Topics in TCS

Appoximate counting

Raphaël Clifford

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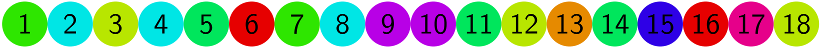

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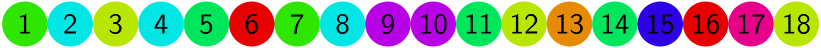

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Set $x = 0$

Morris(a_i)
with probability 2^{-x}
  set x = x + 1

return 2^x-1

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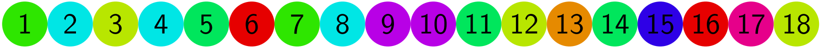

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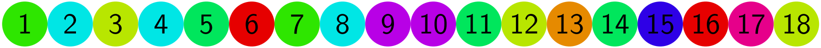

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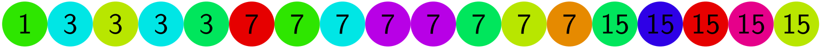

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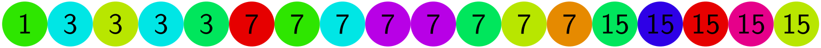

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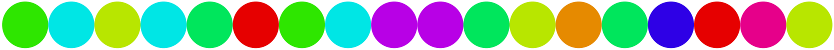
Space is ??? (we will see later).

But how accurate is this going to be?

MORRIS - Quality of estimate



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Consider an equivalent although less space efficient algorithm.

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Set c = 1

SIMPLIFIED-MORRIS(a_i)
with probability 1/c
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Lemma

For all $n \geq 0$, $\mathbb{E}(C_n) = n + 1$

$\mathrm{var}(C_n) = n(n-1)/2$

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We will need the variance for this.

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Therefore $\mathbb{E}(C_n) = n + 1$ since $\mathbb{E}(C_0) = 1$. □

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Lemma (Variance of Morris's algorithm)

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Since $\mathbb{E}(C_0^2) = 1$ we have $\mathbb{E}(C_n^2) = 1 + \frac{3n(n+1)}{2}$.

Finally, $\mathrm{var}(C_n) = \mathbb{E}(C_n^2) - (\mathbb{E}(C_n))^2 = \frac{n(n-1)}{2}$ □

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Return this median as our estimate.

This estimate will be much less likely to be bad.

MORRIS - The main result Ia

Repeat t iterations of k independent runs. Let $X_{i,j}$ be unbiased estimators for the count whose true value we call Q. Let X be distributed identically to $X_{i,j}$. For $\delta, \epsilon > 0$, set

$$t = c \left\lceil \log_2 \frac{1}{\delta} \right\rceil$$

$$k = \frac{3\,\mathrm{var}(X)}{\epsilon^2 (\mathbb{E}(X))^2}$$

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If $\textsc{Morris}$ uses s bits then our (ϵ, δ)-estimate uses

$$O\left(s \cdot \frac{\mathrm{var}(X)}{(\mathbb{E}(X))^2} \cdot \frac{1}{\epsilon^2} \cdot \log \frac{1}{\delta} \right) \text{ bits.}$$

Morris - The main result Ib

Lemma (Preliminary (ϵ, δ) result)

$\Pr(|Z - Q| \geq \epsilon Q) \leq \delta$. That is Z is an (ϵ, δ)-estimate of Q.

Proof.

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Proof.

For each $i \in [t]$ we know $\mathbb{E}(\frac{1}{k} \cdot \sum_{j=1}^{k} X_{i,j}) = Q$ by linearity of expectation.

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From pairwise independence, $\mathrm{var}(\frac{1}{k^2} \cdot \sum_{j=1}^{k} X_{i,j}) = \frac{\mathrm{var}(X)}{k}$.

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Let $Y_i = \frac{1}{k^2} \cdot \sum_{j=1}^{k} X_{i,j}$,

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Now apply the median trick from Lecture 4 (TIDEMARK) to get the desired result.

□

Morris - The main result II

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The probability that any one of the $O(\epsilon^{-2} \log \delta^{-1})$ runs aborts is $o(1)$. (Union bound)

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Is this any good in practice? Think of $\epsilon = 1/2$.

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The theory is however very attractive.

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