Advanced Algorithms – COMS31900

Approximation algorithms part two

more constant factor approximations

Raphaël Clifford

Slides by Benjamin Sach

Approximation Algorithms Recap

An algorithm A is an α-approximation algorithm for problem P if,

- A runs in polynomial time
- A always outputs a solution with value s within an α factor of Opt

Here P is an optimisation problem with optimal solution of value Opt

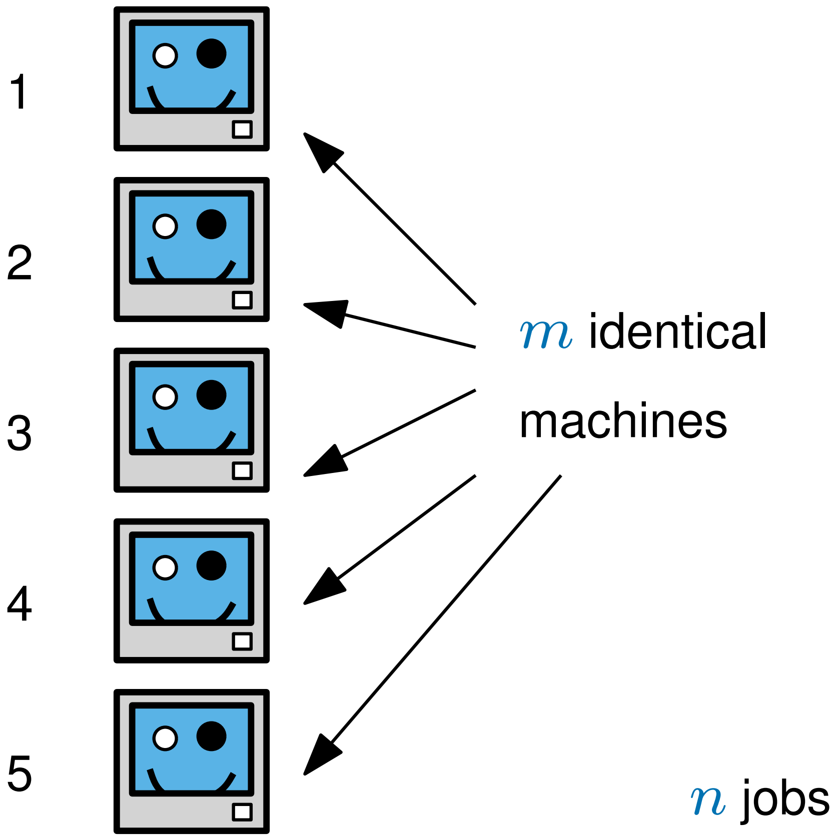
- If P is a *maximisation* problem, $\frac{\mathrm{Opt}}{\alpha} \leqslant s \leqslant \mathrm{Opt}$
- If P is a *minimisation* problem, $\mathrm{Opt} \leqslant s \leqslant \alpha \cdot \mathrm{Opt}$

We have seen

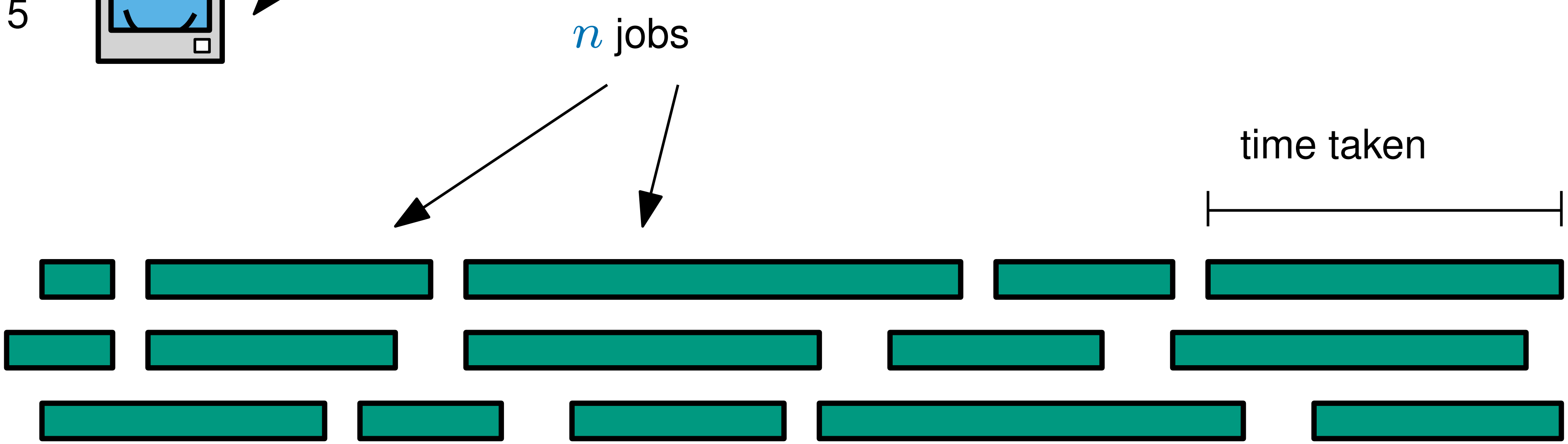
a $3/2$-approximation algorithm for Bin Packing

(and a *faster* 2-approximation)

Scheduling Jobs on Parallel Machines



Goal: minimise the (wall-clock) time taken to process all jobs



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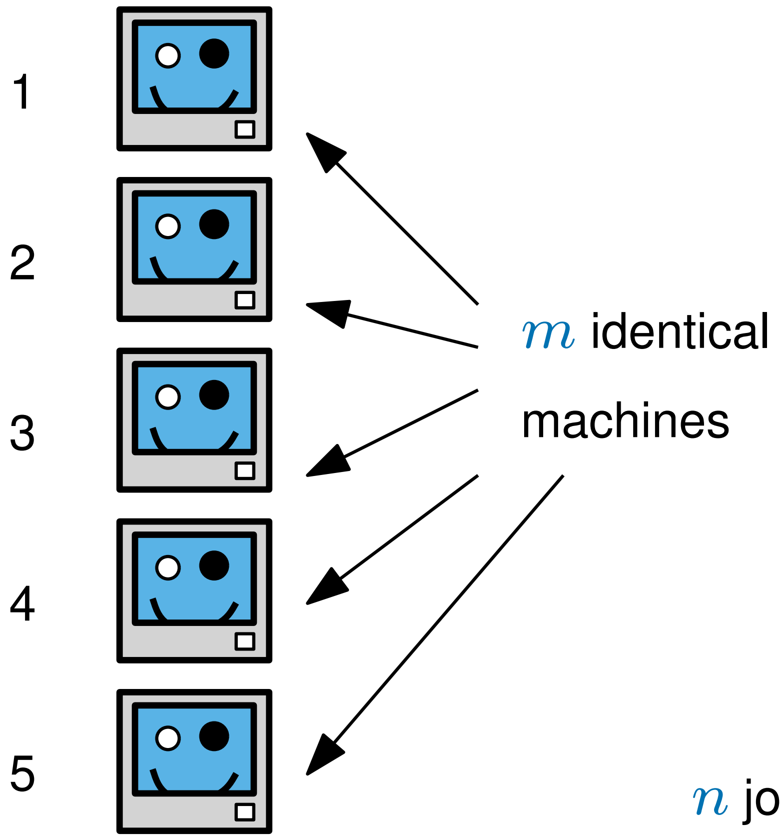

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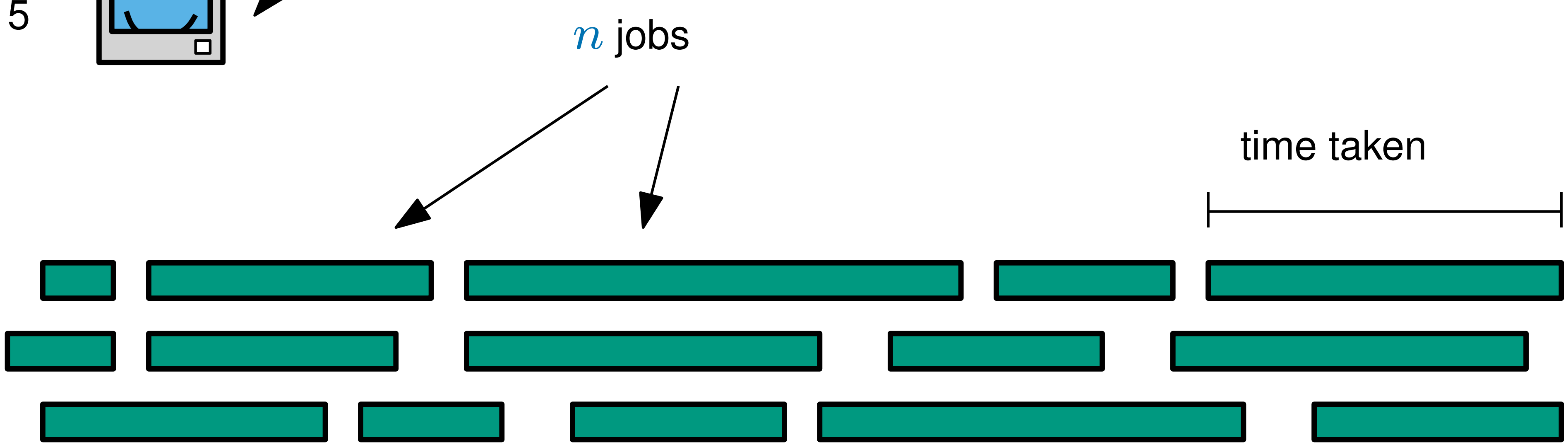
(it's NP-hard)



