Skyrmions and monopoles: D_2-symmetric 3-solitons

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June 1999

The similarity between Skyrmions and Bogomolny-Prasad-Sommerfield (BPS) monopoles has often been remarked. In this talk I will illustrate this similarity by reviewing the rational map ansatz and by discussing the specific example of D_2-symmetric 3-solitons.

The Skyrme model is a nonlinear Lagrangian for a SU_2-valued pion field U. It was proposed by Skyrme as a theory of nuclear interactions: the classical B-nucleon nucleus is a B-Skyrmion, that is, a minimum energy Skyrme field with topological charge B. U is required to attain its vacuum value, the identity, at spacial infinity and so it is a map between topological 3-spheres. This is the origin of the topological charge B.

The 1-Skyrmion is spherical, it is given by the hedgehog ansatz

$$U_1(\mathbf{x}) = \exp(if(r)\,\widehat{\mathbf{n}} \cdot \boldsymbol{\sigma}) \tag{1}$$

where $\widehat{\mathbf{n}}$ is an outward pointing unit normal and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. $f(r)$ is a profile function which is normally determined numerically and is very well approximated by the kink profile [1]: $f(r) = 4 \arctan \exp(-r)$. The 1-Skyrmion has six zero modes: three translational modes and three isospin modes corresponding to global SU_2 transformation.

The product ansatz approximates a configuration of well-separated 1-Skyrmions by the multiplication of individual 1-Skyrmion fields. A Skyrmion has three distinct orthogonal dipoles which are rotated by the global SU_2 transformation. Two well-separated Skyrmions attract or repel depending upon the mutual orientation of these dipoles. Two Skyrmions maximally attract if the difference of orientation of their dipoles is a rotation of π about an axis orthogonal to the line of separation. The product ansatz and the attractive orientation are reviewed in [2, 3]. An attractive orientation is pictured in Figure 1.

Figure 1: Two Skyrmions attracting. The different arrow types correspond to different dipoles.

For B from three to nine the B-Skyrmion has been calculated numerically by evolving an attractive configuration, [4, 5]. The energy of the 3-Skyrmion, for example, is .92 times

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the energy of three 1-Skyrmions. The 3-Skyrmion is tetrahedrally symmetric in the sense that tetrahedral rotations in space are equivalent to isospin rotations.

In [6] a simple ansatz for B-Skyrmions is introduced. It is similar to the hedgehog ansatz (1). In the hedgehog ansatz the outward pointing unit normal maps a 2-sphere identically to a 2-sphere. The ansatz of [6] replaces that map with a more general map, $\widehat{\mathbf{n}}_R$, from 2-sphere to 2-sphere. The 2-sphere has a complex structure given by stereographic projection, it is assumed that $\widehat{\mathbf{n}}_R$ is holomorphic with respect to that complex structure. This ensures the ansatz space is finite dimensional.

The ansatz is usually called the rational map ansatz and is

$$U(r, z) = \exp(if(r)\,\widehat{\mathbf{n}}_R \cdot \boldsymbol{\sigma}) \tag{2}$$

where

$$\widehat{\mathbf{n}}_R = \frac{1}{1 + |R|^2}(2\mathcal{R}e(R), 2\mathcal{I}m(R), 1 - |R|^2) \tag{3}$$

and $R(z)$ is holomorphic map of degree B: $R(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials of degree at most B with at least one of them of degree B. z is the inhomogenous angular coordinate in space and R is the inhomogeneous angular coordinate in $\mathrm{SU}_2 = S^3$. SU_2 Möbius transformation of z corresponds to spatial rotation, SU_2 Möbius transformation of R corresponds to isospin rotation.

The rational map ansatz reduces the field theoretic problem of minimizing the Skyrme energy function to the finite dimensional problem of choosing the rational map R. By substituting the ansatz (2) into the Skyrme energy functional an energy functional on the space of rational maps, usually called $\mathcal{I}$, is derived. The rational map is chosen to minimise $\mathcal{I}$. After $R(z)$ is chosen the profile function $f(r)$ is calculated numerically. In [6] rational maps for the known B-Skyrmions are calculated and the ansatz fields are found to have energies only a few percent higher than the true, numerically determined, minima.

