Topics in TCS

Frequency estimation via sketching

Raphaël Clifford

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They will give us an estimate of the frequency for *every* token.

CountSketch

The sketch is a 2D-array C with t rows and k columns. All hash functions are chosen from a pairwise independent family.

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```
stream ⟨a_1, ..., a_m⟩, a_i ∈ [n]
initialise C[1...t][1...k] = 0
choose hash functions h_1, ... h_t : [n] → [k]
choose hash function g_1, ..., g_t : [n] → {−1, 1}

CountSketch(a_i)
for each j ∈ [t]
      C[j, h_j(a_i)] += c_i g_j(a_i)

return f̂_{a_i} = median{g_j(a_i) C[j, h_j(a_i)]}
```

c_i is the number of instances of a_i. In the turnstile model this can be either positive of negative.

CountSketch - worked example

	1	2	3
h_1			
h_2			

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
	$2, -$	$3, +$

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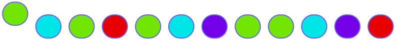

	1	2	3
h_1		+	
h_2	+		

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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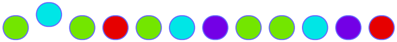

	1	2	3
h_1		$+$	$-$
h_2	$+$	$+$	

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
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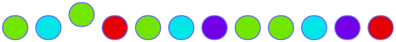

	1	2	3
h_1		$++$	$-$
h_2	$++$	$+$	

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
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CountSketch(a_i)
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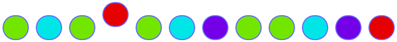

	1	**2**	**3**
h_1		$++-$	$-$
h_2	$++$	$+$	$+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
	$2, -$	$3, +$

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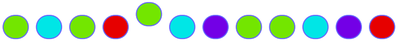
COUNTSKETCH - worked example



	1	2	3
h_1		$++-+$	$-$
h_2	$+++$	$+$	$+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
	$1, +$	$3, -$
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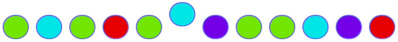

	1	2	3
h_1		$++-+$	$--$
h_2	$+++$	$++$	$+$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
	$3, -$	$2, +$
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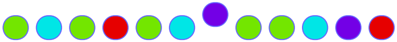

	1	2	3
h_1	$+$	$++-+$	$--$
h_2	$+++$	$++$	$+-$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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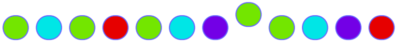

	1	2	3
h_1	$+$	$++-++$	$--$
h_2	$++++$	$++$	$+-$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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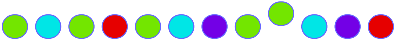

	1	**2**	**3**
h_1	$+$	$++-++$ $+$	$--$
h_2	$+++++$	$++$	$+-$

	h_1, g_1	h_2, g_2
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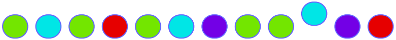

	1	2	3
h_1	$+$	$++-++$ $+$	$---$
h_2	$+++++$	$+++$	$+-$

	h_1, g_1	h_2, g_2
	$2, +$	$1, +$
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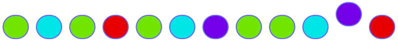

	1	2	3
h_1	$++$	$++-++$ $+$	$---$
h_2	$+++++$	$+++$	$+--$

	h_1, g_1	h_2, g_2
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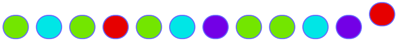

	1	2	3
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
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	$2, +$	$1, +$
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return $\hat{f}_{a_i} =$ median$\{g_j(a_i)C[j, h_j(a_i)]\}$

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	1	2	3
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
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`return` $\hat{f}_{a_i} =$ `median`$\{g_j(a_i) C[j, h_j(a_i)]\}$

$\hat{f} = \text{median}(g_1()C[1, h_1()], g_2()C[2, h_2()]) = \text{median}(1 \cdot 3, 1 \cdot 5)$

CountSketch - worked example

	1	2	3
h_1	$++$	$++-++$ $+-$	$---$
h_2	$+++++$	$+++$	$+--+$

	h_1, g_1	h_2, g_2
$\bigcirc$	$2, +$	$1, +$
$\bigcirc$	$3, -$	$2, +$
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$$\texttt{return } \hat{f}_{a_i} = \texttt{median}\{g_j(a_i)C[j, h_j(a_i)]\}$$

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$$\hat{f}_{\bigcirc} = \text{median}(g_1(\bigcirc)C[1, h_1(\bigcirc)], g_2(\bigcirc)C[2, h_2(\bigcirc)]) = \text{median}(-1 \cdot -3, 1 \cdot 3)$$

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	1	2	3
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	1	2	3
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Therefore

