] / $\alpha \mid\!\sim \gamma$

In words, any hypothesis that is logically weaker than [redacted]given hypothesis confirmed by α is also confirmed by α. This rule will be further analy[redacted]ssible Entailment is an instance of Right Weakening (put β=[redacted]

Equivalence condition (C2.2) translates to Right Lo[redacted]cal Equivalence:

- **Right Logical Equivalence**: [redacted]$\leftrightarrow\gamma \,,\, \alpha \mid\!\sim \beta$ / $\alpha \mid\!\sim \gamma$

Clearly, Right Logical Equivalence follows from Right Weakening[65]. Finally, conjunction condition (C2.3) translates to the rule of Right And:

- **Right And**: $$\frac{\alpha \mid\!\sim \beta \,,\, \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta\wedge\gamma}$$

Right And is a very powerful rule, stating that the set of all confirmed hypotheses (interpreted as a conjunction) is itself confirmed. The combination of Right And and Right Weakening implies Hempel's general consequence condition (C2): if *E* confirms every formula of a set *K*, then it also confirms the conjunction of the formulas in *K* (by Right And), and therefore also every consequence of this conjunction (by Right Weakening)[66].

The next group of adequacy conditions is formed by the consistency conditions.

(C3) *Consistency condition*: every consistent observation report is compatible with the set of all the hypotheses which it confirms.

(C3.1) *Special consistency condition*: an o[redacted]rvation report is compatible with any hypothesis whi[redacted] it confirms.

(C3.2) An observation report does not conf[redacted]n any hypotheses which contradict each other.

Like the general consequence condition (C2), general consis[redacted]e translated directly into a rule, since we have no means t[redacted]efer to the set of confirmed formulas. However, in the light of Right And the conjunc[redacted]n of the formulas in this set is itself confirmed, and therefore it is sufficient to formula[redacted]l consistency condition (C3.1):

- **Consistency**: [redacted]$\mid\!\sim \beta$ / [redacted]$\beta\to\neg\alpha$

Notice that, as a corollary of this rule, we have that contradictory formulas are not admissible (put β=α).

Condition (C3.2) expresses that for any formula β, if β is in the set of confirmed

[65]In fact, (C2.2) is better numbered (C2.1.1), but I follow Hempel's original numbering here.

[66]This holds only for finite *K*, an assumption that I will make throughout.

hypotheses then $\neg\beta$ is not. This principle is expressed [redacted] the following rule:

- **Right Consistency**: $$\frac{\alpha \mathrel{|\!\sim} \beta}{\alpha \not\mathrel{|\!\sim} \neg\beta}$$

Clearly, Right Consistency is implied by Consistency [redacted]d Right And.

Finally, we consider the following equivalence con[redacted]on:

(C4) *Equivalence condition for observations*: [redacted]an observation report *B* confirms a hypothesis *H*, then any obse[redacted] equivalent with *B* also confirms *H*.

The translation of this condition is obvious.

- **Left Logical Equivalence**: [redacted] $$\frac{\alpha \leftrightarrow \beta\ ,\ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$$

It is interesting to note that (C4) is the only condition given by Hempel relating two confirmatory arguments with different observations. In §24 we will consider some additional rules of this important form.

§23. ADEQUACY CONDITIONS FOR EXPLANAT[redacted]RY REASONING

I will proceed by translating the set of adequacy conditions [redacted] explanatory reasoning (§8) into rules for explanatory consequence relations. Throug[redacted]t this section, the intended interpretation of $\alpha \mathrel{|\!\sim} \beta$ is 'hypothesis β is a possible explan[redacted]on of observations α'.

We start with the converse entailment condition:

(E1) *Converse entailment condition*: an observatio[redacted]ort is explained by every consistent formula entailing it.

(E1.1) Any consistent observation report ex[redacted]

The following rule provides a first approximation of (E1):

- **Converse Entailment**: [redacted]

However, an antecedent should be added to the effect that β is [redacted] admissible hypothesis:

- **Admissible Converse Entailment**: [redacted] $$\frac{\beta \rightarrow \alpha\ ,\ \beta \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \beta}$$

As in the case of confirmatory reasoning, condition (E1.1) is expressed by rules of restricted reflexivity. Left and Right Reflexivity are valid for explanatory reasoning as well; thus, the validity of these rules extends to conjectural reasoning in general. In addition we will employ the following rule:

- **Explanatory Reflexivity**: $$\frac{\alpha \mathrel{|\!\sim} \alpha\ ,\ \neg\beta \not\mathrel{|\!\sim} \alpha}{\beta \mathrel{|\!\sim} \beta}$$

Like its confirmatory counterpart, this rule is best understood by rewriting it into its contrapositive:

$$\frac{\alpha \mathrel{|\!\!<} \alpha \;,\; \beta \not\mathrel{|\!\!<} \beta}{\neg\beta \mathrel{|\!\!<} \alpha}$$

This rule states that if β is inadmissible, i.e. too strong a statement with regard to the background knowledge, its negation $\neg\beta$ is so weak that it is explained by arbitrary admissible formulas α.

Next, we consider the converse consequence conditi[redacted]

(E2) *Converse consequence condition*: if an o[redacted]ervation report is explained by a hypothesis *H*, then it is a[redacted] explained by every consistent formula entailing *H*.