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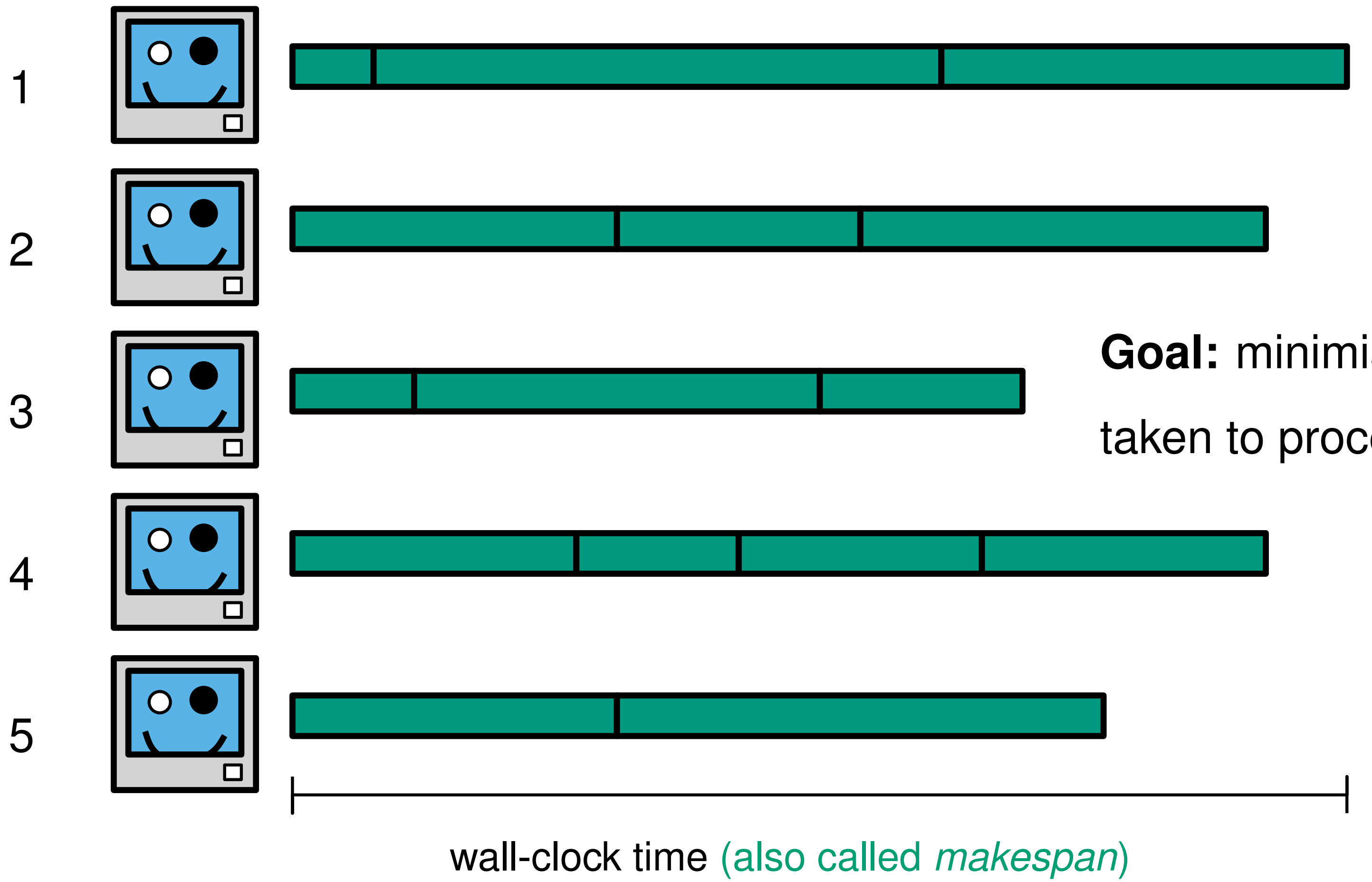

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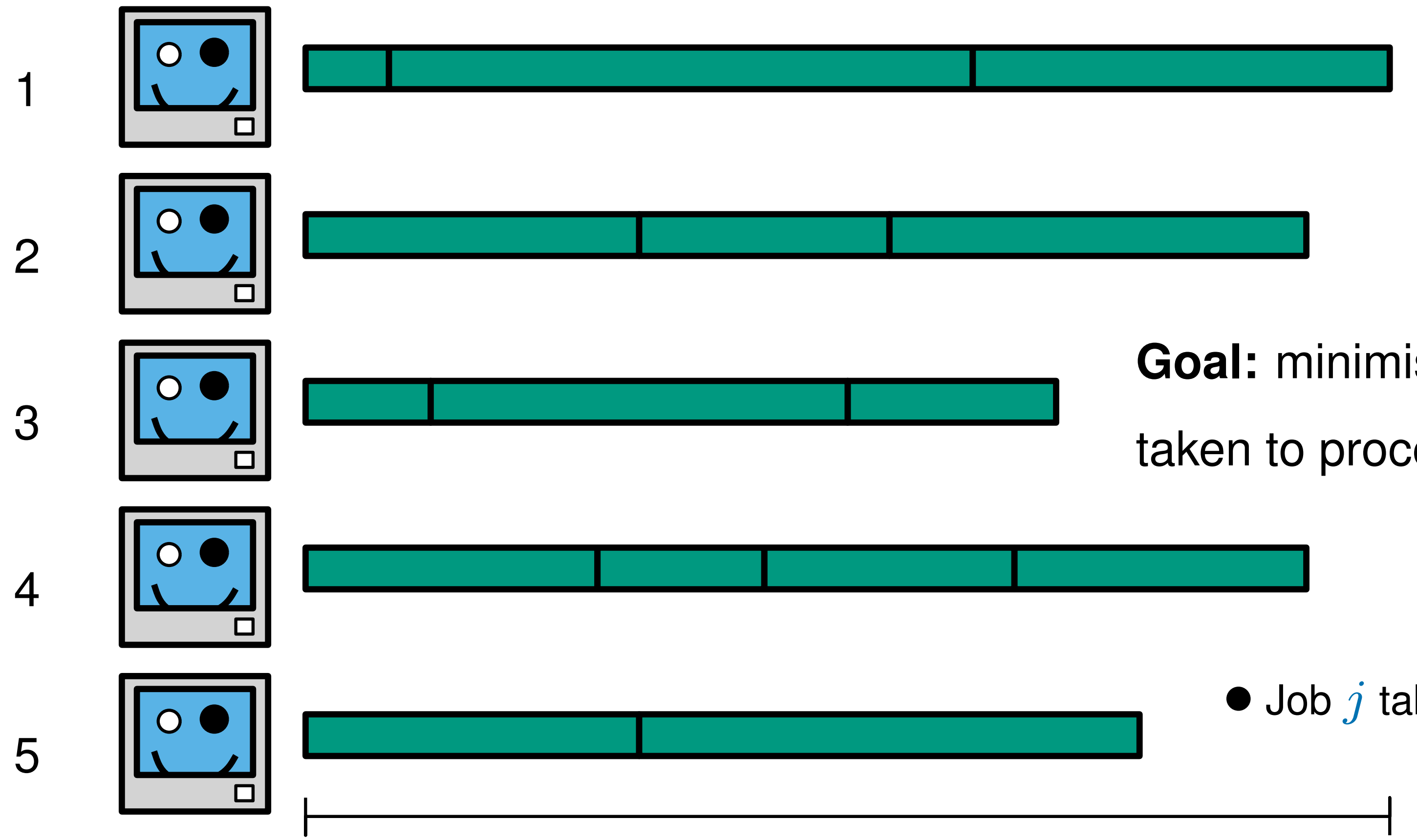

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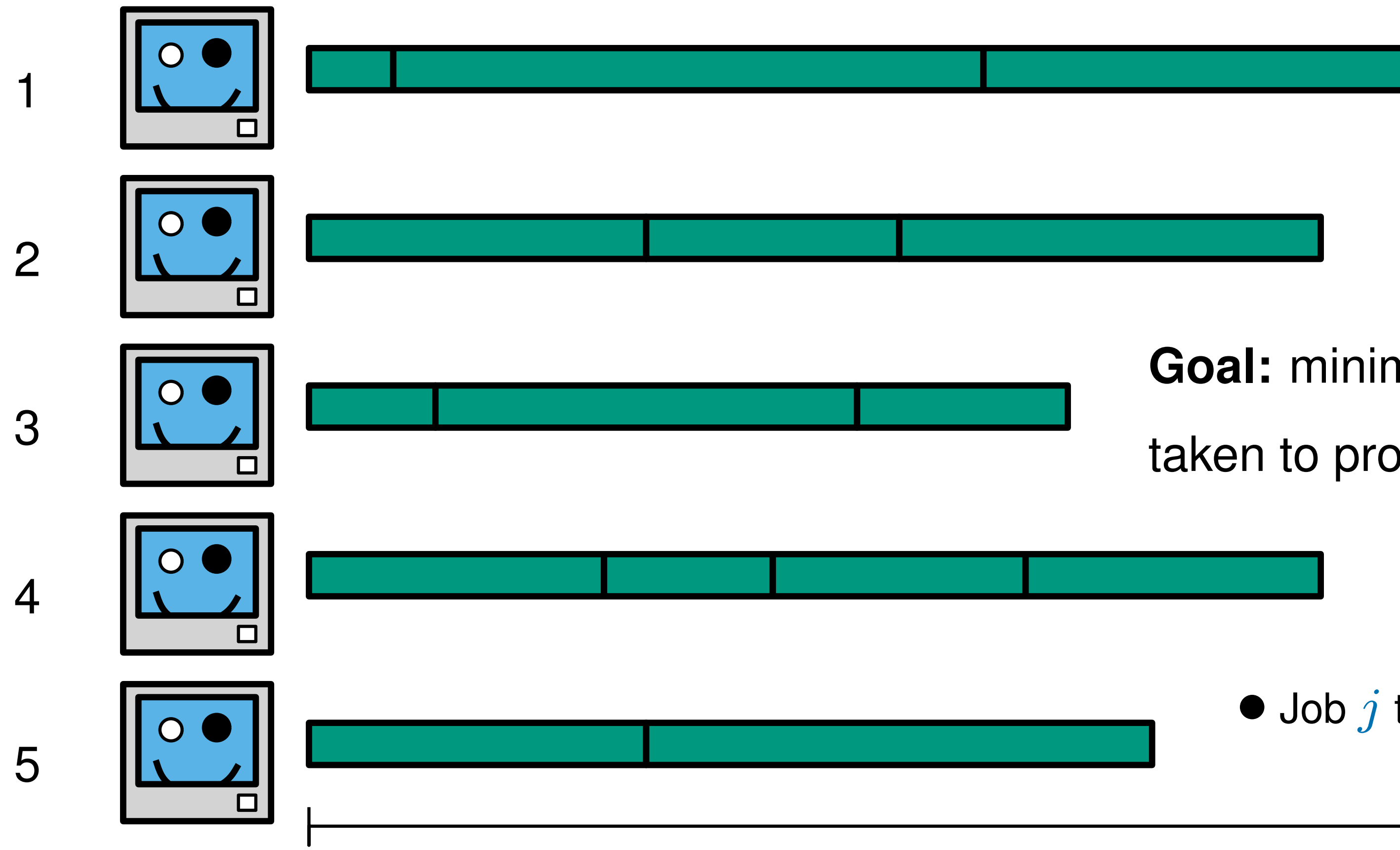

