Topics in TCS

Sparse recovery

Raphaël Clifford

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Imagine that at some point in a stream, there is exactly one token with non-zero frequency. How can we recover it?

1-sparse recovery - simple case

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  3. We know the frequency of token z/ℓ is ℓ.
- If we don't know if the stream is 1-sparse it is much harder and we will have to use randomisation.

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- We will use a polynomial sketch.

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- If there is exactly one non-zero frequency then z/ℓ is the identity of the token as before and $p = \ell r^{z/\ell}$ as all the other coefficients will be zero.

- Otherwise, we show that $p \neq \ell r^{z/\ell}$ with high probability.

1-sparse recovery algorithm

Define $\boldsymbol{e}_i$ to be an all zero vector except for a 1 at index i.

```
initialise (ℓ, z, p) = (0, 0, 0)
choose r to be a uniform random element of F_q

1-SPARSE(j, c)                                 # token, count
set ℓ = ℓ + c
set z = z + cj
set p = p + cr^j                               # fingerprint

OUTPUT
if ℓ = z = p = 0
     return f = 0
else if z/ℓ ∉ [n]
     return ||f||_0 > 1
else if p ≠ ℓr^(z/ℓ)
     return ||f||_0 > 1
else
     return f = ℓe_(z/ℓ)
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- p is updated to $3 \cdot 5^2 = 75$, $75 + (-2)5^1 = 65$, $65 + (-2)5^2 = 15, 15 + 2 \cdot 5^1 = 25$.

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- Output: $\ell \boldsymbol{e}_{z/\ell} = (0, 1)$

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- How likely are we to get false-positives?

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- Now we have that a false positive occurs only when r is a root of the polynomial $q(x) - \ell x^i$.