(E2.1) *Equivalence condition*: if an ob[redacted]vation report is explained by a hypothesis *H*, then it is als[redacted]xplained every hypothesis which is logically equivalent wi[redacted]

Ignoring the requirement of consistency, the followi[redacted]le captures the essence of the converse consequence condition:

- **Right Strengthening**: [redacted] $\alpha \mathrel{|\!\!<} \gamma$

This rule expresses that any hypothesis that is logically [redacted]tion for α also explains α. However, according to our appro[redacted]ed recipe Right Strengthening should be weakened in order to allow only admissible hyp[redacted]theses:

- **Admissible Right Strengthening**: [redacted]$\frac{\rightarrow\beta \;,\; \alpha \mathrel{|\!\!<} \beta \;,\; \gamma \mathrel{|\!\!<} \gamma}{\alpha \mathrel{|\!\!<} \gamma}$

A point that should be stressed here is that Admissible[redacted]ight Strengthening requires certain properties of the underlying explanation mechani[redacted]n (i.e. monotonicity) — this will be elaborated in the next chapter.[67]

The next adequacy condition for explanatory reason[redacted]g is the special consistency condition:

(E3) *Special consistency condition*: an observatio[redacted]report is compatible with every hypothesis by which it is explain[redacted]

This condition is analogous to (C3.1), and hence translated[redacted] the rule of Consistency.

- **Consistency**: [redacted]$\frac{\mathrel{|\!\!<} \beta}{\rightarrow\neg\alpha}$

Consistency is therefore a rule generally valid for conjectural reasoning.

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[67] Notice that Admissible Right Strengthening fails to imply the equivalence condition (E2.1). However, as will be demonstrated in Lemma 6.9, (E2.1) is implied in the presence of some other rules for explanatory reasoning.

Clearly, there are no analogues in explanatory reasoning to the consistency condition (C3), nor to condition (C3.2), because alternative explanations may be incompatible. For instance, if p, q, and $\neg q$ are admissible we have both $p \mathrel{|\!\!<} p \wedge q$ and $p \mathrel{|\!\!<} p \wedge \neg q$, so the set of explanations of p is not consistent. Furthermore, the conjunction of two explanations is not necessarily an explanation, so Right And is invalid. However, notice that, just as the set of hypotheses confirmed by given observations is closed under conjunction by Right And, the set of observations explained by a given hypothesis is also closed under conjunction, giving rise to the rule of Left And or, as I will call it, Additivity.

- **Additivity**: $$\frac{\alpha \mathrel{|\!\!<} \gamma \;,\; \beta \mathrel{|\!\!<} \gamma}{\alpha \wedge \beta \mathrel{|\!\!<} \gamma}$$

This rule is of great importance for practical incremental induction algorithms. To understand its significance, suppose that α denotes the observations seen so far, while β is a new observation. We want to know whether γ, which is known to be an explanation of α, also explains $\alpha \wedge \beta$. The rule of Additivity now states that a sufficient condition for this is that γ explains the new observation β. Notice that this rule is clearly invalid for confirmatory reasoning.

As a corollary to Consistency and Additivity, the following rule is valid for explanatory consequence relations:

- **Left Consistency**: $$\frac{\alpha \mathrel{|\!\!<} \beta}{\neg\alpha \mathrel{|\!\!\not<} \beta}$$

This rule expresses that the set of observations explained by a given hypothesis β is consistent. Notice that Left Consistency is not valid for confirmatory reasoning: there is no inherent reason why the same hypothesis could not be conjectured given evidence $\neg\alpha$ if it can be conjectured given α.

Unlike the previous rules, the rules of Additivity and Left Consistency have not been derived from the adequacy conditions for explanatory reasoning, but represent additional postulates. In effect, this means that adequacy condition (E3) has been strengthened as follows:

(E3) *Explanatory consistency condition*: every consistent hypothesis is compatible with the set of all the observation reports which it explains.

(E3.1) *Special consistency condition*: an observation report is compatible with any hypothesis by which it is explained.

(E3.2) Two incompatible observation reports are not explained by the same hypothesis.

(E3.1) corresponds to the rule of Consistency, which in the presence of Additivity implies (E3). (E3.2) corresponds to the rule of Left Consistency.

Finally, the equivalence condition for observations (E4) has been treated earlier, and corresponds to Left Logical Equivalence.

§24. ADDITIONAL RULES FOR CONJECTURAL REASONING

In the previous two sections the rules of Left and Right Reflexivity, Left and Right Logical Equivalence, and Consistency turned out to be valid for both confirmatory and explanatory reasoning. Thus, these rules express general properties of conjectural reasoning, with an intuitive reading obtained by interpreting $\alpha \mid\!< \beta$ as 'hypothesis β could be conjectured if the evidence is α'[68]. The intention of the present section is to identify a few additional rules that are meaningful for both confirmatory and explanatory reasoning, even if not all of them are actually generally valid.