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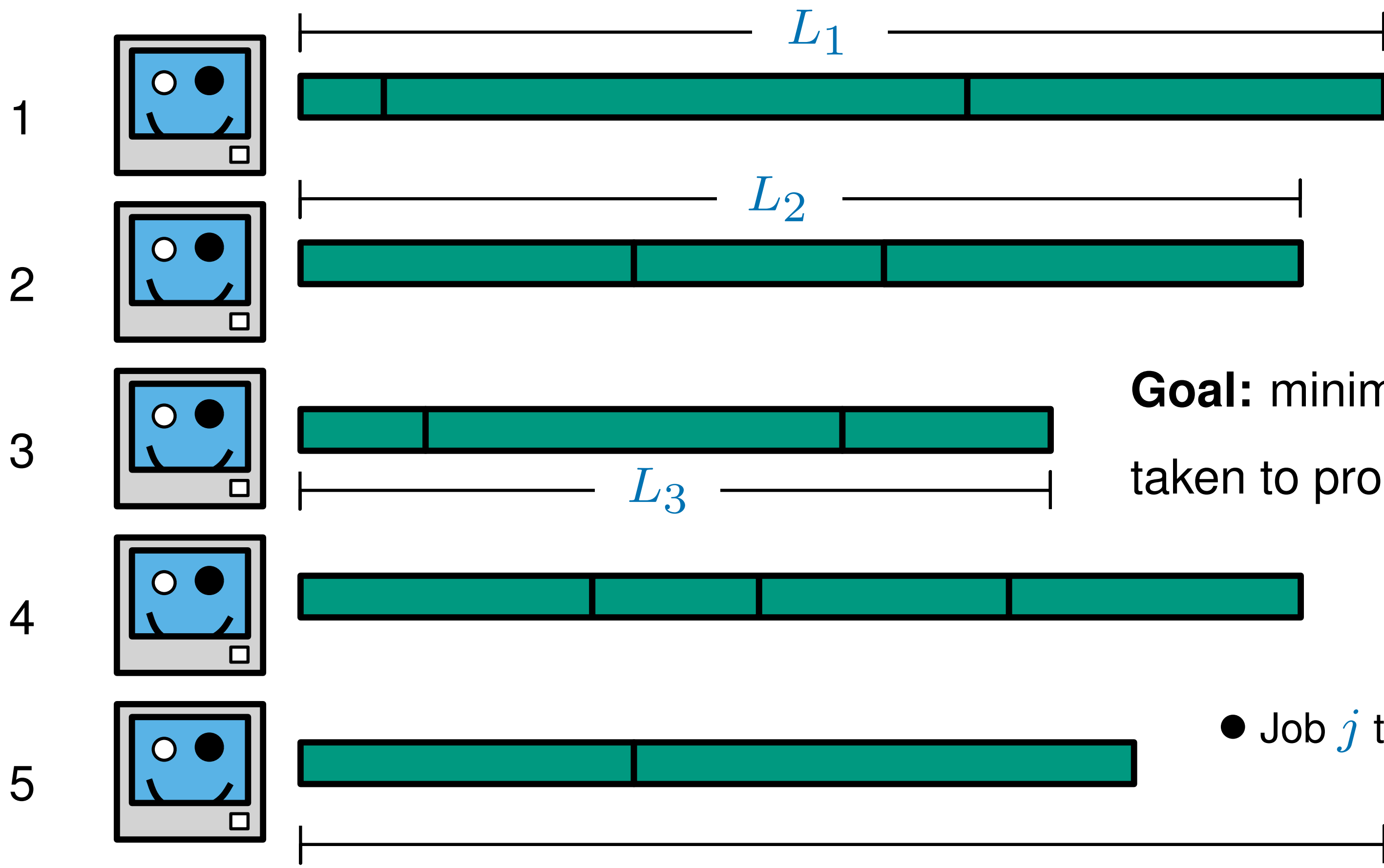

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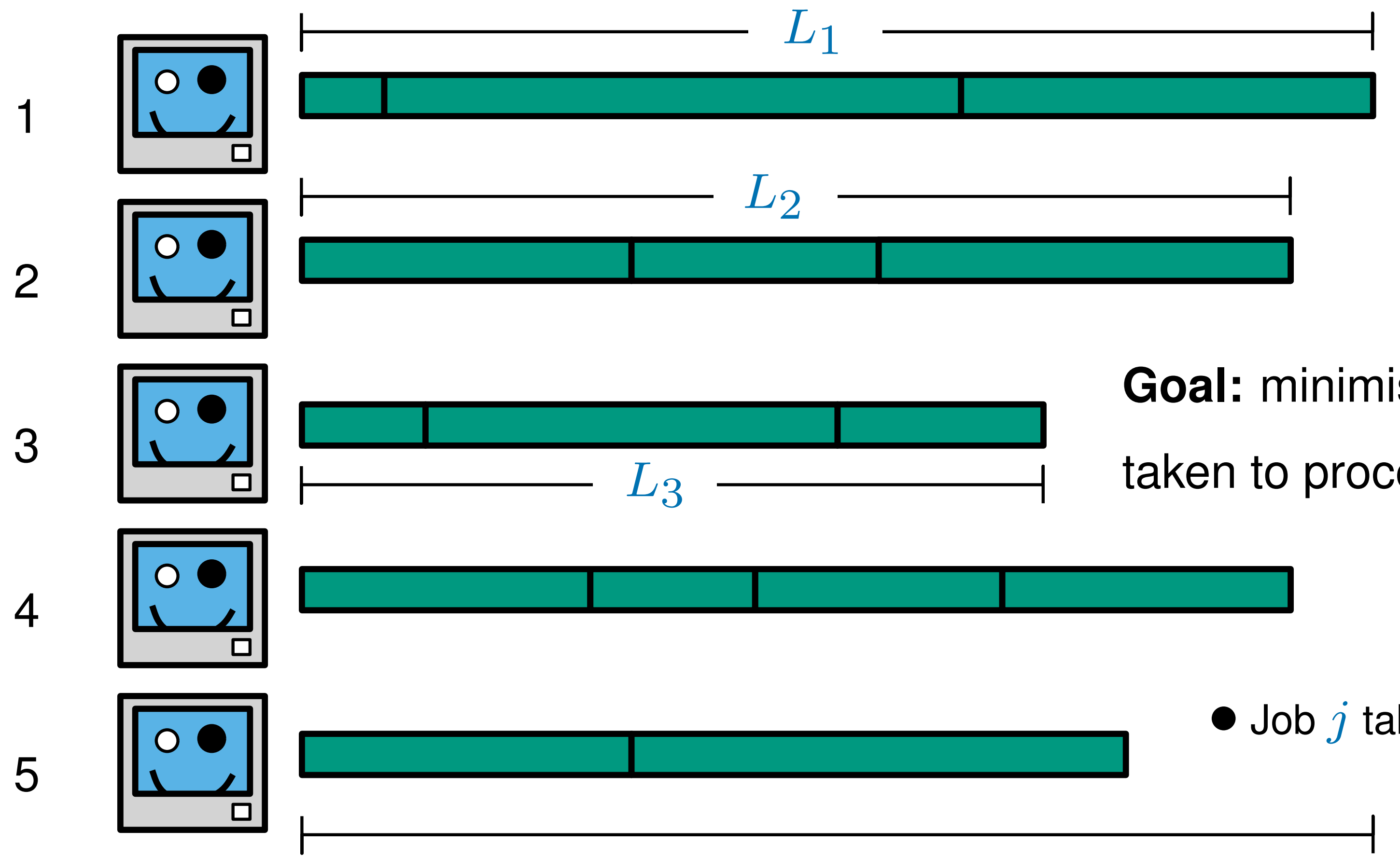

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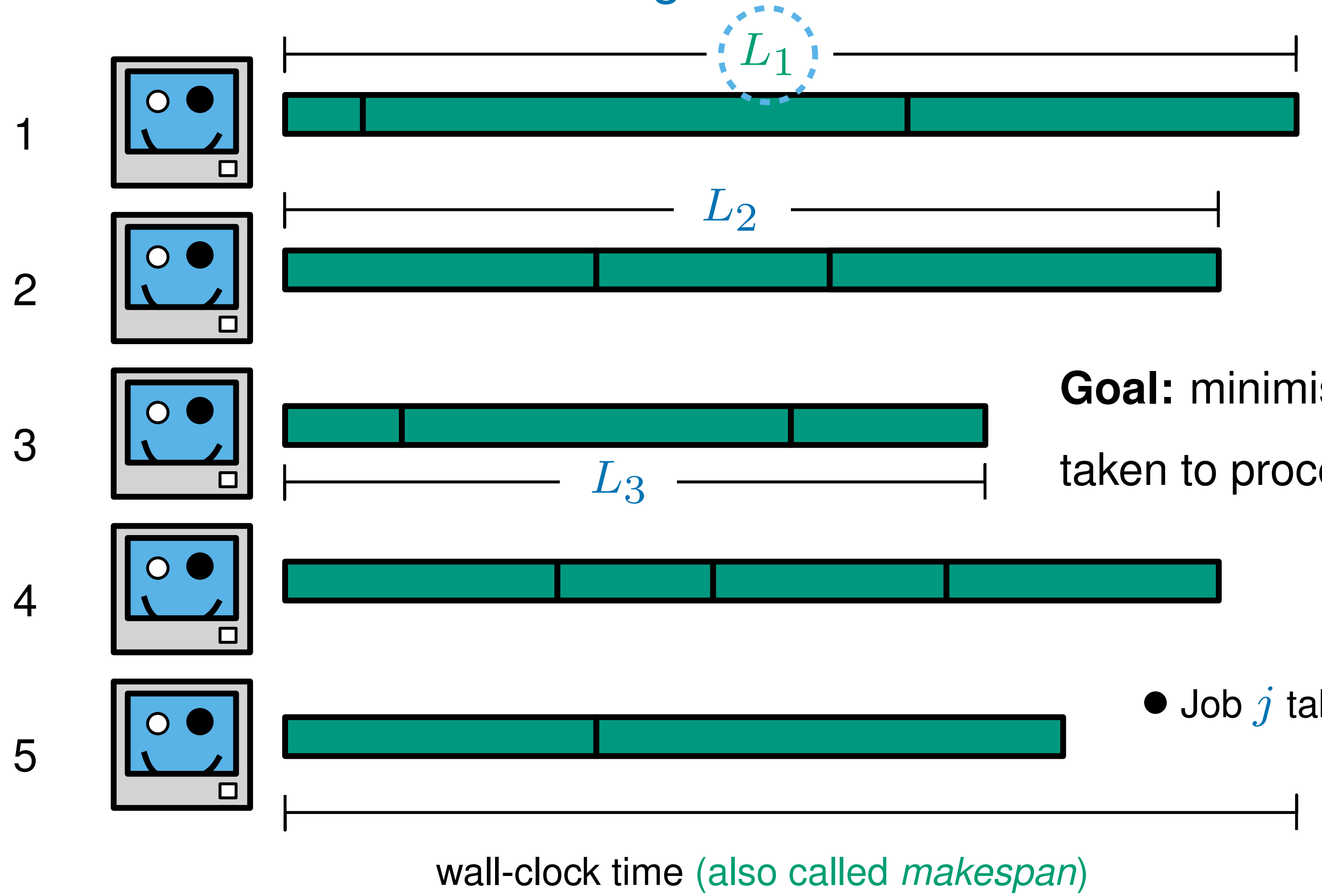