$$X = g(a) \sum_{j=1}^{n} f_j g(j) Y_j = f_a + \sum_{j \in [n] \setminus \{a\}} f_j g(a) g(j) Y_j$$

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As g and h are independent and g is from a pairwise independent family,

$$\mathbb{E}[g(a)g(j)Y_j] = \mathbb{E}(g(a)) \cdot \mathbb{E}(g(j)) \cdot \mathbb{E}(Y_j) = 0 \cdot 0 \cdot \mathbb{E}(Y_j) = 0$$

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By linearity of expectation

$$\mathbb{E}(X) = f_a + \sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(a) g(j) Y_j] = f_a$$

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We will now derive the variance of our estimator $X = \hat{f}$. Recall $Y_j = 1$ iff $h(j) = h(a)$.

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We will now derive the variance of our estimator $X = \hat{f}$. Recall $Y_j = 1$ iff $h(j) = h(a)$.

$$\begin{aligned}
\operatorname{var}(X) &= 0 + \operatorname{var}\left[g(a)\sum_{j\in[n]\setminus\{a\}} f_j \cdot g(j) Y_j\right] \\
&= \mathbb{E}\left[g(a)^2 \sum_{j\in[n]\setminus\{a\}} f_j^2 Y_j^2 + \sum_{\substack{j\in[n]\setminus\{a\} \\ i\neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] - \\
&\qquad \left[\sum_{j\in[n]\setminus\{a\}} f_j \mathbb{E}[g(a) g(j) Y_j]\right]^2
\end{aligned}$$

CountSketch - Analysis IIa

We will now derive the variance of our estimator $X = \hat{f}$. Recall $Y_j = 1$ iff $h(j) = h(a)$.

$$\begin{aligned}
\mathrm{var}(X) &= 0 + \mathrm{var}\left[g(a) \sum_{j \in [n] \setminus \{a\}} f_j \cdot g(j) Y_j\right] \\
&= \mathbb{E}\left[g(a)^2 \sum_{j \in [n] \setminus \{a\}} f_j^2 Y_j^2 + \sum_{\substack{j \in [n] \setminus \{a\} \\ i \neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] - \\
&\qquad \left[\sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(a) g(j) Y_j]\right]^2
\end{aligned}$$

We will need two facts to simplify these terms.

CountSketch - Analysis IIb

$$\mathrm{var}(X) = \mathbb{E}\left[g(a)^2 \sum_{j\in[n]\setminus\{a\}} f_j^2 Y_j^2 + \sum_{\substack{j\in[n]\setminus\{a\} \\ i\neq j}} f_i f_j g(i) g(j) Y_i Y_j\right] - \left[\sum_{j\in[n]\setminus\{a\}} f_j \mathbb{E}[g(a)g(j)Y_j]\right]^2$$

CountSketch - Analysis IIb

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Now, the two facts:

1. $\mathbb{E}(Y_j^2) = \mathbb{E}(Y_j) = \Pr(h(j) = h(a)) = \frac{1}{k}$.
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$$= \frac{\|\boldsymbol{f}\|_2^2 - f_a^2}{k} \quad \text{where } \boldsymbol{f} \text{ is the array of frequencies}$$

CountSketch - Analysis III

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&\leq \frac{\mathrm{var}(X)}{\epsilon^2(\|\boldsymbol{f}\|_2^2 - f_a^2)} \\
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&= \frac{1}{3} \qquad \text{(set } k = 3/\epsilon^2\text{)}
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$$\Pr(|\hat{f}_a - f_a| \geq \epsilon\,\|\boldsymbol{f}_{-a}\|_2) \leq \frac{1}{3}$$

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Using Chernoff's bound with $\mu = t/3$

$$\Pr\left(\sum_{i=1}^{t} Z_i \geq (1+\delta)\mu\right) \leq \exp(-\delta^2\mu/3) \qquad = \exp(-\delta^2 t/9)$$

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For an arbitrary token a, the probability of being further than $\epsilon \left\| \boldsymbol{f}_{-a} \right\|_2$ from the correct frequency is at most $\exp(-t/36)$.

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With $k = \lceil 3/\epsilon^2 \rceil$ and $t = \lceil \ln 1/\delta \rceil$, this equals

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Running time: one-pass and $O(t)$ time per token.

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CountSketch is a one-pass randomised algorithm to estimate the frequency of the tokens in a stream.

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Assuming we set $k = 3/\epsilon^2$, for an arbitrary token a, the probability that CountSketch's estimate is further than $\epsilon \|\boldsymbol{f}_{-a}\|_2$ from the correct frequency is at most $\exp(-t/36)$.

Count-Min sketch

The sketch is a 2D-array C with t rows and k columns. All hash functions are chosen from a pairwise independent family.

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```
stream ⟨a_1, ..., a_m⟩, a_i ∈ [n]
initialise C[1...t][1...k] = 0
choose hash functions h_1, ... h_t : [n] → [k]

Count-Min(a_i)
for each j ∈ [t]
      C[j, h_j(a_i)] += c_i



return f̂_a = min_{1≤i≤t} C[i, h_i(a)]
```

