COMS21202: Symbols, Patterns and Signals
Deterministic Data Models

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Data Modelling

▶ From Data to a Model
Data Modelling

- From Data to a Model
Data Modelling

▶ Models are descriptions of the data
▶ They encode our assumptions about the data
▶ Enabling us to:
  ▶ design ‘optimal’ algorithms
  ▶ compare and contrast methods
  ▶ quantify performance
▶ A model is ‘more than’ the data - a ‘generalisation’ of the data
Data Modelling

e.g. build a model of Messi as he rolls the ball across the pitch
Data Modelling

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**Data:** collect data of body joints during action from multiple examples
Data Modelling

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**Model:** ?
Data Modelling

- No need to play God
- Models do not have to exactly describe the ‘real world’, nor correctly model how data was generated
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- In some cases, we may approximate an underlying physical process as part of our model
- In others, this may be impossible and/or impractical
Data Modelling

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- Models do not have to exactly describe the ‘real world’, nor correctly model how data was generated
- In some cases, we may approximate an underlying physical process as part of our model
- In others, this may be impossible and/or impractical
- Models only need to enable us to define a method to tackle the task at hand
- Performance of the method then depends on how well the model ‘maps’ the data onto the required solution
- choice of model is often dictated by practicality of method, as well as by our assumptions about the data

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Fish Again,

- When classifying, we wish to find the model that achieves maximum discrimination.
Fish Again,

- When classifying, we wish to find the model that achieves maximum discrimination
- Model selected here is a linear classifier
Model Parameters

- Models are defined in terms of parameters (one or more)
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\[ x = t \]
Model Parameters

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One parameter needed: $x = t$

Two parameters needed: $y = mx + c$

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Two parameters needed: \( y = mx + c \)
Model Parameters

- Models are defined in terms of **parameters** (one or more)
- These may be empirically obtained e.g. by trial and error
- or from training data by **tuning** or **training** the model

one parameter needed $x = t$

two parameters needed $y = mx + c$
Generalisation vs. Overfitting

- **Generalisation** is the *probably* the most fundamental concept in machine learning.
Generalisation vs. Overfitting

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- We care about whether we can take a decision on *future* data.
- A good performance on training data is only a means to an end, not a goal in itself.
- In fact trying *too hard* on training data leads to a damaging phenomenon called overfitting.

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Generalisation vs. Overfitting

Example

Imagine you are trying to prepare for *Symbols, Patterns and Signals* exam this June.

source: Flach (2012), Machine Learning
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Imagine you are trying to prepare for Symbols, Patterns and Signals exam this June. You have access to previous exam papers and their worked answers available online. You begin by trying to answer the previous papers and comparing your answers with the model answers provided.

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Generalisation vs. Overfitting

Simpler models often give good performance and can be more general.

Highly complex models over-fit the training data.

Two parameters needed:

\[ y = mx + c \]

A large number of parameters needs to be tuned.

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Generalisation vs. Overfitting

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![Diagram](image)

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Deterministic Models

Deterministic models produce an output without a confidence measure. For example, for the fishy model, prediction of whether the fish is salmon or sea bass is given, without an estimate of how good the prediction is. Deterministic models do not encode the uncertainty in the data. This is in contrast to probabilistic models (next lecture).
Deterministic Models

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- e.g. For the *fishy* model, prediction of whether the fish is salmon or sea bass is given, without an estimate of how good the prediction is.
- Deterministic models do not encode the uncertainty in the data.
- This is in contrast to **probabilistic models** (next lecture).
Deterministic Models

To build a deterministic model,

1. Understand the task
2. Hypothesise the model’s type
3. Hypothesise the model’s complexity
4. Tune/Train the model’s parameters
Another Fish Problem

**Data:** a set of data points $D = \{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\}$ where $x_i$ is the length of fish $i$ and $y_i$ is the weight of fish $i$.

**Task:** build a model that can predict the weight of a fish from its length.
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**Model Type:** assume there exists a polynomial relationship between length and weight

**Model Complexity:** assume the relationship is linear

$\text{weight} = a + b \times \text{length}$

\[ y_i = a + bx_i \]  \hspace{1cm} (1)
Another Fish Problem

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weight = a + b \times length

\[ y_i = a + bx_i \quad (1) \]

Model Parameters: model has two parameters \( a \) and \( b \) which should be estimated.