Predictions

Given his background as a member of the Wiener Kreis, and his familiarity with Popper's work, it is surprising that Hempel did not include the principles of verification and falsification among his adequacy conditions. The principle of verification can be formulated as follows:

> *A predicted [redacted]ervation verifies the hypothesis.*

Here, verification s[redacted]uld be interpreted qualitatively: the hypothesis is still a possible conjecture after obs[redacted]vation of a predicted observation. The principle of falsification, on the other hand, can b[redacted] formulated as follows:

> *An observati[redacted] the negation of which was predicted, falsifies the hypothesis.*

[redacted]early, if a hypothe[redacted] is falsified, it ceases to be a possible conjecture.
In order to formu[redacted]e these principles as rules, we need to define what a prediction is.

> DEFINITION [redacted]2. Given a conjectural argume[redacted] $\alpha \mid\!< \beta$, a formula $\gamma \in L$ is *predicted* iff [redacted]$\alpha\wedge\beta\rightarrow\gamma$.

[redacted]is definition stresses the fact that the 'epistemic [redacted]tcome' of a conjectural [redacted]gument [redacted] $\mid\!< \beta$ is $\alpha\wedge\beta$ rather than just β. Of course, $\alpha\wedge\beta$ is lo[redacted]cally equivalent to β if β explains [redacted] by deductive entailment, and the definition of a[redacted]fied to [redacted]$\rightarrow\gamma$. However, $\alpha\wedge\beta$ is logically stronger than β if[redacted] or if β [redacted]merely confirmed by α. Definition 6.2 covers all th[redacted]e cases.
It is now straightforward to formalise the principle[redacted]on:

- **Verification**: $\dfrac{[redacted]\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!< \beta}{\alpha\wedge\gamma \mid\!< \beta}$

- **Falsification**: $\dfrac{[redacted]\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!< \beta}{\alpha\wedge\neg\gamma \not\mid\!< \beta}$

[68]Notice that, with the exception of Consistency, each of these rules is also valid when $\alpha \mid\!< \beta$ is interpreted as 'β is a plausible consequence of α' (they are all valid in KLM's system **C**).

§24. Additional rules for conj[redacted]

In analogy with Verification and Falsification, if γ i[redacted] predicted formula, it can also be added to the hypothesis, but its negation cannot. This [redacted] rules:

- **Right Extension**: [redacted] $\dfrac{\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \beta\wedge\gamma}$

- **Right Excess**: [redacted] $\dfrac{\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!\sim \beta}{\alpha \not\mid\!\sim \beta\wedge\neg\gamma}$

In the next two subsect[redacted]s, I will investigate the implications of Verification and [redacted] Extension, and those of [redacted]lsification and Right Excess.

Verification, Right Exte[redacted]

The significance of Ver[redacted]cation and Right Extension exceeds conjectural reasonin[redacted] are also valid in KLM's [redacted]

> LEMMA 6.3. *Ve*[redacted]*fication and Right Extension are derived rules of the system* ***C****.*
> *Proof.* Suppose [redacted]$\wedge\beta\rightarrow\gamma$, then by Entailment $\alpha\wedge\beta \mid\!\sim \gamma$. Furthermore, if $\alpha \mid\!\sim \beta$, then by [redacted]t $\alpha \mid\!\sim \gamma$. By using Cautious Monotonicity we derive $\alpha\wedge\gamma \mid\!\sim \beta$; by using Right And we obtain $\alpha \mid\!\sim \beta\wedge\gamma$.

The proof of Lemma 6.3 suggests that Verification is related to Cautious Monotonicity, while Right Extension is related to Right And; I will now argue that this is indeed the case.

Cautious Monotonicity states that, if β and γ are two tentative conclusions from α, adding one of them to α still allows the other one as a tentative conclusion:

- **Cautious Monotonicity**: $\dfrac{\alpha \mid\!\sim \beta\ ,\ \alpha \mid\!\sim \gamma}{\alpha\wedge\beta \mid\!\sim \gamma}$

[redacted]nce β and γ are tentative conclusions, they can only be reached by means of additional [redacted] (coded in e.g. the preference ordering employed by the [redacted]asoner). Cautious Monotonicity states that these additional assumptions can be [redacted]mbined without problem: the assumptions on which β is based do not contradict the [redacted] based. Now, one can imagine, at least in principle, that [redacted] are incompatible — in such a case, we need to state that [redacted]th tentative conclusions are based on the same assumptions. This, of course, is exactly [redacted]at is stated by Verification: $\alpha\wedge\beta$ includes all the assumptions needed to derive β, and [redacted]$\wedge\beta\rightarrow\gamma$ states that, given these assumptions, γ can be obtained deductively rather than [redacted]ntatively. In a similar fashion, Right Extension represents a weakening of Right And. A concrete example of a form of confirmatory reasoning in which Cautious Monotonicity is replaced by its weaker versions Verification is given in §28.

Right Extension shows certain ways of strengthening the hypothesis. Although this rule is rather trivial for explanatory reasoning, it interacts in an interesting way with Right Weakening, a property of confirmatory reasoning.