c_i is the number of instances of a_i. In the turnstile model this can be either positive of negative.

Count-Min - worked example

	1	2	3
h_1			
h_2			
h_3			

	h_1	h_2	h_3
	1	2	3
	2	1	1
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	3	3	2

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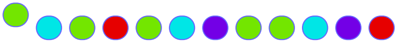

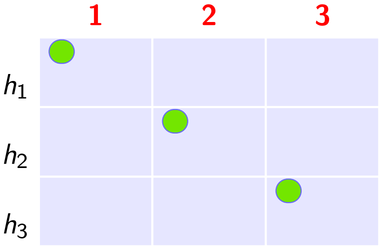

	h_1	h_2	h_3
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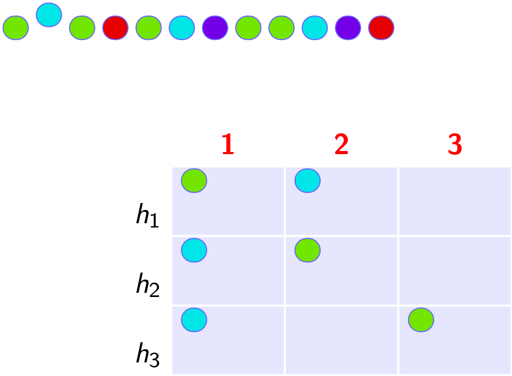

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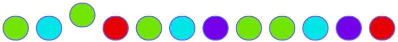

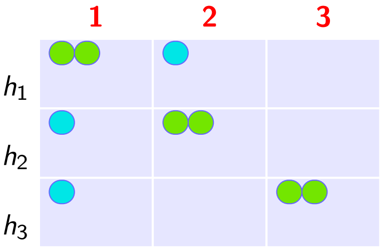

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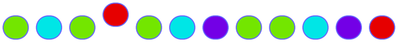

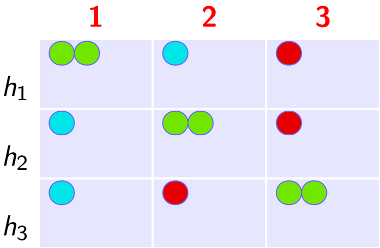

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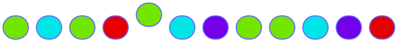

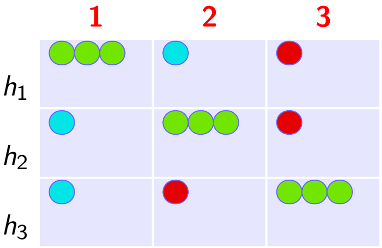

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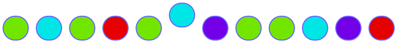

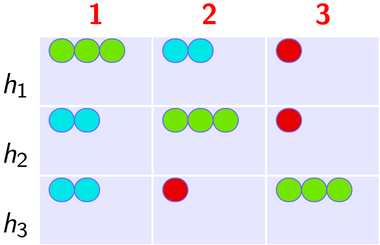

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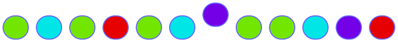

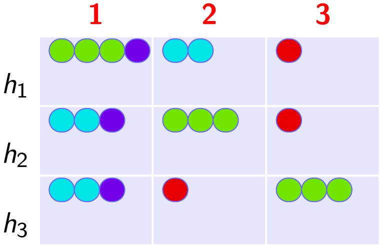

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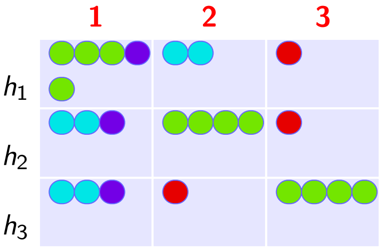