- \( a \) is the y-intercept
- \( b \) is the slope of the line
Determinist Model - Line Fitting

- Finding the linear model parameters amounts to finding the *best fitting line* given the data.

- **criterion:** The best fitting line is that which minimises a distance measure from the points to the line.

![Graph showing a line fitting through points]
Determinist Model - Line Fitting

- Find $a, b$ which minimises

\[ R(a, b) = \sum_{i=1}^{N} (y_i - (a + bx_i))^2 \]

- This is known as the residual

- A method which gives a closed form solution is to minimise the sum of squared vertical offsets of the points from the line **Method of Least-Squares**
Least Squares Solution

Example

The Ceres Orbit of Gauss:

On Jan 1, 1801, the Italian astronomer G. Piazzi discovered the asteroid Ceres. He was able to track the asteroid for six weeks but it was lost due to interference caused by the sun.

A number of leading astronomers published papers predicting the orbit of the asteroid. Gauss also published a forecast, but his predicted orbit differed considerably from the others.

Ceres was relocated by one observer on Dec 7, 1801 and by another on Jan 1, 1802. In both cases the position was very close to that predicted by Gauss. Needless to say Gauss won instant fame in astronomical circles and for a time was more well known as an astronomer than as a mathematician.

One of the keys to Gauss’s success was his use of the method of least squares.

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Least Squares Solution

- A least squares problem is an overdetermined linear system of equations (i.e. number of equations $>>$ number of unknowns)
- Such systems are usually inconsistent
Least Squares Solution

Minimise residual by taking the partial derivatives, and setting them to zero (using chain rule)

\[ R(a, b) = \sum_i (y_i - (a + bx_i))^2 \]

\[ \partial R / \partial a = -2 \sum_i (y_i - (a + bx_i)) = 0 \]

\[ \partial R / \partial b = -2 \sum_i x_i (y_i - (a + bx_i)) = 0 \]

\[ a_{LS} = \bar{y} - b_{LS} \bar{x} \]

\[ b_{LS} = \frac{\sum_i x_i y_i - N \bar{x} \bar{y}}{\sum_i x_i^2 - N \bar{x}^2} \]

\[ \bar{z} \equiv \text{mean of } \{x_i\} \]
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Minimise residual by taking the partial derivatives, and setting them to zero (using chain rule)

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\frac{\partial R}{\partial a} = -2 \sum (y_i - (a + bx_i)) = 0
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# Least Squares Solution Example

## Example

Find the best least squares fit by a linear function to the data

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
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<tbody>
<tr>
<td>y</td>
<td>0</td>
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## Calculation

\[
\bar{x} = 0.5, \quad \bar{y} = 3.25
\]

\[
\begin{align*}
\sum_{i=1}^{N} x_i y_i &= 21 - 4 \times 0.5 \times 3.25 \\
\sum_{i=1}^{N} x_i^2 &= 6 - 4 \times 0.5^2
\end{align*}
\]

\[
\sum_{i=1}^{N} x_i y_i = 2.9
\]

\[
\hat{a}_{LS} = \bar{y} - \hat{b}_{LS} \bar{x} = 3.25 - 0.5 \times 1.8 = 1.8
\]

\[
y = 1.8 + 2.9 x
\]
Example

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|c|c|c|c}
  x & -1 & 0 & 1 & 2 \\
  y & 0 & 1 & 3 & 9 \\
\end{array}
\]

\[\bar{x} = 0.5, \quad \bar{y} = 3.25\]
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\[ b_{LS} = \frac{\sum_i x_i y_i - N \bar{x} \bar{y}}{\sum_i x_i^2 - N \bar{x}^2} \]
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\[ b_{LS} = \frac{\sum_i x_i y_i - N \bar{x} \bar{y}}{\sum_i x_i^2 - N \bar{x}^2} = \frac{21 - 4 \times 0.5 \times 3.25}{6 - 4 \times 0.5^2} = 2.9 \]
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$$a_{LS} = \bar{y} - b_{LS}\bar{x}$$
**Least Squares Solution Example**

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$a_{LS} = \bar{y} - b_{LS} \bar{x} = 3.25 - 0.5 b_{LS} = 1.8$

$y = 1.8 + 2.9x$
Least Squares Solution - Outliers

- Outliers can have disproportionate effects on parameter estimates when using least squares.