LEMMA [redacted]4. *The combination of Right Exten[redacted]on and Right We[redacted]ng is equivale[redacted]to the following rule:*

- **P[redacted]dictive Right Weakening**: [redacted] $\wedge\beta\rightarrow\gamma\ ,\ \alpha\ |\!\!<$ [redacted] $\alpha\ |\!\!<\ \gamma$

Proof. I[redacted]order to derive Predictive Right Weakening, suppose [redacted] $\rightarrow\gamma$ and $\alpha\ |\!\!<$ [redacted] then by Right Extension $\alpha\ |\!\!<\ \beta\wedge\gamma$, and the result f[redacted]s by Right W[redacted]kening.
Predicti[redacted] Right Weakening implies Right Weakening, sin[redacted] $\beta\rightarrow\gamma$ implies [redacted] $\wedge\beta\rightarrow\gamma$.
Predicti[redacted] Right Weakening implies Right Extension, since [redacted] $\wedge\beta\rightarrow\gamma$ implies [redacted] $\wedge\beta\rightarrow\beta\wedge\gamma$.

In words, Predictive Right Weakening expresses that[redacted]iven a conjectural argument, any predicted formula is confirmed by the same evide[redacted]e. The manner in which Right Weakening is strengthened to Predictive Right We[redacted]ening reflects the idea that the 'epistemic outcome' of a conjectural argument $\alpha\ |\!\!<$ [redacted]is $\alpha\wedge\beta$ rather than just β. The adjective 'predictive' will be used whenever the conseq[redacted]dicted formula. Notice that by putting $\gamma=\alpha$ in Predictive [redacted]ght Weakening we obtain Left Reflexivity.

Lemma 6.4 results from the delicate interplay betw[redacted]n Right Extension, which can be seen as a restricted form of Right Strengthening, and [redacted] result can be obtained by combining Verification and the foll[redacted]perty of Left Weakening[69]:

- **Incrementality**: [redacted] $\rightarrow\beta\ ,\ \alpha\ |\!\!<\ \gamma$ [redacted]

The significance of this rule becomes perhaps mo[redacted] apparent when considering its contrapositive (an equivalent formulation):

[redacted] $\rightarrow\beta\ ,\ \beta\not|\!\!<\ \gamma$ / $\alpha\not|\!\!<\ \gamma$

That is, hypotheses that are refuted by certain evidence stay refuted when the evidence is strengthened. In other words, the set of refuted hypotheses is monotonically non-decreasing with the evidence, or equivalently, the set of possible hypotheses (the Version Space) is monotonically non-increasing. This is exactly the property that was mentioned in §12 as a necessary condition for performing incremental induction, which justifies its name.

A conjectural consequence relation is said to be *incremental* whenever it satisfies Incrementality. Note that Admissible Converse Entailment is an instance of Incrementality (put $\gamma=\alpha$). This implies that an incremental confirmatory consequence relation satisfies both Admissible Entailment and Admissible Converse Entailment, and one may wonder whether this combination of a rule and its converse re-introduces the

[69]Clearly, Left Weakening is invalid for deductive or plausible reasoning.

confirmation paradox through the back door. This q[redacted]stion will be answered in the next chapter: the semantics for incremental confirmatory re[redacted]soning proposed in §28, formulated in terms of partial models, satisfies both Right Weak[redacted]ing (hence Admissible Entailment) and Incrementality (hence Admissible Converse Ent[redacted]ment)[redacted]his is not at variance with Hempel's analysis (§6), since he considered the joint[redacted]ffect o[redacted]the consequence condition (Right Weakening) and the converse consequence [redacted]g), the latter o[redacted]which is much stronger than Converse Entai[redacted]ent.

I w[redacted]now prove a result that is similar to Lemm[redacted].4.

[redacted]EMMA 6.5. *The combination of Verific[redacted]on a[redacted]* [redacted]*e:*

[redacted]**ntality**: $\alpha\wedge\gamma\rightarrow$[redacted] β[redacted]

[redacted]*oof*. To derive Predictive Incrementality, suppose [redacted]
[redacted]n by Verification $\alpha\wedge\beta \mid\!\sim \gamma$, and by Incrementality β[redacted]
[redacted]edictive Incrementality implies Incrementality, sinc[redacted]$\rightarrow\beta$ implies [redacted]$\wedge\gamma\rightarrow\beta$.
[redacted]edictive Incrementality implies Verification, since [redacted]$\wedge\beta\rightarrow\gamma$ implies [redacted]$\wedge\beta\rightarrow\alpha\wedge\gamma$.

Predictive Incrementality can be seen as a strengthening of Incrementality, in the sense that β is not merely a weakening of evidence α, but can be any set of predicted observations. Since Verification is considered to be valid for arbitrary conjectural consequence relations, I will consider Incrementality and Predictive Incrementality interchangeable. Note that Right Reflexivity is an instance of Predictive Incrementality (put $\gamma=\beta$).

Falsification, Right Excess,[redacted]*nd Consisten*[redacted]

Falsification is not universa[redacted]y valid: for i[redacted]nce,[redacted] ($\alpha\wedge\neg\gamma$ may be unsatisfiabl[redacted]in which ca[redacted]nythi[redacted]g can b[redacted]dedu[redacted]d from it). Sim[redacted] Right Excess is not va[redacted]f[redacted] it turns ou[redacted] Falsification and Right [redacted]c[redacted]s are ref[redacted]mu[redacted]ons o[redacted]Consis[redacted]ncy[redacted]