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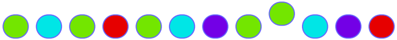

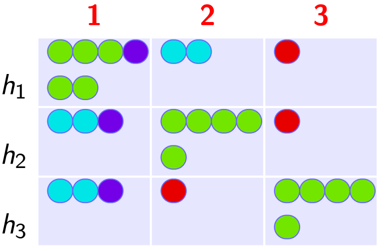

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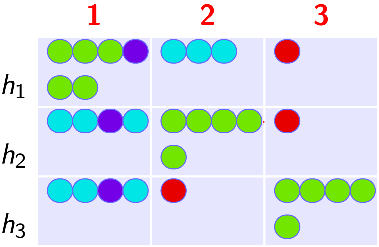

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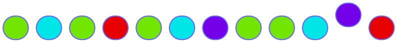

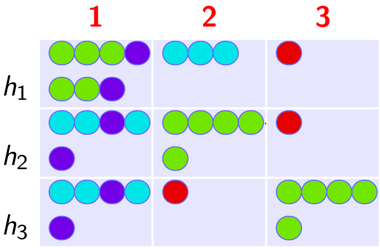

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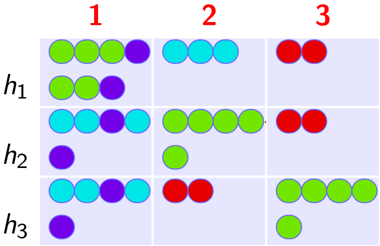

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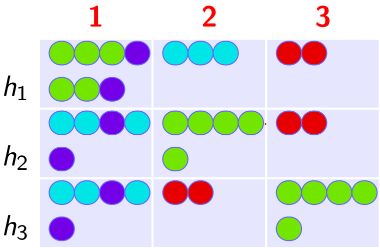

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$\hat{f}_{\bigcirc} = \min(C[1, h_1(\bigcirc)], C[2, h_2(\bigcirc)], C[3, h_3(\bigcirc)]) = \min(7, 5, 5) = 5\checkmark$

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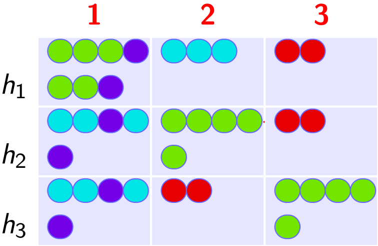

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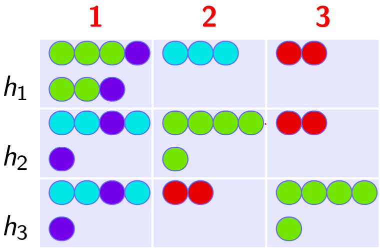

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	2	1	1
	1	1	1
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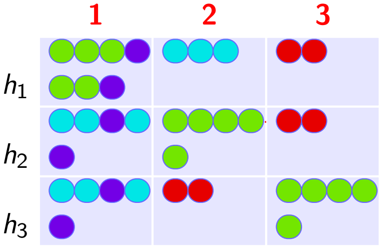
COUNT-MIN - worked example



	h_1	h_2	h_3
$\bullet$	1	2	3
$\bullet$	2	1	1
$\bullet$	1	1	1
$\bullet$	3	3	2

`return` $\hat{f}_a = \min_{1 \le i \le t} C[i, h_i(a)]$

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By Markov's inequality

$$\Pr(X_i \geq \epsilon \|\boldsymbol{f}_{-a}\|_1) \leq \frac{\|\boldsymbol{f}_{-a}\|_1}{k\epsilon \|\boldsymbol{f}_{-a}\|_1} = \frac{1}{2} \qquad \text{set } k = 2/\epsilon$$

Count-Min - Analysis II

We have a bound for a single counter. Over t counters the reported excess is the minimum over all X_i. We can now derive the probability that all the excesses are at least $\epsilon \left\| \boldsymbol{f}_{-a} \right\|_1$ directly.

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bits if $k = \lceil 2/\epsilon \rceil$. This is a factor of $1/\epsilon$ improvement.

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CountSketch: with $k = \lceil 3/\epsilon^2 \rceil$ and $t = \lceil \log_2(1/\delta) \rceil$,

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$\textsc{Misra-Gries}$ gives a lower bound on frequency where $\textsc{Count-Min}$ gives an upper bound.

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MISRA-GRIES uses $O((1/\epsilon)(\log m + \log n)$ bits but does not work in the turnstile model (with deletions).