Because residual is defined in terms of squared differences, 'Best line' moves closer to outliers.
Least Squares Solution - Outliers

- Outliers can have disproportionate effects on parameter estimates when using least squares.
Least Squares Solution - Outliers

- Outliers can have disproportionate effects on parameter estimates when using least squares
- Because residual is defined in terms of squared differences
- ‘Best line’ moves closer to outliers
Least Squares Solution - Outliers

- Outliers can have disproportionate effects on parameter estimates when using least squares.
- Because residual is defined in terms of squared differences.
- ‘Best line’ moves closer to outliers (Lab - week 15)
Least Squares Solution - matrix form

- Least squared solution can be defined using matrices and vectors
- Easier when dealing with variables

\[ R(a, b) = \sum_i (y_i - (a + bx_i))^2 = \|y - Xa\|^2 \]

where \( y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \), \( X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} \), \( a = \begin{bmatrix} a \\ b \end{bmatrix} \)

\[ y - Xa = \begin{bmatrix} y_1 - a - bx_1 \\ \vdots \\ y_N - a - bx_N \end{bmatrix} \]
Least Squares Solution - matrix form

- To solve least squares in matrix form, find $a_{LS}$;

**WARNING:** This is not a derivation! It merely intends to give you intuition of the solution. For accurate understanding please refer to: this derivation - p8
Least Squares Solution - matrix form

To solve least squares in matrix form, find $a_{LS}$:

$$\|y - X a_{LS}\|^2 = 0$$

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(minimise vector’s length)

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To solve least squares in matrix form, find \( a_{LS} \);

\[
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\[
y - X a_{LS} = 0 \quad \text{(optimal vector is of length 0)}
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(minimise vector’s length)

$$y - X a_{LS} = 0$$
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$$X a_{LS} = y$$
(re-arrange)

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\[
y - X a_{LS} = 0 \quad \text{(optimal vector is of length 0)}
\]
\[
X a_{LS} = y \quad \text{(re-arrange)}
\]
\[
X^T X a_{LS} = X^T y \quad \text{(to get a square matrix)}
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Least Squares Solution - matrix form

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\[ \| y - X a_{LS} \|^2 = 0 \]  
\[ y - X a_{LS} = 0 \]  
\[ X a_{LS} = y \]  
\[ X^T X a_{LS} = X^T y \]  
\[ a_{LS} = (X^T X)^{-1} X^T y \]

(minimise vector’s length)  
(optimal vector is of length 0)  
(re-arrange)  
(to get a square matrix)  
(matrix inverse)

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Least Squares Solution Example - again

Example

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|cccc}
  x & -1 & 0 & 1 & 2 \\
  y & 0 & 1 & 3 & 9 \\
\end{array}
\]

\[
y = ax + bx
\]

\[
X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\
\end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 9 \\
\end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \\
\end{bmatrix}
\]

\[
a_{LS} = (X^T X)^{-1} X^T y = \begin{bmatrix} 1.8 \\ 2.9 \\
\end{bmatrix}
\]

\[
y = 1.8 + 2.9 x
\]
Least Squares Solution Example - again

Example

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|cccc}
  x & -1 & 0 & 1 & 2 \\
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\end{array}
\]

\[
y = \begin{bmatrix}
  0 \\
  1 \\
  3 \\
  9 \\
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
  1 & -1 & 1 & 1 \\
  -1 & 0 & 1 & 2 \\
\end{bmatrix}
\]

\[
x^T x = \begin{bmatrix}
  4 & 2 \\
  2 & 6 \\
\end{bmatrix}
\]

\[
x^T y = \begin{bmatrix}
  1 & 20 \\
  18 & 24 \\
\end{bmatrix}
\]

\[
y_{LS} = x^T \left( x^T x \right)^{-1} x^T y = \begin{bmatrix} 1.9 & 2.9 \end{bmatrix}^T 
\]

\[
y = 1.9 + 2.9 x
\]
Least Squares Solution Example - again

Example

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|cccc}
  x & -1 & 0 & 1 & 2 \\
  \hline
  y & 0 & 1 & 3 & 9 \\
\end{array}
\]

\[
y = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} \quad x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}
\]

\[
y = ax + b = \begin{bmatrix} 1.8 \\ 2.9 \end{bmatrix}
\]
Least Squares Solution Example - again