LEMMA 6.6. *E*[redacted]*ch*[redacted]*f Falsif*[redacted]*equivalent to Consistency in th*[redacted]*ence.*
Proof. To deri[redacted]F[redacted]sificatio[redacted]s[redacted]pose[redacted] i.e.[redacted] then by Consiste[redacted]cy[redacted]$\wedge\neg\gamma \not\mid\!\sim$[redacted]T[redacted]rive Consiste[redacted]y from Falsification, suppose $\alpha \mid\!\sim \beta$ [redacted]hen by Falsification $\alpha\wedge\neg$**false** $\not\mid\!\sim$[redacted]and by Left[redacted]ogi[redacted] Equivalence [redacted]β, a contradiction.
To derive Righ[redacted]Excess, sup[redacted]se [redacted]$\wedge\beta\rightarrow\gamma$, i.e. [redacted]$\beta\wedge\neg\gamma\rightarrow\neg\alpha$, then by Consistency α [redacted]$\beta\wedge\neg\gamma$. To d[redacted]ve Consistency fro[redacted] Right Excess, suppose $\alpha \mid\!\sim \beta$ and [redacted]$\beta\rightarrow\neg\alpha$, i.e. [redacted]$\wedge\beta\rightarrow$**false**, then by Right Excess $\alpha \not\mid\!\sim$ $\beta\wedge\neg$**false**, an[redacted]y Right Log[redacted]al Equivalence $\alpha \not\mid\!\sim \beta$, a contradiction.

Of these three equivalent rules, Consisten[redacted] will be cons[redacted]ered the most fundament[redacted]
A consequence relation is said to be *consis*[redacted]

The next two results show that Consis[redacted]ncy is equiv[redacted]ent to Right Consistency [redacted] case of confirmatory reasoning, and to [redacted]eft Consist[redacted]cy in the case of expla[redacted] reasoning.

> LEMMA 6.7. *In the presence of Ad[redacted]ssible Entai[redacted]ent and Left Reflexivity, Right Consistency implies Consist[redacted]cy.*
> *Proof.* For Consistency, suppose [redacted]$\to\neg\alpha$, i.e. [redacted]$\to\neg\beta$. Now, either we have $\alpha \mid\!\sim \alpha$, or else $\alpha \not\mid\!\sim \alpha$. In t[redacted] former case, $\alpha \mid\!\sim \neg\beta$ by Admissible Entailment, and we conclude by [redacted]ght Consistency. In the latter case, we have $\alpha \not\mid\!\sim \delta$ for any δ by Left Refl[redacted]ivity.

As has already been remarked, Consiste[redacted]y in the prese[redacted] Right And. As a corollary to Lemm[redacted] 6.7, we have that Right Consistenc[redacted] Consistency are equivalent in the prese[redacted]e of Left Reflexivity, Admissible Entai[redacted] and Right And.

> LEMMA 6.8. *In the presence of R[redacted]ht Reflexivity and Admissible Converse Entailment, Left Consistency imp[redacted]s Consistency.*
> *Proof.* For Consistency, suppose [redacted]$\to\neg\alpha$. Now, either $\beta \mid\!\sim \beta$ or $\beta \not\mid\!\sim \beta$; in the former case, $\neg\alpha \mid\!\sim \beta$ by Ad[redacted]issible Converse Entailment, and we conclude by Left Consistency. In the latter case, we have $\delta \not\mid\!\sim \beta$ for any δ by Right Reflexivity.

Since Left Consistency follows from Consistency in the presence of Additivity, as has been noted above, it follows that Left Consistency an[redacted]Consistency are equivalent in the presence of Right Reflexivity, Admissible Converse En[redacted]ilment, and Additivity.

Convex consequence relations

As we have seen above, confirmatory reasoning obeys [redacted] while certain forms of explanatory reasoning satisfy the rule [redacted]Admissible Right Strengthening. I will now show that these two properties can be seen a[redacted]special cases of the more general property of convexity, thus providing a clear link with [redacted] §9).

In confirmatory reasoning, the rule of Right We[redacted]ening expresses that confirmed hypotheses can be arbitrarily weakened:

- **Right Weakening**: $$\frac{\text{[redacted]}\to\gamma\ ,\ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

Thus, if we order the set of hypotheses by logical implication, there will be an upper boundary of hypotheses confirmed by α, and every hypothesis below this boundary is also confirmed by α (fig. 6.1). Analogously, if the rule of Right Strengthening would be valid for explanatory reasoning, the set of explanations of given observations would have a

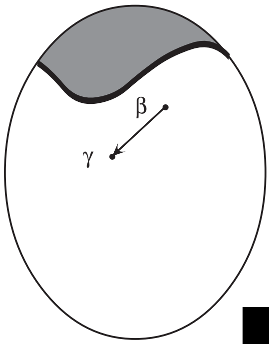

Figure 6.1. Graphical representation of t[redacted] set of hypotheses confirmed by given evidence α, when [redacted] holds. Arrows point from stronger to w[redacted]ker hypotheses.

lower boundary with respect to the ordering of logical[redacted]mplication, and every hypothesis above this boundary is an explanation of α (fig. 6.2).

- **Right Strengthening**:
$$\frac{\text{[redacted]} \rightarrow \delta\ ,\ \alpha \mid\!< \delta}{\alpha \mid\!< \gamma}$$

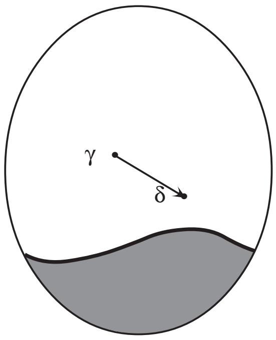

Figure 6.2. Graphical representation of the set of explanations of given evidence α, if Right Strengthening would hold.