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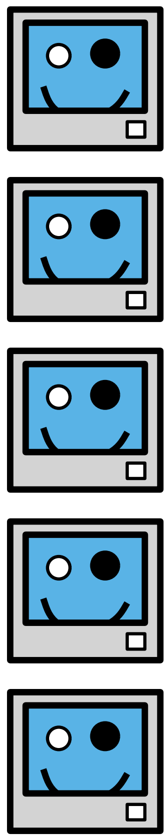
1
2
3
4
5



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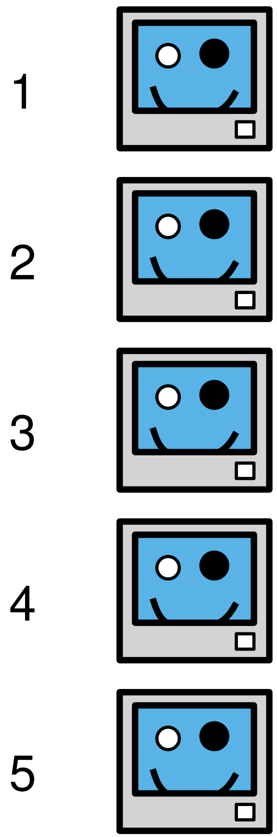

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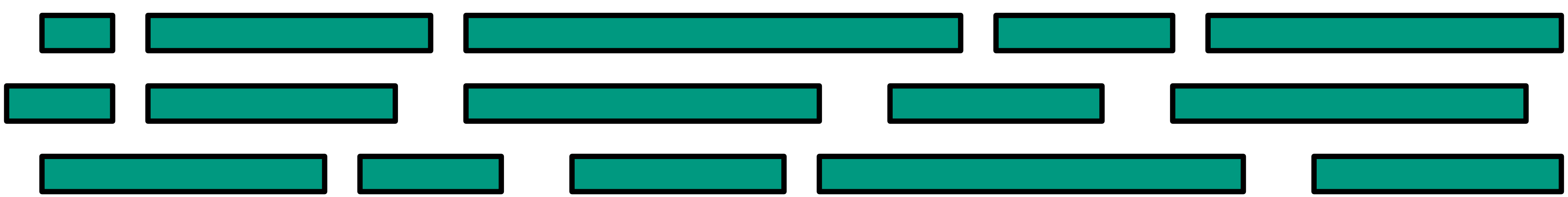

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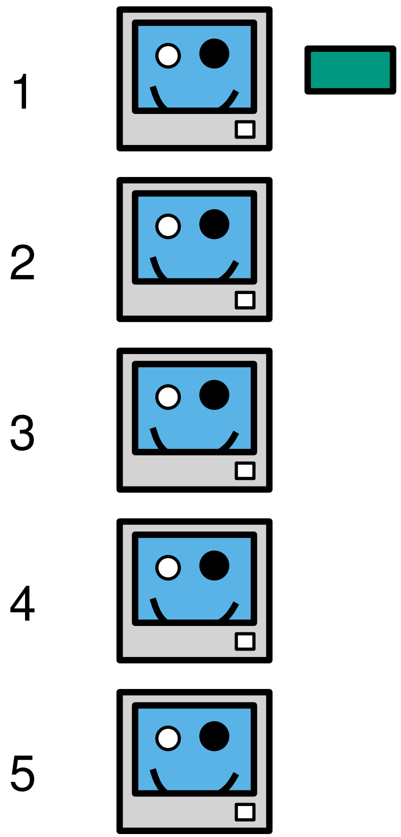

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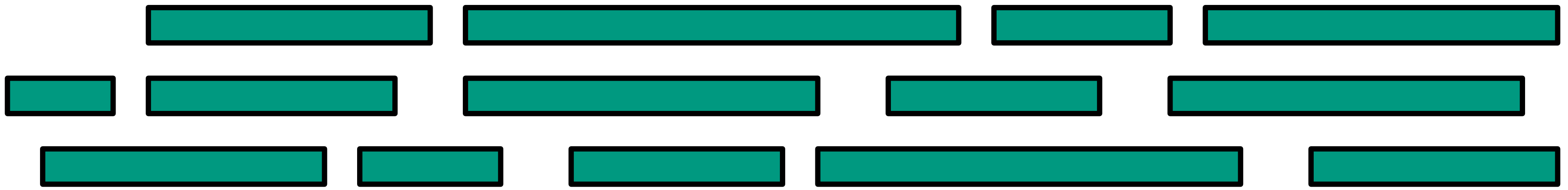

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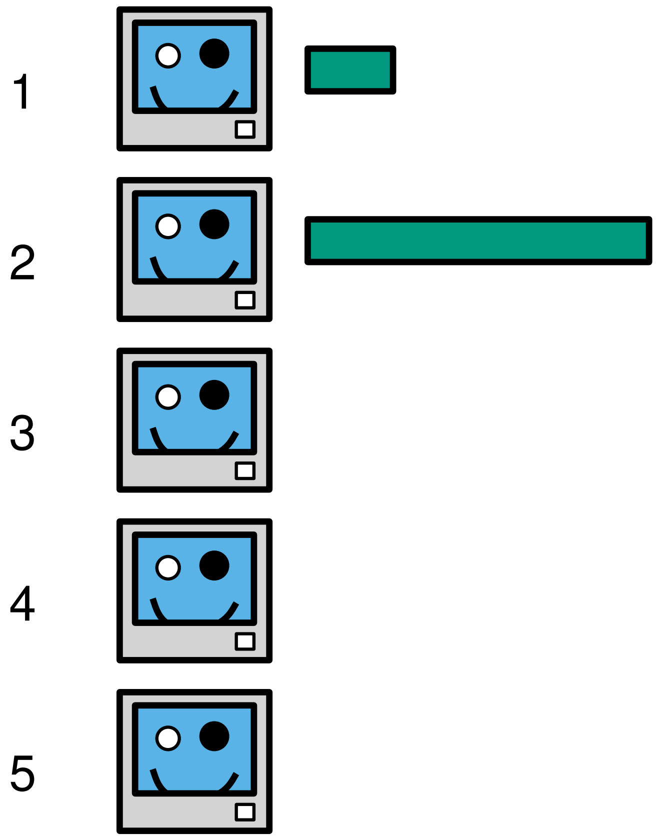

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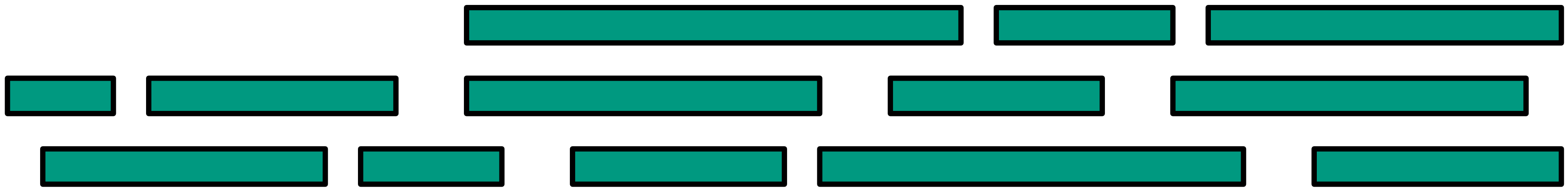

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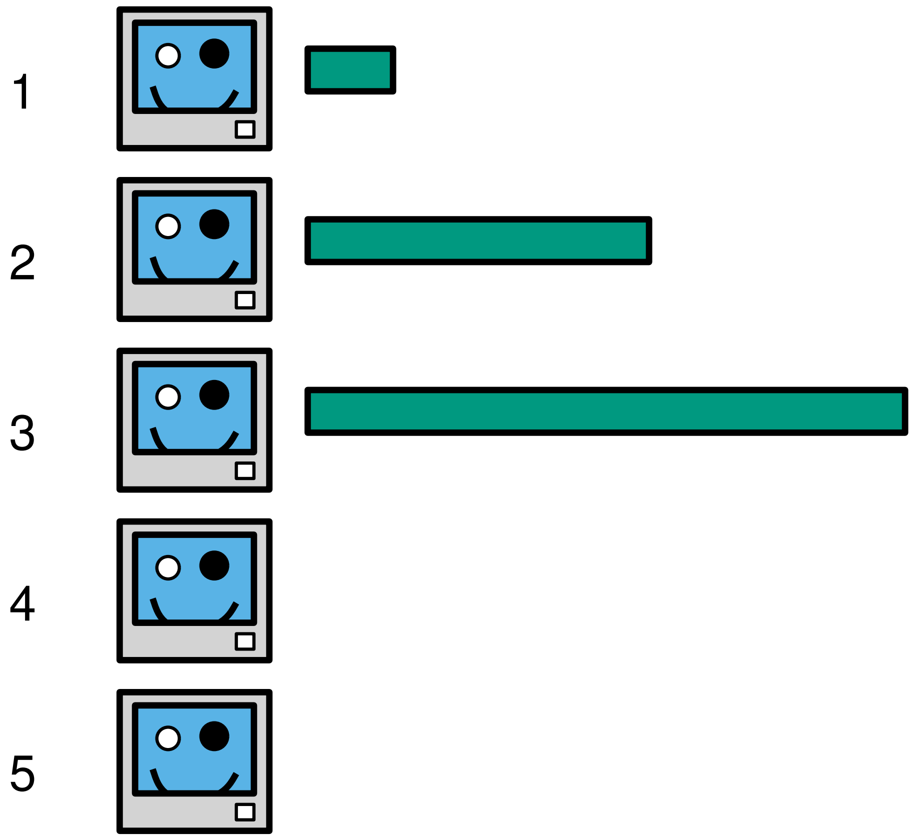