The rational map approximation to the 3-Skyrmion is pictured in Figure 2. The corresponding rational map is

Figure 2: A surface of baryon density for the rational map approximation to the 3-Skyrmion

$$R_3(z) = \frac{\sqrt{3}iz^2 - 1}{z(z^2 - \sqrt{3}i)}. \tag{4}$$

This rational map is tetrahedrally symmetric in the sense that a tetrahedral transformation of z is equivalent to a SU_2 Möbius transformation of R_3:

$$\begin{aligned}
R_3(-z) &= -R_3(z) \\
R_3\left(\frac{iz + 1}{-iz + 1}\right) &= \frac{iR_3(z) + 1}{-iR_3(z) + 1}
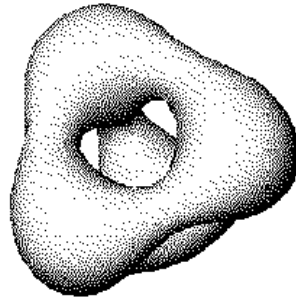\end{aligned} \tag{5}$$

$R_3(z)$ is thought to minimize $\mathcal{I}$ for degree three and the corresponding ansatz field is a good approximation of the 3-Skyrmion. More general degree three rational maps give Skyrme fields with rather high energy.

A BPS monopole is a solution to the Bogomolny equation: $D_i\Phi = \frac{1}{2}\epsilon_{ijk}F_{jk}$, where F_{ij} is an $\mathfrak{su}_2$ field strength and Φ is an $\mathfrak{su}_2$ scalar field. Φ is required to have unit length at spatial infinity and because of this it maps the large 2-sphere at infinity to a 2-sphere in $\mathfrak{su}_2$. The degree of this map gives a topological classification of the solution and a k-monopole is a solution with degree k. There is a $(4k-1)$-dimensional space of gauge inequivalent k-monopoles. Solutions to the Bogomolny equation are minimal energy solitons in a Yang-Mills-Higgs theory, the full Lagrangian gives a metric for the space of k-monopoles and the corresponding geodesic flow is known to approximate low-energy dynamics in the full Yang-Mills-Higgs theory [7, 8].

The striking thing is that that gauge inequivalent monopoles are classified by rational maps, or, more precisely, a k-monopoles is classified by an SU_2 orbits in the space of degree k rational maps [9]. This relationship between monopoles and rational maps is natural under rotation and so the tetrahedrally symmetric map R_3 corresponds to a tetrahedrally symmetric monopole. Furthermore, the space of D_2 symmetric degree three maps is two dimensional. D_2 is generated by π rotations about orthogonal axes taken to be the Cartesian axes. The rational map that is symmetric under these transformations is

$$R(z) = \frac{\alpha z^2 - 1}{z(z^2 - \alpha)}, \tag{6}$$

where α is complex. This rational map degenerates when $\alpha = \infty$ or $\alpha = \pm 1$. At these points, one monopole is infinitely far along each direction of a Cartesian axis and the third is at the origin.

There are exceptional values of α where the symmetry is larger than D_2. The obvious example is $\alpha = 0$ where R has axial symmetry about the x_3-axis. There is axial symmetry about the x_1-axis if $\alpha = -3$ and about the x_2-axis if $\alpha = 3$.

If α is real, the rational map is symmetric under inversion and since this is a one-parameter fixed point set of a group action it is a geodesic. In this case, the 3-monopoles have the same symmetries as 2-monopoles. The point $\alpha = \infty$ is the same as $\alpha = -\infty$ and the one-parameter family is represented by the equator in Figure 3. Around the equator are three points with axial symmetry and three degenerate points. Geodesics run from one degenerate point, through a point with axial symmetry and then on to another degenerate point. The geodesics correspond to $\pi/2$ scattering mimicking the classic 2-monopole head-on scattering except with an an extra monopole at the origin [10, 11].

If α is imaginary, the rational map is symmetric under S_4 rotary-reflection in the x_3-axis. This transformation is a rotation of $\pi/2$ about the x_3-axis followed by a reflection in the x_1x_2-plane. This geodesic intersects the α real geodesic at the torus $\alpha = 0$. If $\alpha = \pm\sqrt{3}i$, then there is the additional C_3 symmetry about $x_1 = x_2 = x_3$ which, along with the D_2 symmetry, generates the tetrahedral group. This geodesic is one of the twisted line scattering geodesics discussed in [12]. Monopoles approach along the x_3-axis, coalesce to form a tetrahedron, then a torus and finally the dual tetrahedron, before separating

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again along the same axis. These are also twisted line scattering geodesics in the x_1-axis with $|\alpha|^2 + 2\mathcal{R}e(\alpha) = 3$ and in the x_2-axes with $|\alpha|^2 - 2\mathcal{R}e(\alpha) = 3$.

In Figure 3, α is represented by a sphere and the various exceptional values are marked. The real α circle of inversion symmetry is the horizontal equator. The other great circle is the geodesic of twisted line scattering in the x_3-axis.