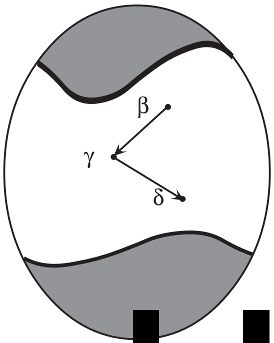

Figure 6.3. Graphical repres[redacted]ation of [redacted]e set of possible hypotheses given evidence [redacted]

We are interested in what these two rule[redacted]g, the answer is simple: supply the set of possible [redacted]ypothes[redacted] with both a lower and an upper boundary (fig. 6.3). The formal analogue of [redacted]e:

- **Right Interval**: [redacted]$\beta\rightarrow\gamma$, [redacted]$\rightarrow\delta$, $\alpha \mathrel{|\!\!<} \beta$, $\alpha \mathrel{|\!\!<} \delta$ / $\alpha \mathrel{|\!\!<} \gamma$

Clearly, both Right Weakening and Rig[redacted] Right Interval. A consequence relation is said to be *convex* [redacted]henever it satisfies Right Interva[redacted] justificati[redacted] for this terminology is that Rig[redacted] Interval expresses that the set {β | [redacted] is convex [redacted]rt. the ordering established by [redacted](if β,δ are elements, then so is any [redacted] that β≥γ≥[redacted].

Howe[redacted]r, no consistent explanatory consequence relation satisfies [redacted] Strengthe[redacted]hows that explanatory consequence relatio[redacted] satisfy A[redacted]issible Right Strengthening are convex.

LE[redacted] *Admissible Converse Entailment and Left* [redacted] *an*[redacted]*Right Reflexivity, Right Interval is implied by Admissible Right* [redacted] *Str*[redacted]*gthening.*
Pr[redacted]*f*. First of all, if $\alpha \mathrel{|\!\!<} \beta$ then $\beta \mathrel{|\!\!<} \beta$ by Right Reflexivity. Furthermore, if [redacted]$\beta\rightarrow\gamma$ then $\gamma \mathrel{|\!\!<} \beta$ by Admissible Converse Entailment, and $\gamma \mathrel{|\!\!<} \gamma$ by [redacted] Le[redacted]Reflexivity. We conclude by Admissible Right Strengthening.

An interesting corollary of Lemma 6.9 is that, in the presence of the same three rules, Admissible Right Strengthening implies Right Logical Equivalence, since the latter is an instance of Right Interval (put δ=β).

As suggested by the similarity between fig. 6.3 and fig. 3.1, the rule of Right Interval can be construed as expressing the Version Space model of concept learning from examples. This becomes immediate once the condition $\gamma \mathrel{|\!\!\sim} \gamma$ is read as 'γ does not cover any negative example'. For instance, an adequate reading of Admissible Right Strengthening in the context of learning from positive and negative examples is 'if β is a possible explanation[70] of positive examples α, and γ implies β without covering any negative examples, then γ is also a possible explanation of α'.

One may note that if the set of possible hypotheses is closed under conjunction, the upper boundary in fig. 6.3 is represented by a single hypothesis. The same can be said about the lower boundary if the set of possible hypotheses is closed under disjunction. This justifies the following definitions.

DEFINITION 6.10. A consequence relation is said to be *conjunctively closed* if it satisfies the following rule:

- **Right And**: $$\frac{\alpha \mathrel{|\!\!\sim} \beta\ ,\ \alpha \mathrel{|\!\!\sim} \gamma}{\alpha \mathrel{|\!\!\sim} \beta\wedge\gamma}$$

A consequence relation is said to be *disjunctively closed* if it satisfies the following rule:

- **Right Or**: $$\frac{\alpha \mathrel{|\!\!\sim} \beta\ ,\ \alpha \mathrel{|\!\!\sim} \gamma}{\alpha \mathrel{|\!\!\sim} \beta\vee\gamma}$$

As we will see in the next chapter[redacted]xplanatory consequence relations are typically disjunctively closed, while confirmato[redacted]consequence relations are conjunctively closed.

The following property of increme[redacted]tal convex explanatory consequence relatio[redacted] prove useful in the next chapter.

LEMMA 6.11. *In the presence [redacted]Predictive Incrementality and Explanatory Reflexivity, Admissible Right [redacted]engthening implies the following rule:*

- **Consistent Right [redacted]engthening**: $$\frac{\alpha \mathrel{|\!\!\sim} \gamma\ ,\ \neg\beta \mathrel{|\!\!\not\sim} \gamma}{\alpha \mathrel{|\!\!\sim} \beta\wedge\gamma}$$

Proof. Suppose $\neg\beta \mathrel{|\!\!\not\sim} \gamma$; since [redacted]$\neg(\beta\wedge\gamma)\wedge\gamma\rightarrow\neg\beta$, we have $\neg(\beta\wedge\gamma) \mathrel{|\!\!\not\sim} \gamma$ by Predictive Incrementality. Fu[redacted]ermore, suppose $\alpha \mathrel{|\!\!\sim} \gamma$, then by Right Reflexivity (which is an instance of Predictive Incrementality) $\gamma \mathrel{|\!\!\sim} \gamma$, so by Explanatory Reflexivity we have $\beta\wedge\gamma \mathrel{|\!\!\sim} \beta\wedge\gamma$. We conclude by Admissible Right Strengthening.

