Solutions for exercise sheet 1

Solution 1.
The number X of heads on the second round is the same as if we toss all the coins twice and count the number that show heads on both occasions. Each coin shows heads twice with probability p^2, so

$$P(X = k) = \binom{n}{k} p^{2k}(1-p^2)^{n-k}$$

Solution 2.
We have that

$$P(X = 1, Z = 1) = P(X = 1, Y = 1) = \frac{1}{4} = P(X = 1)P(Z = 1)$$

This, together with three similar equations, show that X and Z are independent. Likewise, Y and Z are independent. However,

$$P(X = 1, Y = 1, Z = -1) = 0 \neq \frac{1}{8} = P(X = 1)P(Y = 1)P(Z = -1)$$

so that X, Y and Z are not independent.

Solution 3.
a. No!

b. Let X have mass function: $f(-1) = \frac{1}{9}$, $f(\frac{1}{2}) = \frac{4}{9}$, $f(2) = \frac{4}{9}$. Then

$$\mathbb{E}(X) = -\frac{1}{9} + \frac{2}{9} + \frac{8}{9} = 1 = -\frac{1}{9} + \frac{8}{9} + \frac{2}{9} = \mathbb{E}(1/X)$$

Solution 4.*
Let I_j be the indicator function of the event that the outcome of the $(j+1)$th toss is different from the outcome of the jth toss. The number R of distinct runs is given by $R = 1 + \sum_{j=1}^{n-1} I_j$. Hence

$$\mathbb{E}(R) = 1 + (n-1)\,\mathbb{E}(I_1) = 1 + (n-1)2pq$$

where q = 1-p. Now remark that I_j and I_k are independent if $|j-k| > 1$, so that

$$\mathbb{E}(R-1)^2 = \mathbb{E}\left\{\left(\sum_{j=1}^{n-1} I_j\right)^2\right\} = (n-1)\,\mathbb{E}(I_1) + 2(n-2)\,\mathbb{E}(I_1 I_2) + \{(n-2)(n-3)\}\,\mathbb{E}(I_1)^2.$$

This follows from two observations. First we need to separate the independent and non-independent cases so that

$$\left(\sum_{j=1}^{n-1} I_j\right)^2 = \sum_{j=1}^{n-1} I_j^2 + 2\sum_{1\le j<k<n} I_j I_k = \sum_{j=1}^{n-1} I_j^2 + 2\sum_{j=1}^{n-2} I_j I_{j+1} + 2\sum_{j=1}^{n-3}\sum_{k=j+2}^{n-1} I_j I_k.$$

Second all the I_j are identically distributed so that $\mathbb{E}(I_j) = \mathbb{E}(I_1)$ and $\mathbb{E}(I_j I_{j+1}) = \mathbb{E}(I_1 I_2)$. Now $\mathbb{E}(I_1) = 2pq$ and $\mathbb{E}(I_1 I_2) = p^2 q + pq^2 = pq$, and therefore

$$\begin{aligned}\mathrm{var}(R) &= var(R-1)\\ &= (n-1)\,\mathbb{E}(I_1) + 2(n-2)\,\mathbb{E}(I_1 I_2) - (n-2)(n-3)\,\mathbb{E}(I_1)^2 - (\mathbb{E}(R-1))^2\\ &= (n-1)2pq + (n-2)2pq + (n-2)(n-3)(2pq)^2 - ((n-1)2pq)^2\\ &= 2pq(2n-3-2pq(3n-5)).\end{aligned}$$

Solution 5.

(a) We have that

$$\begin{aligned} y\,\mathbb{E}(aY + bZ \mid X = x) &= \sum_{y,z}(ay+bz)P(Y=, Z=z \mid X=x)\\ &= a\sum_{y,z} yP(Y=y, Z=z \mid X=x) + b\sum_{y,z} zP(Y=y, Z=z \mid X=x)\\ &= a\sum_{y} yP(Y=y \mid X=x) + b\sum_{z} zP(Z=z \mid X=x).\end{aligned}$$

The remaining parts are similarly straightforward once you write out expectation as a sum. Recall also that $\mathbb{E}(Y \mid X)$ is itself a random variable that is a function of X.