The rational map (6) only approximates the 3-Skyrmion when $\alpha = \pm\sqrt{3}i$. The ansatz probably gives good approximations to any D_2 $B = 3$ Skyrmion lying in the descent manifold from the axially symmetric saddle-point of the energy functional. For general α, however, the rational map ansatz does not give Skyrme fields which resemble the corresponding 3-monopole. Nonetheless there are Skyrme fields resembling these monopoles.

Three Skyrmions, superimposed so that they are equally spaced along a line, move towards each other if the two outlying Skyrmions maximally attract the middle one. The relative orientation of the outlying Skyrmions does not affect the fact of overall dipole attraction. However, the orientation is significant in the interaction region where the product ansatz is not valid. If the outlying Skyrmions are in the same orientation, Figure 4a, they form a pretzel configuration [13]. If there is a relative rotation of π about the separation axis, Figure 4b, they form the tetrahedral configuration, [14].
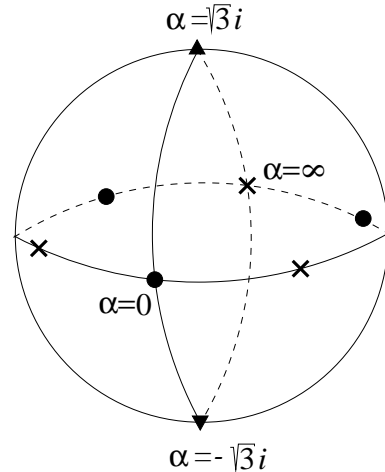

Figure 3: The α-sphere of D_2 symmetric rational maps. Crosses denote degenerate maps, dots denote axially symmetric maps and triangles denote the tetrahedral maps.
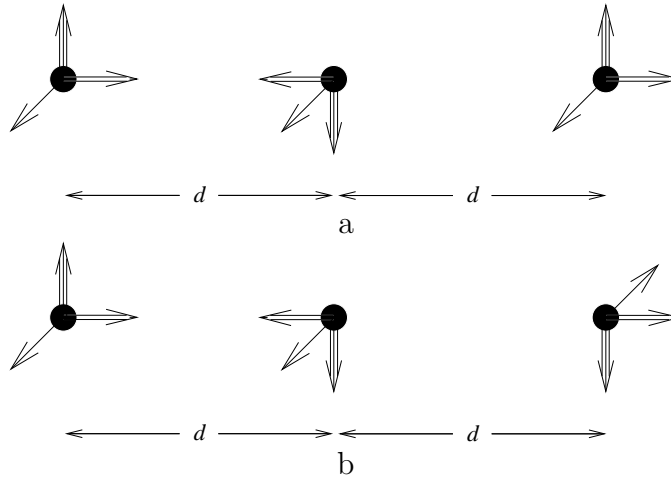
The similarities of this behaviour to that of D_2 symmetric 3-monopoles is apparent. It is imagined that the unstable manifold of three well-separated D_2 Skyrmions is two dimensional. When the three Skyrmions are well-separated these two dimensions correspond to separation and a relative dipole orientation. In the monopole analogue, the two dimensions are also imagined to correspond to separation and relative phase orientation. The important feature of the analogy is that attraction partially fixes the relative dipole orientation of the individual Skyrmions.

Acknowledgements

Figure 4: Three equally spaced collinear Skyrmions.

References

[1] P.M. Sutcliffe, Phys. Lett. **292B** (1992) 104.

[2] N.S. Manton, Acta Phys. Polon. **B25** (1994) 1757.

[3] B.J. Schroers, Z. Phys. **C61** (1994) 479.

[4] E. Braaten, S. Townsend and L. Carson, Phys. Lett. **235B** (1990) 147.

[5] R.A. Battye and P.M. Sutcliffe, Phys. Rev. Lett. **79** (1997) 363.

[6] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, Nucl. Phys. **B510** (1998) 507.

[7] N.S. Manton, Phys. Lett. **110B** (1982) 54.

[8] D. Stuart, Commun. Math. Phys. **166** (1994) 149.

[9] S. Jarvis, A rational map for Euclidean monopoles via radial scattering, Oxford preprint, (1996).

[10] C.J. Houghton and P.M. Sutcliffe, Nonlinearity **9** (1996) 1609.

[11] R. Bielawski, Nonlinearity **9** (1996) 1463.

[12] C.J. Houghton and P.M. Sutcliffe, Nucl. Phys. **B464** (1996) 59.

[13] R.A. Battye and P.M. Sutcliffe, Bound states and dynamics of Skyrmions in (3+1) dimensions, in preparation (1997).

[14] R.A. Battye and P.M. Sutcliffe, Phys. Lett. **391B** (1997) 150.