Consistent Right Strengthening is a powerful rule, which states that an explanation γ can be extended with any formula β of which the negation is not explained by γ.

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[70] By Right Reflexivity, this means that β itself is admissible, i.e. does not cover any negative example.

§25. SUMMARY AND CONC[redacted]SIONS

I will now summarise the main results of this chapt[redacted] The following rules have been found valid for either explanatory or confirmatory [redacted], any conjectural consequence relation will be assumed to sat[redacted]y them.

- **Left Reflexivity**: [redacted]
- **Right Reflexivity**: [redacted]
- **Right Logical Equivalence**: [redacted]
- **Left Logical Equivalence**: [redacted]
- **Consistency**: [redacted]
- **Verification**: $\dfrac{[redacted]\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!\!< \beta}{\alpha\wedge\gamma \mid\!\!< \beta}$
- **Right Extension**: $\dfrac{[redacted]\wedge\beta\rightarrow\gamma\ ,\ \alpha \mid\!\!< \beta}{\alpha \mid\!\!< \beta\wedge\gamma}$

It may be argued that neither of these rules seems typic[redacted]for conjectural reasoning: each of them could also occur in, say, the context of plausib[redacted]reasoning. Therefore, additional properties are needed to characterise different forms of c[redacted]jectural reasoning.

Adding the following three rules ensures satisfact[redacted]n of each of Hempel's adequacy conditions for confirmation[71].

DEFINITION 6.12. A conjectural conseque[redacted]e relation is said to be *confirmatory* if it satisfies the following rules:

- **Confirmatory Reflexivity**: $\dfrac{\alpha [redacted]< \alpha\ ,\ \alpha \not\mid\!\!< \neg\beta}{\beta \mid\!\!< \beta}$
- **Right Weakening**: $\dfrac{[redacted]\beta\rightarrow\gamma\ ,\ \alpha \mid\!\!< \beta}{\alpha \mid\!\!< \gamma}$

It is called *Hempelian* if, in addition, it satisfies the following rule:

- **Right And**: $\dfrac{\alpha \mid\!\!< \beta\ ,\ \alpha \mid\!\!< \gamma}{\alpha \mid\!\!< \beta\wedge\gamma}$

[71] That is, my version of those adequacy conditions (H1–4).

In the presence of Right Extension, Right Weakening is equivalent to Predictive Right Weakening. Furthermore, confirmatory consequence relations satisfy Admissible Entailment, which means that Right Consistency implies Consistency (Lemma 6.7). For Hempelian consequence relations, Right Consistency and Consistency are equivalent. The distinction between confirmatory and Hempelian consequence [redacted]ations is motivated by the fact that in the next chapter I will identify a form of confirm[redacted]tory reasoning that is not conjunctively closed.

Analogously, I have given rules that express the adequacy [redacted]onditions for explanatory reasoning.

DEFINITION 6.13. A conjectural consequence rel[redacted]ion is said to be *explanatory* if it satisfies the following rules:

- **Explanatory Reflexivity**: α [redacted] $\alpha , \neg\beta \not\mathrel{|\!\!<} \alpha$ / $\beta \mathrel{|\!\!<} \beta$
- **Admissible Converse Entailment**: [redacted] / $\alpha \mathrel{|\!\!<} \beta$
- **Additivity**: $\alpha \mathrel{|\!\!<} \gamma , \beta \mathrel{|\!\!<} \gamma$ / [redacted]

It is called *Peircean* if, in addition, it satisfies the [redacted]lowing rule:

- **Admissible Right Strengthening**: [redacted]$\rightarrow\beta , \alpha \mathrel{|\!\!<} \beta , \gamma \mathrel{|\!\!<} \gamma$ / $\alpha \mathrel{|\!\!<} \gamma$

For explanatory consequence relations Left Consistency and Consistency are equivalent. The distinction between explanatory and Peircean consequence relations is needed because not every explanatory consequence relation satisfies Admissible Right Strengthening — most notably, inference of plausible explanations does not.

I have furthermore identified the concepts of incremental and convex conjectural consequence relations. An incremental consequence relation only considers a hypothesis to be falsified when it is explicitly contradicted by available observations. This is expressed by the rule of Incrementality, or equivalently (in the presence of Verification) by the rule of Predictive Incrementality. Any incremental conjectural consequence relation satisfies Admissible Converse Entailment.

Convex consequence relations satisfy the property of Right Interval; this holds for confirmatory and Peircean consequence relations alike. Hempelian consequence relations are conjunctively closed, hence there is a single strongest hypothesis. Alternatively, consequence relations may be disjunctively closed, giving rise to a single weakest hypothesis — this is trivially so for confirmatory consequence relations (because of Right Weakening), but also for certain explanatory consequence relations, as we will see in the next chapter.