**Example**

Find the best least squares fit by a linear function to the data:

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

\[ y = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}, \quad x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \]

\[ a_{LS} = (X^TX)^{-1}X^Ty = \begin{bmatrix} 1.2 \ 2.9 \end{bmatrix} \]

\[ y = 1.2x + 2.9 \]
Least Squares Solution Example - again

**Example**

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|c|c|c|c}
\text{x} & -1 & 0 & 1 & 2 \\
\hline
\text{y} & 0 & 1 & 3 & 9 \\
\end{array}
\]

\[
y = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}, \quad x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad x^T x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}
\]

\[
a_{LS} = (x^T x)^{-1} x^T y
\]
Example

Find the best least squares fit by a linear function to the data

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\]

\[
a_{LS} = (x^T x)^{-1} x^T y = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}
\]

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COMS21202: Data Acquisition
Example

Find the best least squares fit by a linear function to the data

\[
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\]
Least Squares Solution Example - again

Example

Find the best least squares fit by a linear function to the data

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\]

The least squares solution is:

\[
a_{LS} = (x^T x)^{-1} x^T y = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}
\]

\[
a_{LS} = \begin{bmatrix} 0.8 \\ 2.9 \end{bmatrix}
\]

\[
y = 1.8 + 2.9 x
\]
Example

Find the best least squares fit by a linear function to the data

\[
\begin{array}{c|c|c|c|c}
  x & -1 & 0 & 1 & 2 \\
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\[
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Least Squares Solution Example - again

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\[ y = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} \quad x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \]

\[ X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \]

\[ a_{LS} = (X^T X)^{-1} X^T y = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2.9 \end{bmatrix} \]

\[ y = 1.8 + 2.9x \]
K-D Least Squares - matrix form

- Matrix formulation allows least squares method to be easily extended to data points in higher dimensions
K-D Least Squares - matrix form

- Matrix formulation allows least squares method to be easily extended to data points in higher dimensions
- Consider set of points $D = \{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\}$ where $x_i$ has K dimensions
K-D Least Squares - matrix form

- Matrix formulation allows least squares method to be easily extended to data points in higher dimensions
- Consider set of points $D = \{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\}$ where $x_i$ has $K$ dimensions
- For a model where $y_i$ is linearly related to $x_i$

$$y_i = a_0 + a_1 x_{i1} + a_2 x_{i2} + \cdots + a_K x_{iK} \quad (2)$$
K-D Least Squares - matrix form

- Solved in the same manner

\[
\begin{align*}
y(N \times 1) &= \begin{bmatrix} y_1 & \ldots & y_N \end{bmatrix}, \\
X(N \times (K + 1)) &= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\
& & \vdots \\
& & 1 & x_{NK} \end{bmatrix}, \\
a((K + 1) \times 1) &= \begin{bmatrix} a_0 & a_1 & \cdots & a_K \end{bmatrix}, \\
R(a) &= \| y - Xa \|_2 \\
a_{LS} &= (X^T X)^{-1} X^T y
\end{align*}
\]
K-D Least Squares - matrix form

- Solved in the same manner

\[
\begin{bmatrix}
y_1 \\
\vdots \\
y_N
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
y_1 \\
\vdots \\
y_N
\end{bmatrix}
\]

\[
R(a) = \| y - Xa \|_2
\]

\[
a_{LS} = (X^T X)^{-1} X^T y
\]

where \( (X^T X) \) is a \((K+1) \times (K+1)\) square matrix
K-D Least Squares - matrix form