- But $q(x) - \ell x^{z/\ell}$ has degree at most n and hence at most n roots and so

$$\Pr(r \text{ is a root of } q(x) - \ell x^{z/\ell}) \leq \frac{n}{|\mathbb{F}_q|} \in O\left(\frac{1}{n^2}\right)$$

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- We get a false positive if the stream is not 1-sparse and either $\ell = z = p = 0$ or $z/\ell \in [n]$ and $p = \ell r^{z/\ell}$. This occurs with probability at most $O(1/n^2)$.
- Each (j, c) pair is processed in constant time so the total running time is $O(m)$.
- If M is the largest frequency of any item, the total space is $O(\log n + \log M)$ bits. This is because $|\ell| \leq nM$ and $|z| \leq n^2 M$ and $p, r \in \mathbb{F}_q$ with $q \leq 2n^3$.

s-sparse recovery

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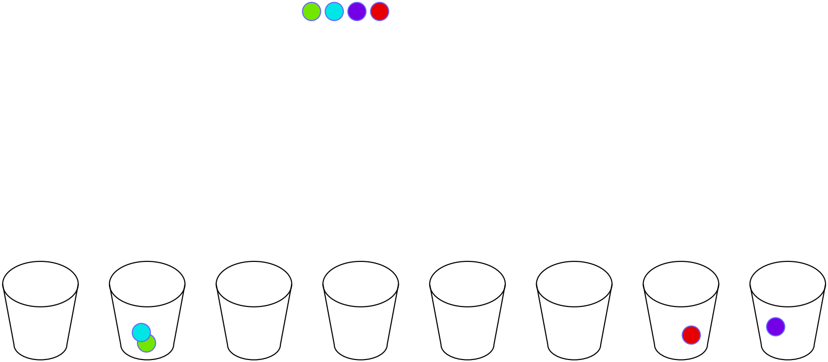
- We repeat the whole process to decrease the error probability.

Throwing s balls in $2s$ bins

What happens when you throw s balls into $2s$ bins?

Attempt 1:



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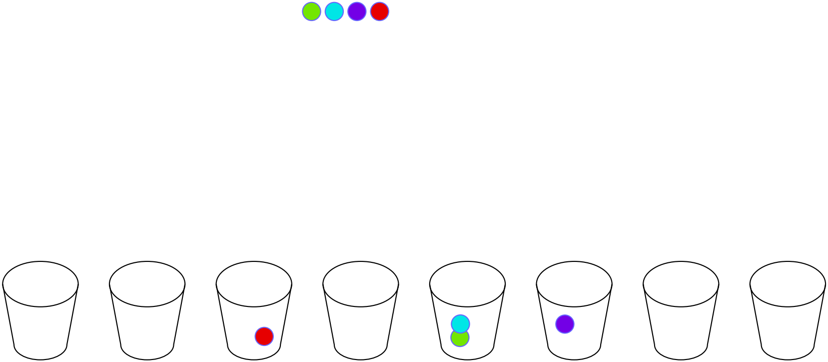
Attempt 2:



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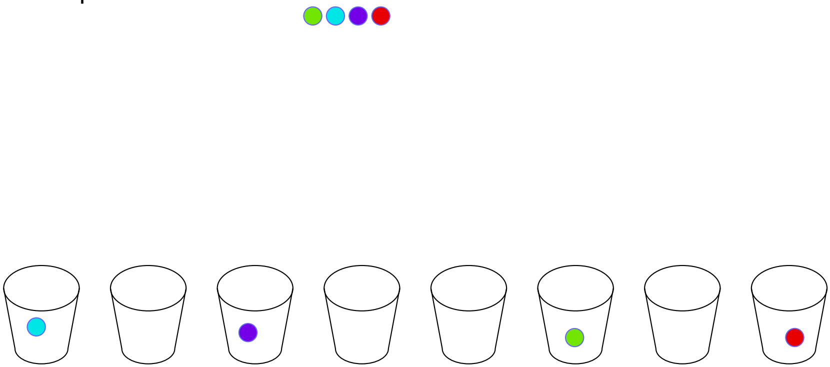

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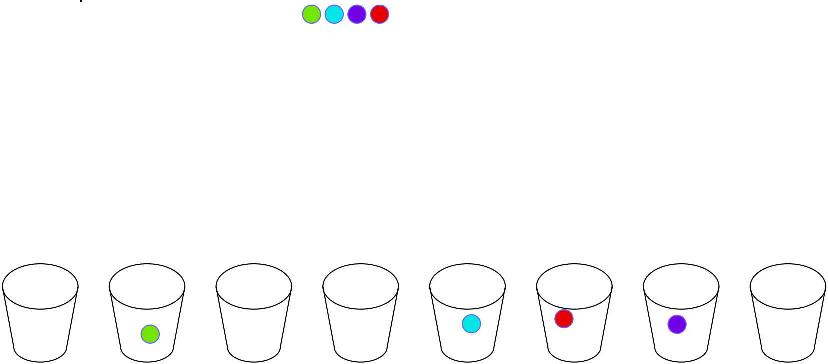

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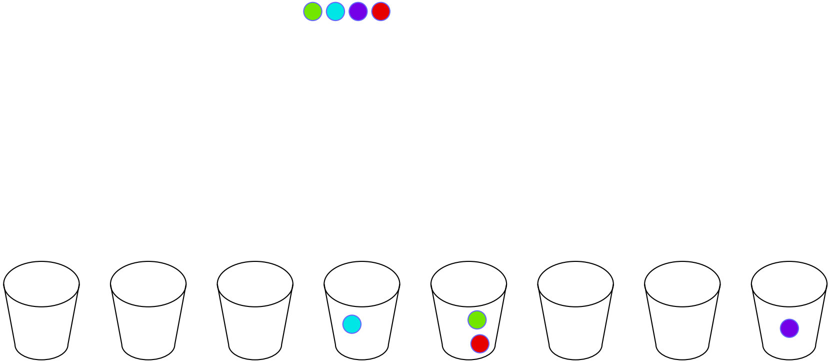

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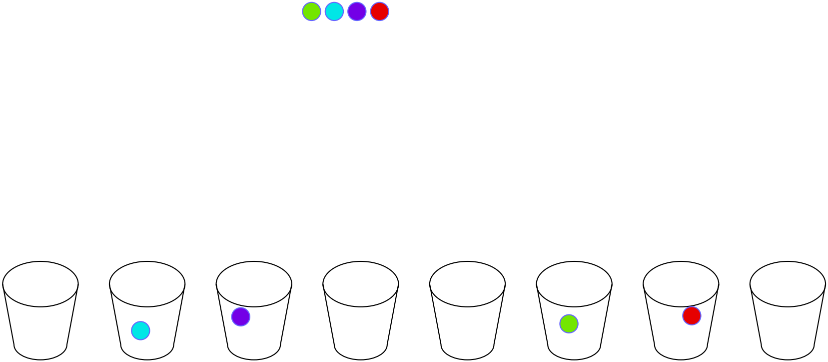
Attempt 6:



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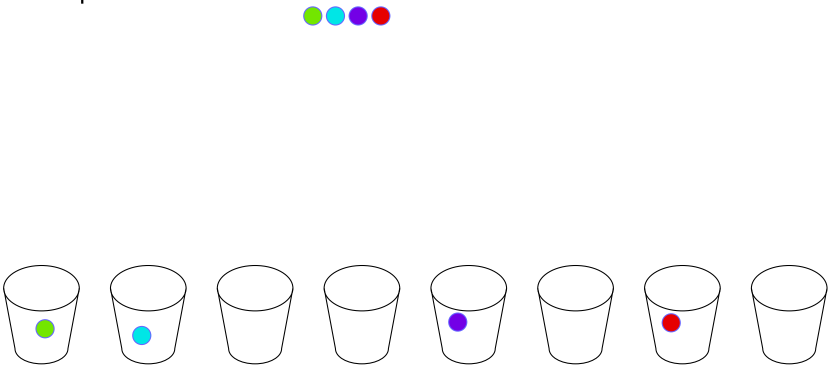
Attempt 7:



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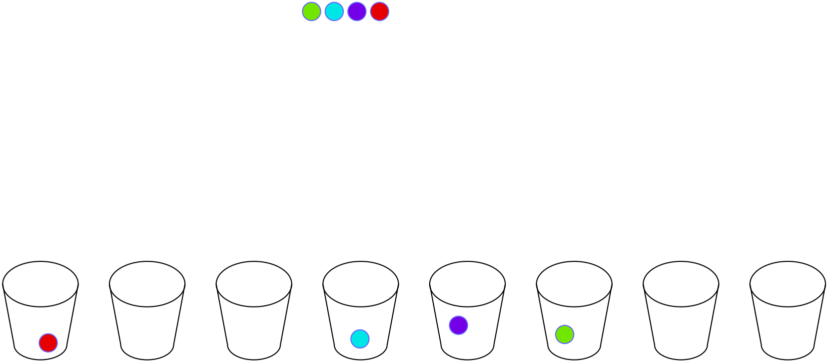
Attempt 8:



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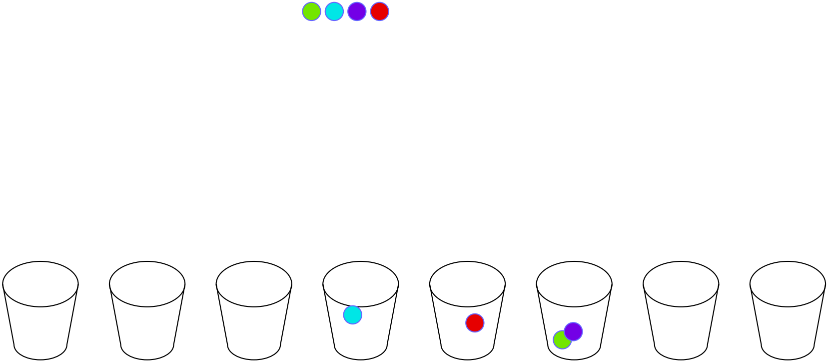
Attempt 9:



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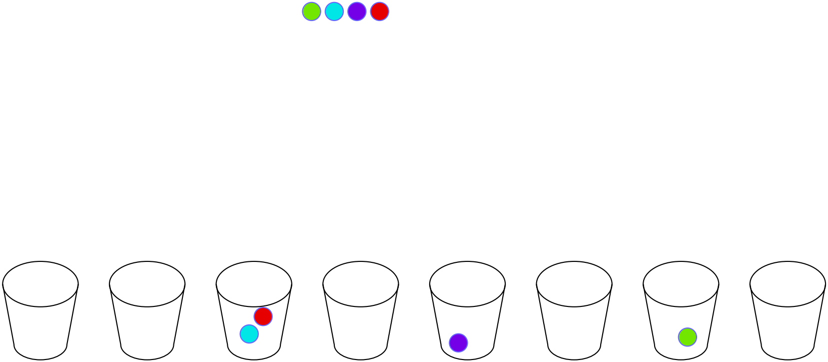

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initialise t = ⌈log(s/δ)⌉
initialise D[1...t][1...2s] = 0
choose t independent hash functions h_1,...,h_t : [n] → [2s]

s-SPARSE(j, c)                                # c can be negative
for each i ∈ [t]
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- `update` runs one step of the 1-sparse recovery and detection algorithm with the new incoming token pair.

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- The t rows of D are for the t independent hash functions.

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- Output all frequency/token pairs inferred from the stored information.

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Proof for SR_1. Consider a particular item $j \in \operatorname{supp} \boldsymbol{f}$. For each $i \in [t]$, let $\boldsymbol{\sigma}_i(j)$ be the substream generated by h_i containing elements of the form (j, c). Note that $f_j \neq 0$ necessarily for this stream.

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$\Pr(\text{SR}_1 \text{ fails for item } j) = \prod_{i=1}^{t} \Pr(\boldsymbol{\sigma}_i(j) \text{ is not 1-sparse}) \leq \left(\frac{1}{2}\right)^t \leq \frac{\delta}{s}$

End of s-sparse proof

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- By another union bound over the entries of D,

$$\Pr(SR_2 \text{ fails}) \leq 2st \cdot O\left(\frac{1}{n^2}\right) \in o(1),$$

because $st \leq n$. By yet another union bound the probability that the recovery fails is at most $\delta + o(1)$.

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- How can we check if the stream really was s-sparse?