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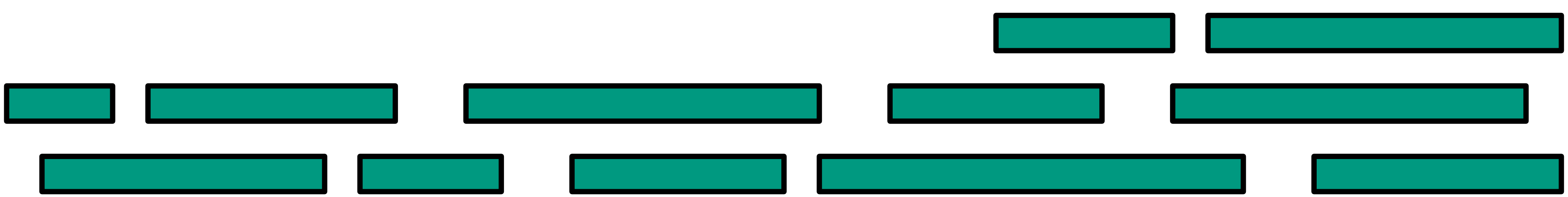

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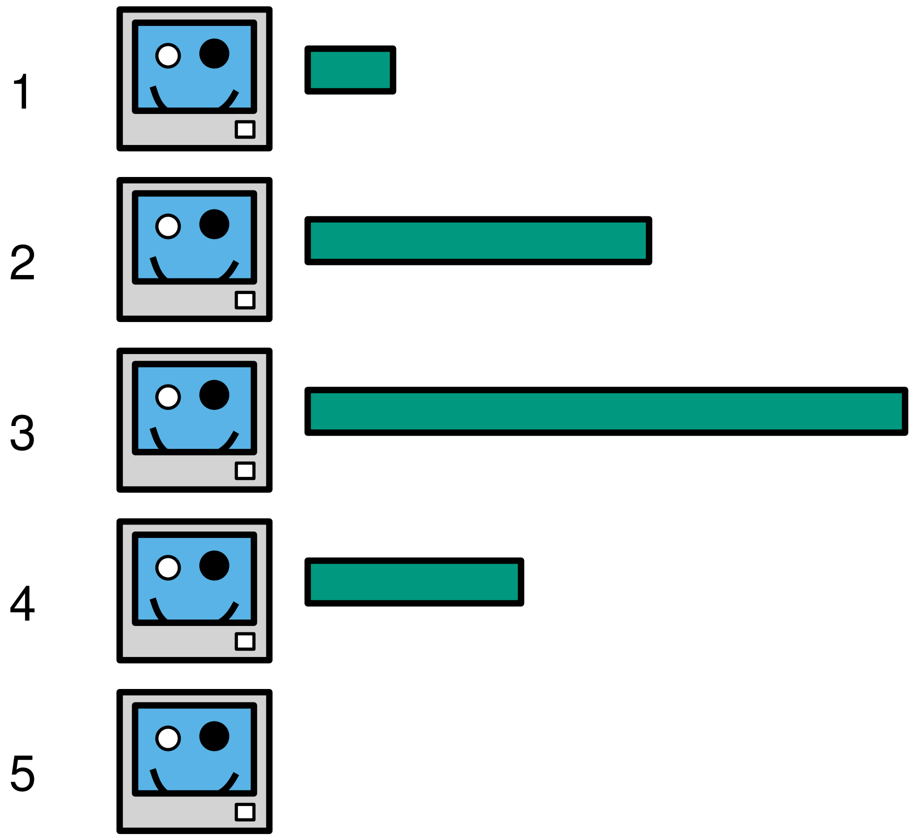

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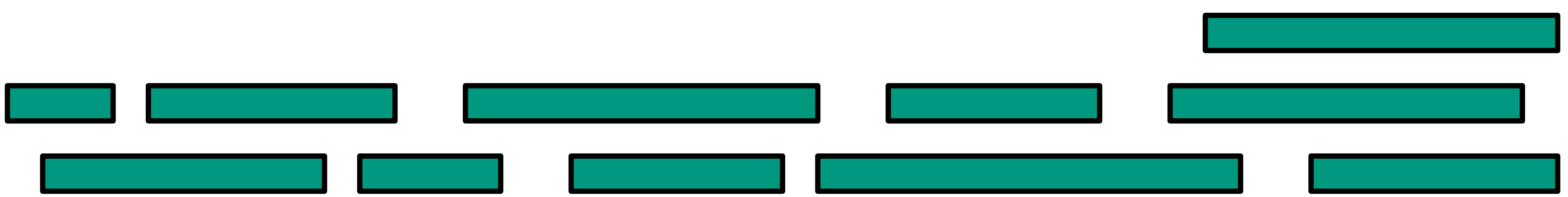

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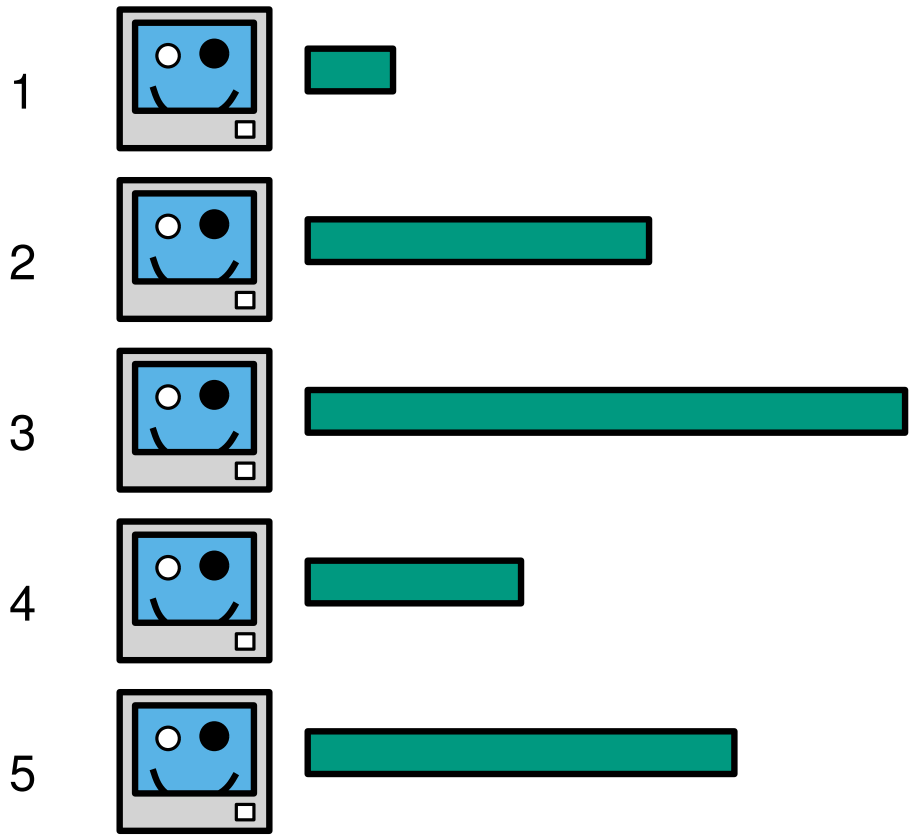

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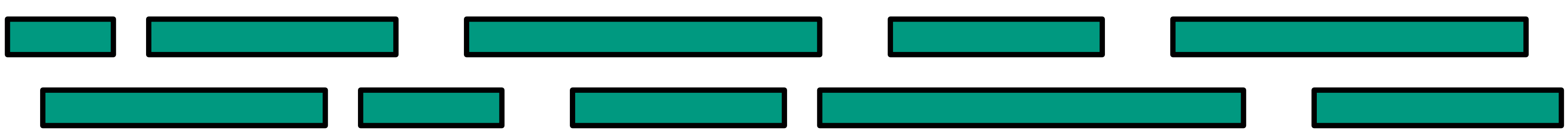

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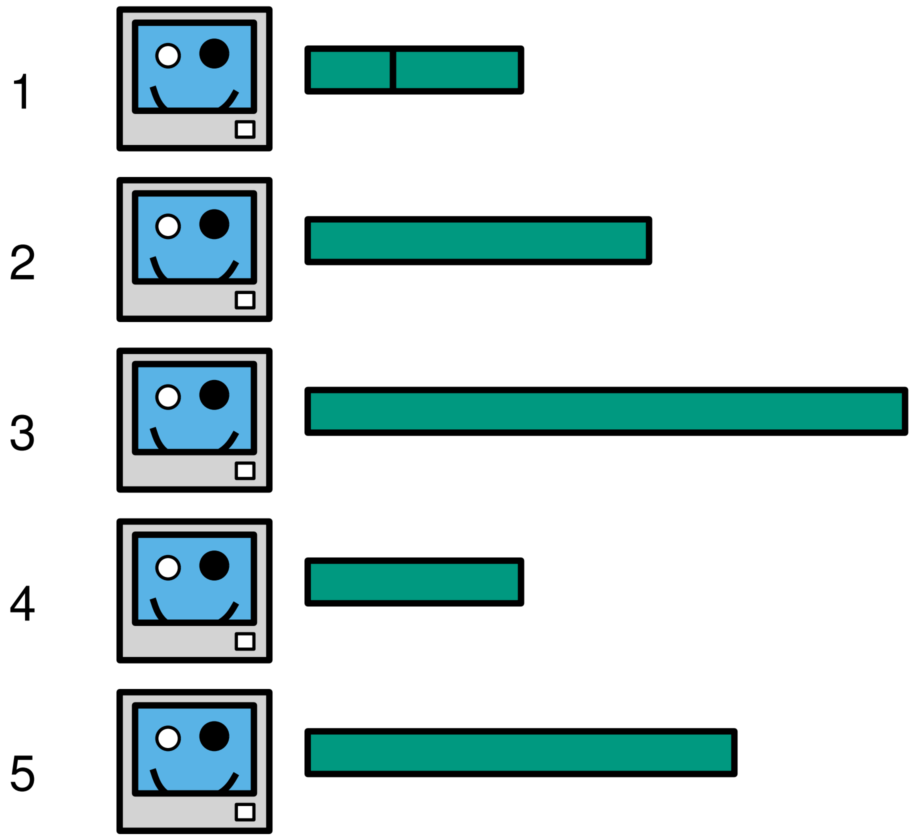