- Solved in the same manner

\[
\mathbf{y}_{(N \times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix},
\]

\[
\mathbf{y} \in \mathbb{R}^{N \times 1}, \quad \mathbf{X} \in \mathbb{R}^{N \times (K+1)},
\]

\[
\mathbf{a} \in \mathbb{R}^{(K+1) \times 1},
\]

\[
R(a) = \| \mathbf{y} - \mathbf{Xa} \|_2^2,
\]

\[
a_{\text{LS}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y},
\]

where \((\mathbf{X}^T \mathbf{X})\) is a \((K+1) \times (K+1)\) square matrix.
K-D Least Squares - matrix form

- Solved in the same manner

\[ y_{(N \times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & \cdots & x_{NK} \end{bmatrix}, \]
K-D Least Squares - matrix form

- Solved in the same manner

\[
\begin{align*}
y_{(N\times1)} &= \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \\
x_{(N\times(K+1))} &= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & \cdots & x_{NK} \end{bmatrix},
\end{align*}
\]
K-D Least Squares - matrix form

Solved in the same manner

\[
y_{(N \times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad x_{(N \times (K+1))} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & \cdots & x_{NK} \end{bmatrix}, \quad a_{((K+1) \times 1)} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix}
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\[
R(a) = \|y - Xa\|_2
\]

\[
a_{LS} = (X^T X)^{-1} X^T y
\]
K-D Least Squares - matrix form

- Solved in the same manner

\[ y_{(N \times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad x_{(N \times (K+1))} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & \cdots & x_{NK} \end{bmatrix}, \quad a_{((K+1) \times 1)} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix} \]

\[ R(a) = \| y - Xa \|^2 \]
K-D Least Squares - matrix form

- Solved in the same manner

\[
\begin{align*}
\mathbf{y}_{(N \times 1)} &= \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, & \mathbf{x}_{(N \times (K+1))} &= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1K} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & \cdots & x_{NK} \end{bmatrix}, & \mathbf{a}_{((K+1) \times 1)} &= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_K \end{bmatrix}
\end{align*}
\]

\[
R(\mathbf{a}) = \| \mathbf{y} - \mathbf{Xa} \|^2
\]

\[
\mathbf{a}_{\text{LS}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}
\]

where \((\mathbf{X}^T \mathbf{X})\) is a \((K + 1) \times (K + 1)\) square matrix
General Least Squares - matrix form

- Matrix formulation also allows least squares method to be extended to polynomial fitting
- For a polynomial of degree $p + 1$

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_p x_i^p$$
General Least Squares - matrix form

- Solved in the same manner

\[
y_{(N \times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad x_{(N \times (p+1))} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^p \\ 1 & x_2 & x_2^2 & \cdots & x_2^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^p \end{bmatrix}, \quad a_{((p+1) \times 1)} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}
\]

\[
a_{LS} = (X^T X)^{-1} X^T y
\]

where \((X^T X)\) is a \((p + 1) \times (p + 1)\) square matrix
Generalisation and Overfitting - again

Data

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Generalisation and Overfitting - again

Data

\[ p = 3 \]
\[ \text{Residual} = 3.5744 \]

\[ p = 1 \]
\[ \text{Residual} = 4.7557 \]

\[ p = 4 \]
\[ \text{Residual} = 3.4236 \]

\[ p = 2 \]
\[ \text{Residual} = 3.7405 \]

\[ p = 5 \]
\[ \text{Residual} = 3.4217 \]
Generalisation and Overfitting - again

Data

$p = 1$
Residual = 4.7557

$p = 2$
Residual = 3.7405
Generalisation and Overfitting - again

Data

$p = 1$
Residual = 4.7557

$p = 2$
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$p = 3$
Residual = 3.5744

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Generalisation and Overfitting - again

Data

\( p = 1 \)
\text{Residual} = 4.7557

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Generalisation and Overfitting - again

Data

\[
p = 1
\]
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Generalisation and Overfitting - again

Strong effect on how to generalise to future data
Further Reading

- Linear Algebra and its applications
  Lay (2012)
  - Section 6.5
  - Section 6.6
  - Available online
    http://www.math.usu.edu/powell/pseudoinverses.pdf