Advanced topics in TCS

Exercise sheet 1.

Probability.[†]

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These questions only concern discrete random variables.

Question 1. Probability mass functions

We toss n coins and each one shows heads with probability p, independently of each of the others. Each coin which shows heads is tossed again. What is the mass function of the number of heads resulting from the second round of tosses?

Question 2. Independence

Let X and Y be independent random variables, each taking the values -1 or 1 with probability $\frac{1}{2}$, and let $Z = XY$. Show that X, Y and Z are pairwise independent. Are they independent?

Question 3. Expectation

Is it generally true that $\mathbb{E}(1/X) = 1/\mathbb{E}(X)$? Is it ever true that $\mathbb{E}(1/X) = 1/\mathbb{E}(X)$?

Question 4.* Indicator random variables

A biased coin is tossed n times and heads shows up with probability p on each toss. A *run* is a sequence of throws which result in the same outcomes, so that for example, the sequence HHTHTTH contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Find the variance of the number of runs.

[†]Thank you to Grimmett and Stirzaker for the questions on this sheet.

Question 5. Conditional expectation
Show the following:

(a) $\mathbb{E}(aY + bZ \mid X) = a\,\mathbb{E}(Y \mid X) + b\,\mathbb{E}(Z \mid X)$ for $a, b \in \mathbb{R}$.

(b) $\mathbb{E}(Y \mid X) \geq 0$ if $Y \geq 0$.

(c) $\mathbb{E}(1 \mid X) = 1$.

(d) if X and Y are independent then $\mathbb{E}(Y \mid X) = \mathbb{E}(Y)$.