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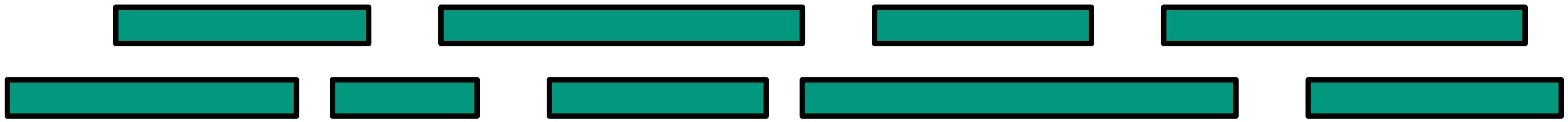

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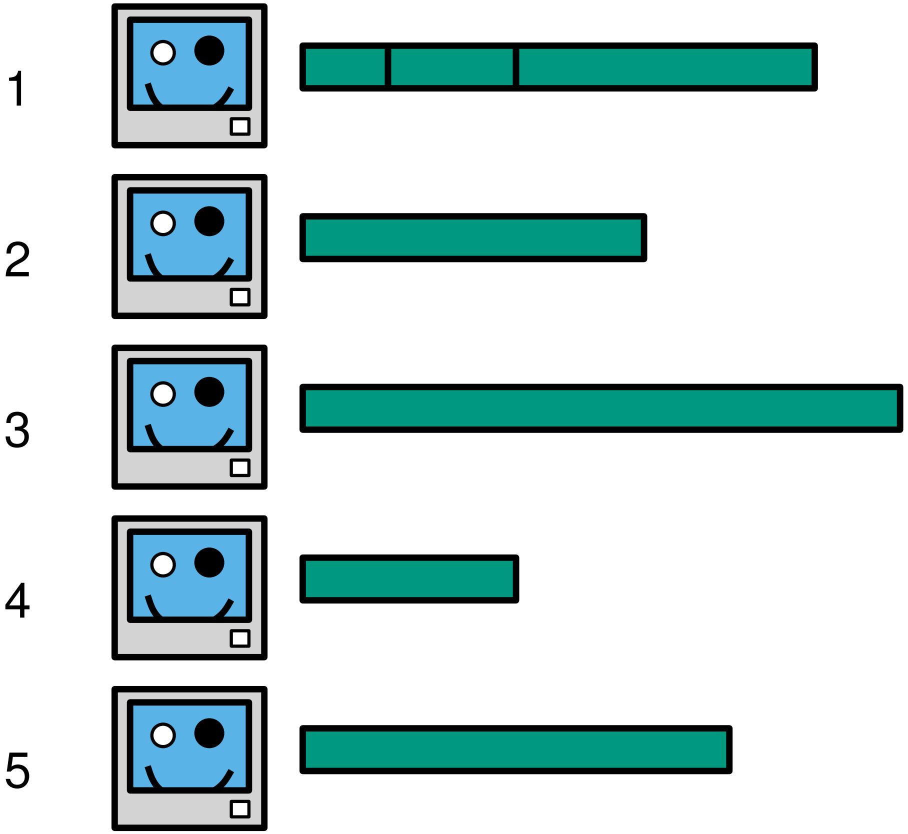

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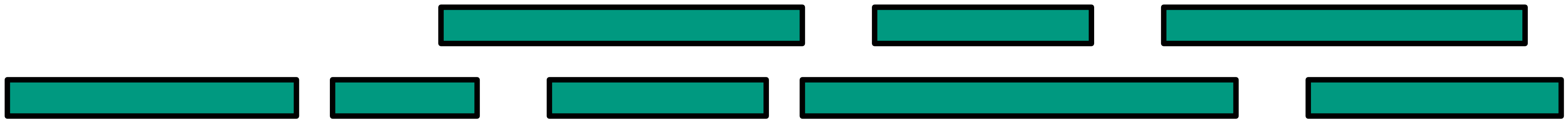

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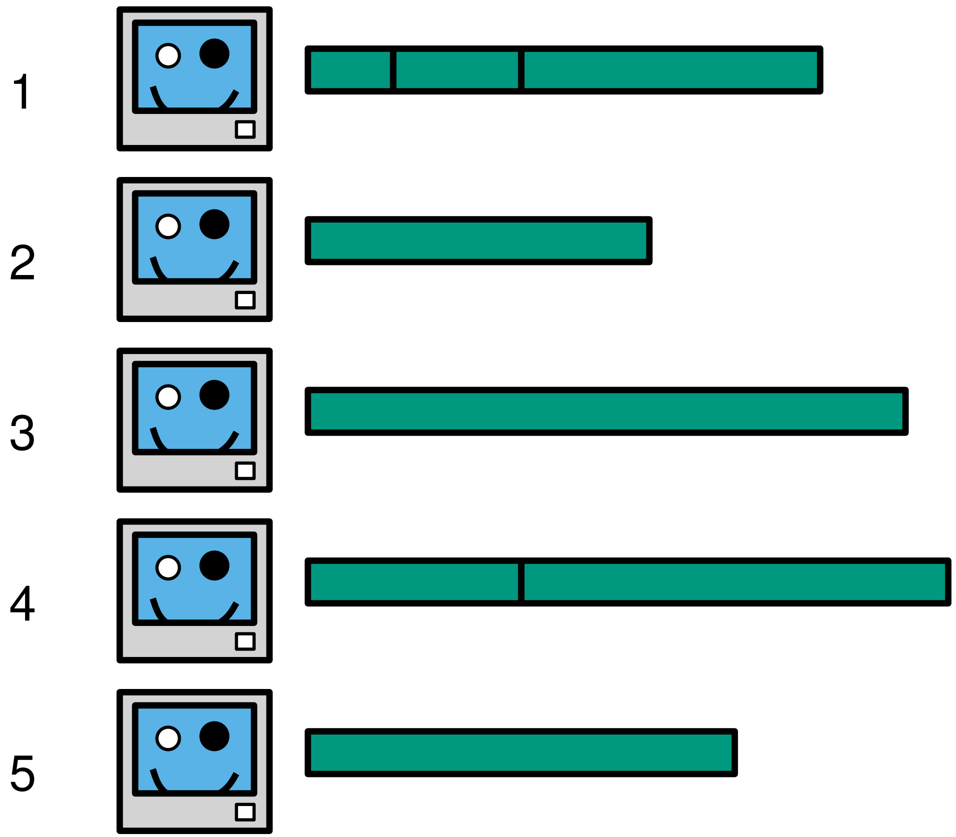

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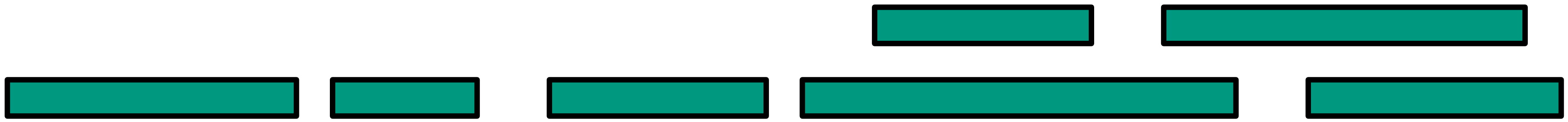

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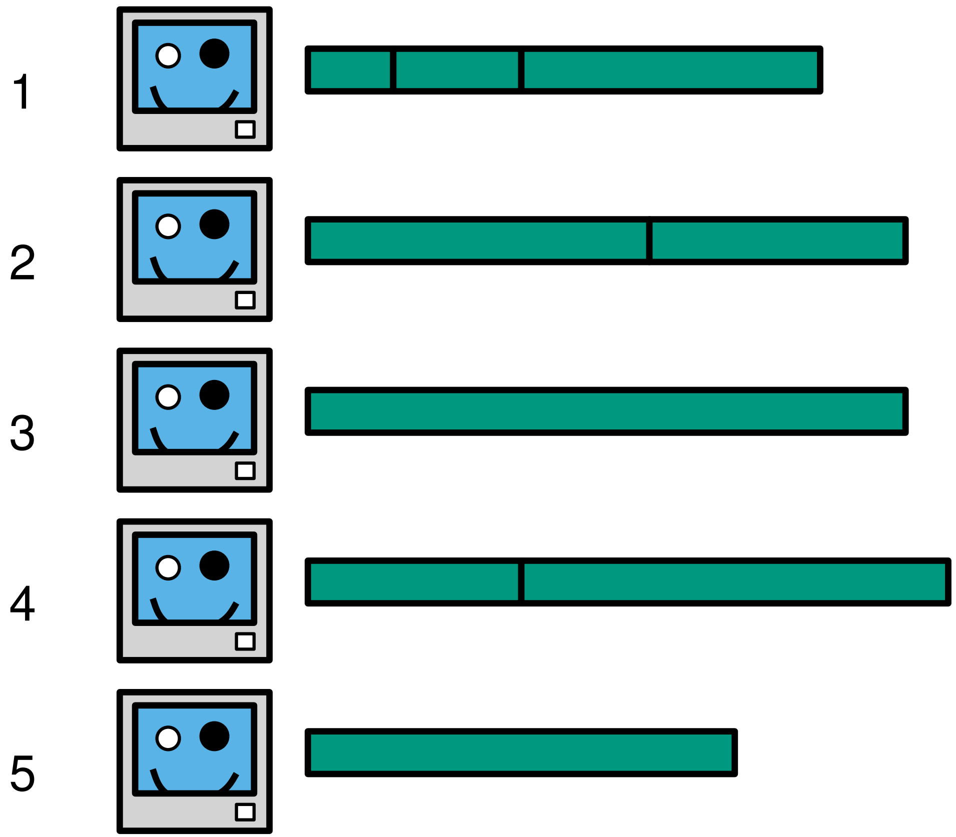