In principle, the results in this chapter have been formulated with an eye for the subsequent formulation and proof of a number of representation theorems in the next chapter. However, I would like to stress that the results obtained in this chapter also have a certain independent value: they provide a vocabulary with which existing approaches to and models of computational induction can be described and analysed. The following correspondences are particularly significant:

(*i*) the notion of convexity is the formal analogue of the well-known Version Space model of learning concepts from examples;

(*ii*) the notion of an admissible hypothesis can be construed as one not covering any negative example;

(*iii*) the property of Incrementality formalises a practical distinction between so-called interactive and empirical approaches to computational induction;

(*iv*) the property of Additivity represents a characteristic of explanatory induction that is computationally employed in many induction algorithms.

Thus, this chapter can also be seen as a contribution to the machine learning literature, in the sense that it provides some tools for the conceptual analysis of machine learning problems and algorithms.

APPENDIX: PROPERTIES OF CONJECTURAL DISCOVERY PROCEDURES

Although the central concern of this thesis is the notion of inductive proof procedures rather than discovery procedures, I will briefly demonstrate that some of the rules discussed in this chapter are meaningful also when interpreted as properties of discovery procedures. This is primarily meant to illustrate that the concept of a conjectural consequence relation may have significance outside the scope of this thesis.

I will assume that the discovery procedure operates as follows: it receives a sequence of observations $\alpha_1,\alpha_2,\alpha_3,\ldots$, and after each observation α_i it outputs a **single** hypothesis β_i, also referred to as the *current* hypothesis. This is denoted by $\alpha_1\wedge\ldots\wedge\alpha_i \mathrel{|\!\!<} \beta_i$; i.e. in statements of the form $\alpha \mathrel{|\!\!<} \beta$, α refers to the **complete** set of observations presented to the discovery algorithm up to a certain moment.

Clearly, the fact that the discovery procedure outputs, at any moment, a single hypothesis means that rules of the form

$$\frac{\ldots\ \alpha \mathrel{|\!\!<} \beta\ \ldots}{\alpha \mathrel{|\!\!<} \gamma}$$

are meaningless if $\gamma\neq\beta$ (and tautologies otherwise). This rules out a number of rules (Right Logical Equivalence, Left Reflexivity, Right Strengthening and its derivations, (Predictive) Right Weakening, Right And, and Admissible Entailment). Furthermore, rules of the form

$$\frac{\ldots\quad\ldots}{\alpha \mathrel{|\!\!<} \gamma}$$

are meaningless if γ allows for several instantiations for fixed α (this rules out (Admissible) Converse Entailment and Entailment).

Of the remaining rules, (Predictive) Incremental [redacted] ed use, since it expresses that the discovery procedure alwa [redacted] outputs the same hypothesis (for suppose that β is the current hypothesis, then β was [redacted] s step); Confirmatory and Explanatory Reflexivity do not [redacted] Right Consistency is true but trivial, since if $\alpha \mid\!< \beta$ then [redacted] or any $\gamma \neq \beta$. This leaves us with the following set of rules:

- **Verification**: [redacted] $\to\gamma$, $\alpha \mid\!< \beta$ / [redacted] $\gamma \mid\!< \beta$
- **Falsification**: [redacted] / α [redacted] $\neg\gamma \not\mid\!< \beta$
- **Consistency**: [redacted] $\mid\!< \beta$ / [redacted] $\beta\to\neg\alpha$
- **Right Reflexivity**: [redacted] / $\beta \mid\!< \beta$
- **Additivity**: [redacted] $\mid\!< \gamma$, $\beta \mid\!< \gamma$ / [redacted]
- **Left Consistency**: $$\frac{\alpha \mid\!< \beta}{\neg\alpha \not\mid\!< \beta}$$
- **Left Logical Equivalence**: [redacted] $\leftrightarrow\beta$, $\alpha \mid\!< \gamma$ / $\beta \mid\!< \gamma$

In the context of a discovery procedure, Verification expresses that the procedure will not change its current hypothesis if the next observation is a predicted one. In the terminology of (Angluin & Smith, 1983), Verification indicates that the discovery procedure is *conservative*. On the other hand, Falsification expresses that the current hypothesis must be abandoned if the next observation contradicts a prediction. The rule of Consistency expresses the stronger property that the current hypothesis is (logically) compatible with the observations.

Right Reflexivity seems a reasonable rule: if a discovery algorithm conjectures β on the basis of α, wouldn't it still conjecture β if it finds out that β is actually true? Additivity expresses another (but much weaker) 'conser [redacted] tive' property: the current hypothesis is retained if it would also have been outp [redacted] on the basis of the next observation alone. Left Consistency seems reasonable in [redacted] context of an explanatory discovery procedure. Left Logical Equivalence expresses t [redacted] the syntactical form of the observations (including their order) is irrelevant, which [redacted] discovery procedure.

However, note that a discovery procedure might als [redacted] satisfy some rules that are meaningless for proof procedures. For instance, the follo [redacted] hypotheses are never logically weaker than old ones:

- **Generalisation**: $\alpha \mid\!< \gamma$ [redacted] $\alpha\wedge\beta \mid\!< \delta$ / [redacted] $\to\gamma$

Clearly, this property does not make much sense for a proof procedure: if γ is a possible conjecture on the basis of α, and δ is a possible conjecture on the basis of $\alpha\wedge\beta$, it does not follow that there exists any relationship between γ and δ.

* * * * * *