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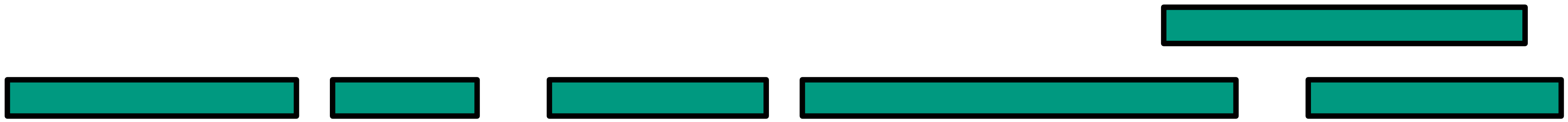

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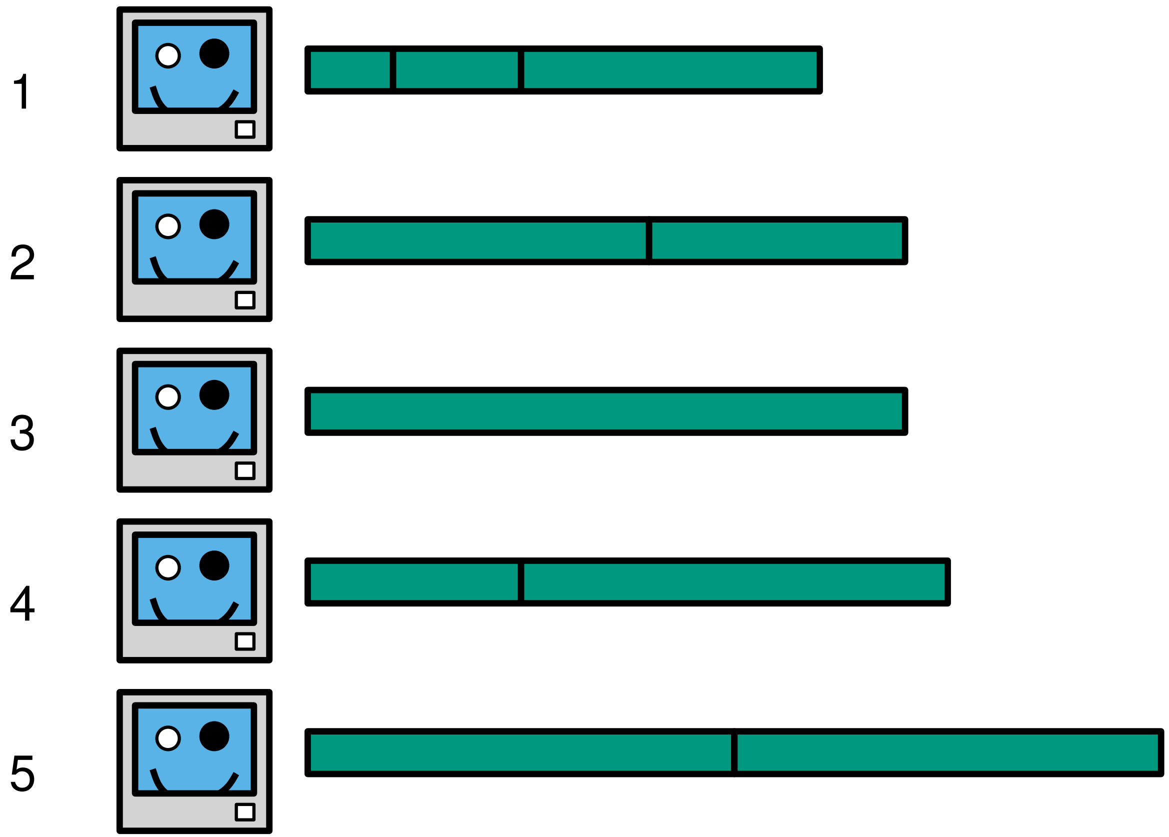

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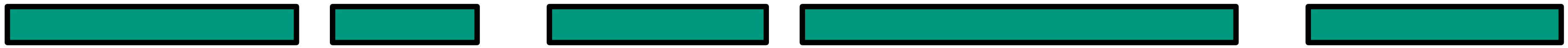

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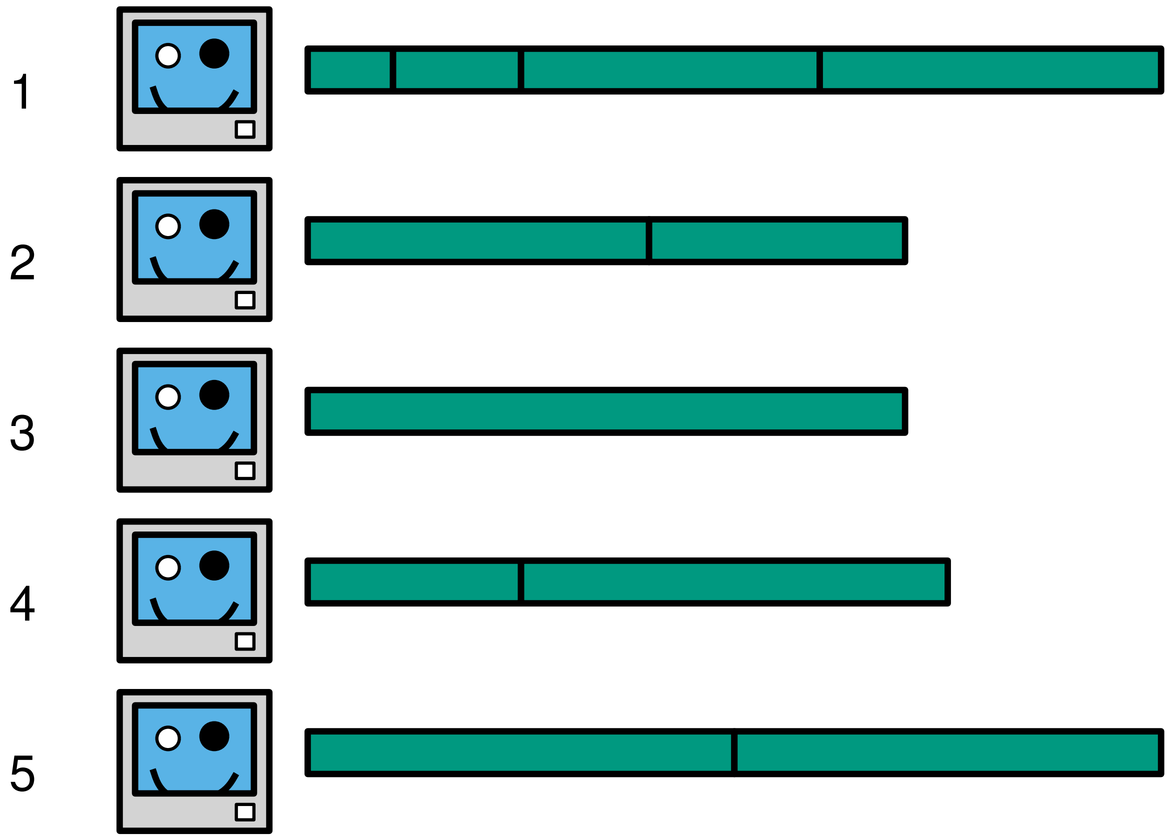

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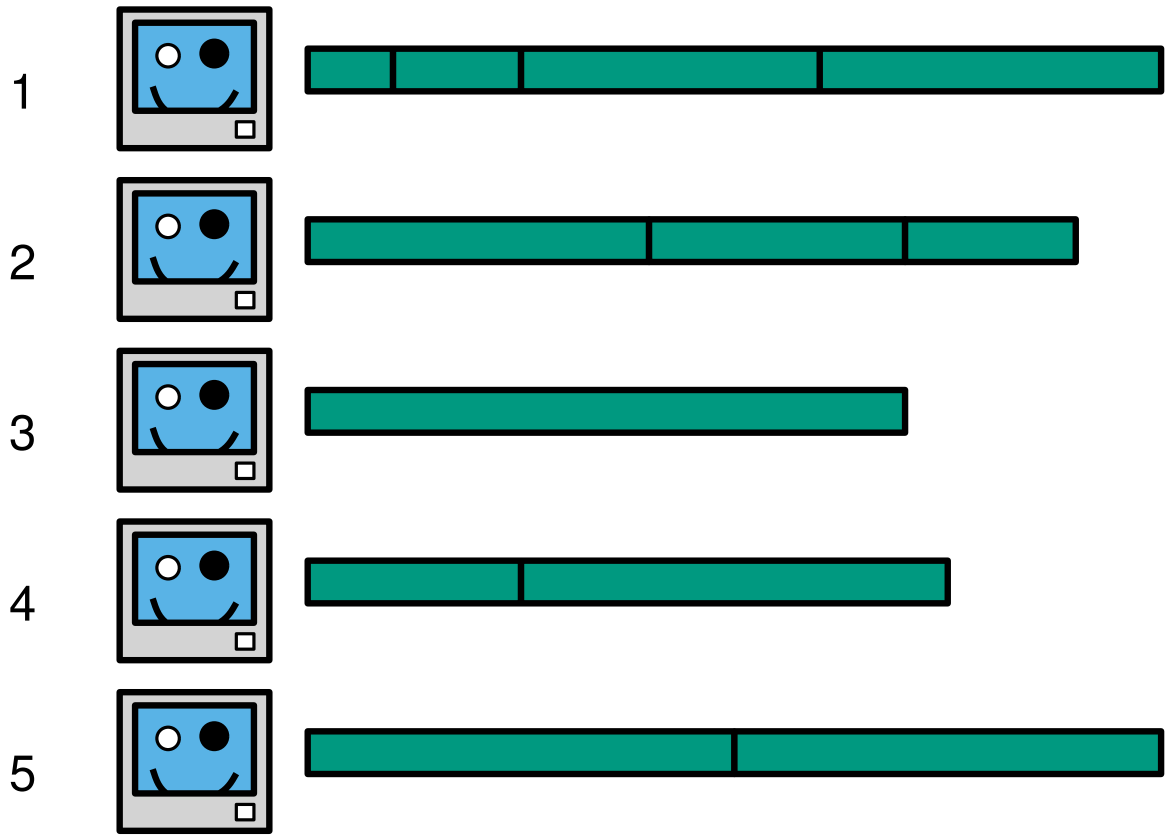

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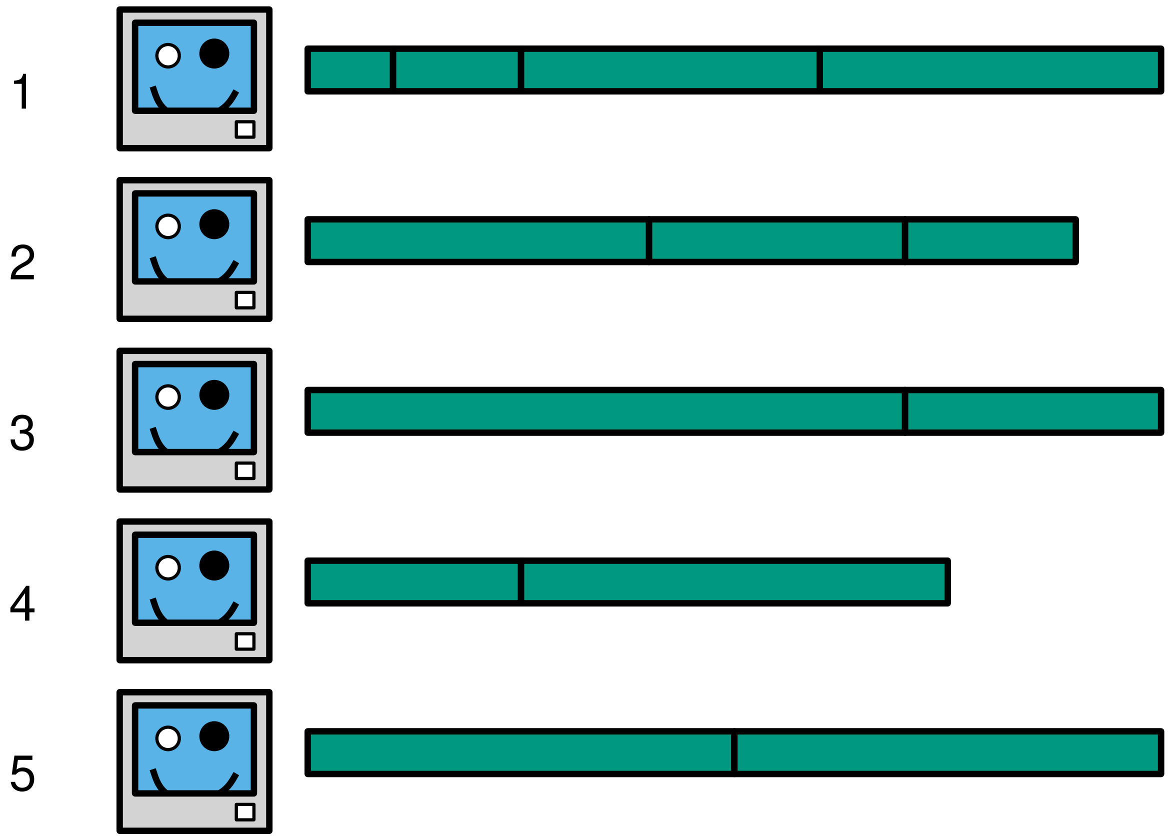

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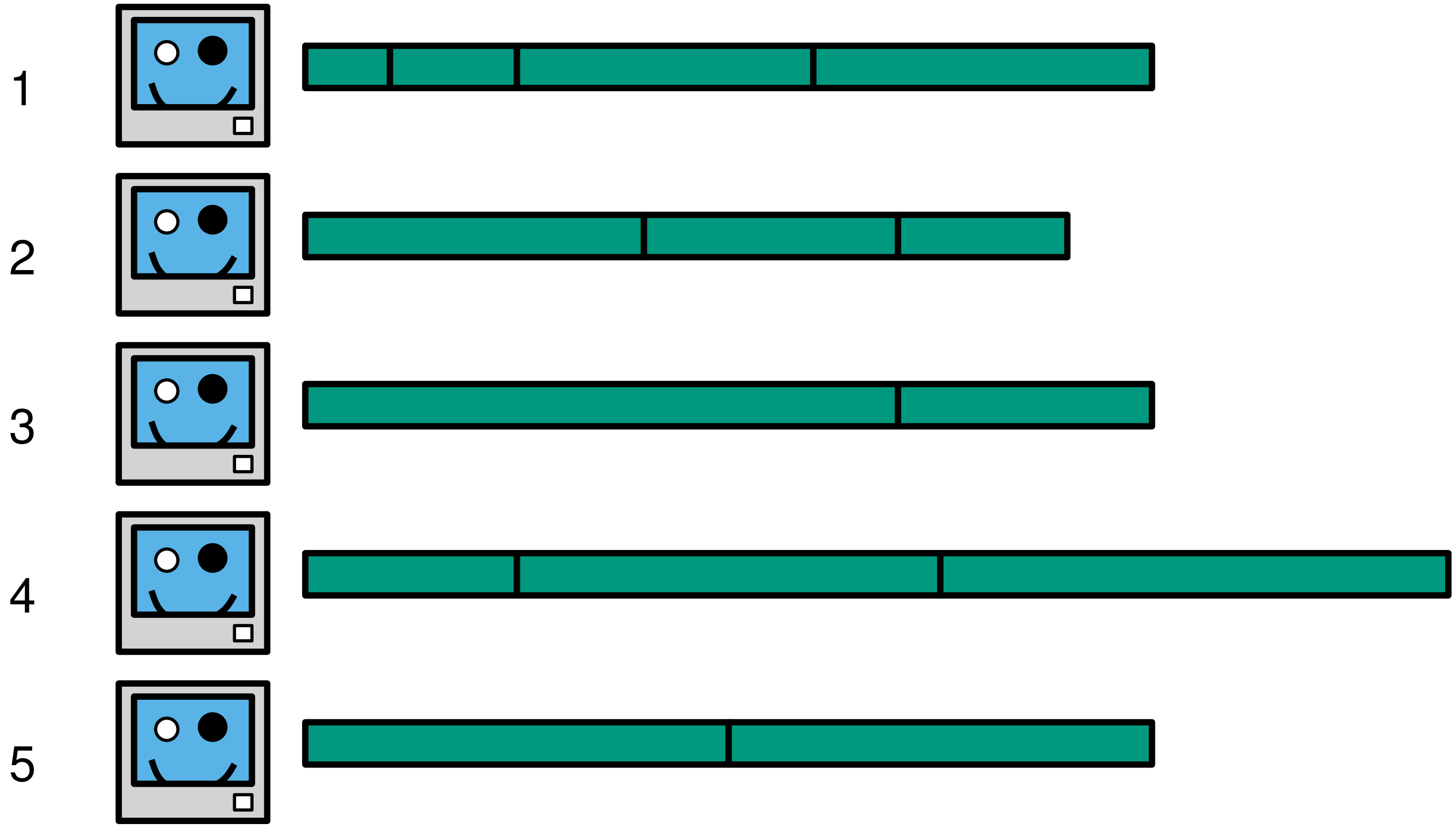

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Scheduling Jobs on Parallel Machines

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How long does it take to compute this schedule?

Scheduling Jobs on Parallel Machines

m machines

n jobs

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$O(nm)$ time naively, $O(n \log m)$ time using a priority queue

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(it's also an online solution)

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How good is it?

The greedy approximation

Let Opt denote the time taken by the optimal scheduling of jobs

Let T_g denote the time taken by the greedy schedule

Theorem The greedy algorithm given is a 2-approximation algorithm

L_i is the *load* of machine i

Job j takes t_j time units

m machines
n jobs

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(the m machines can't **all** *have below average load)*

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Proof Consider the machine i with largest load $T_g = L_i$

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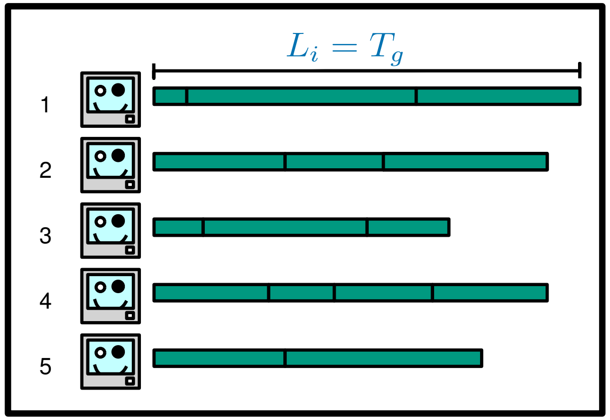

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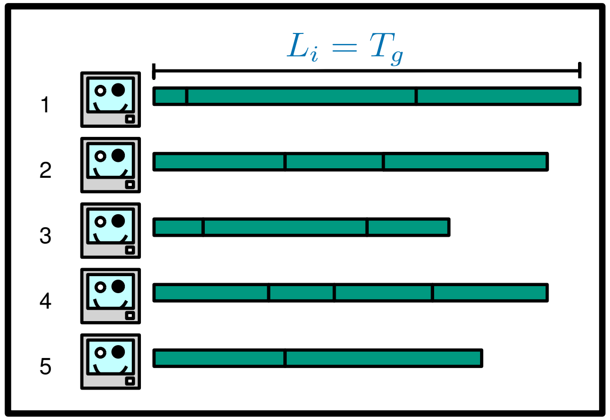

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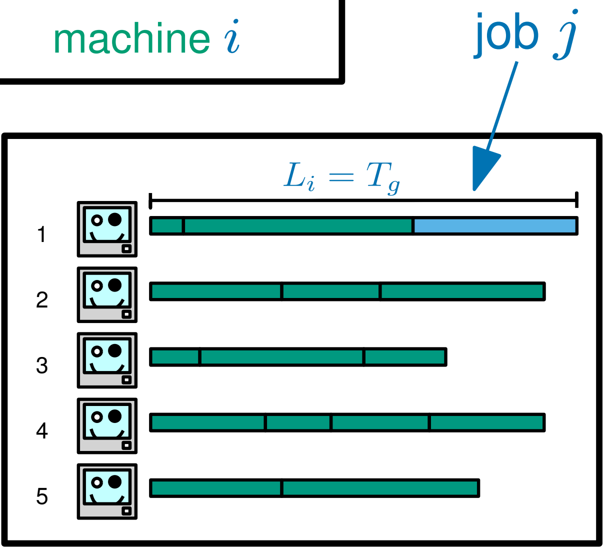

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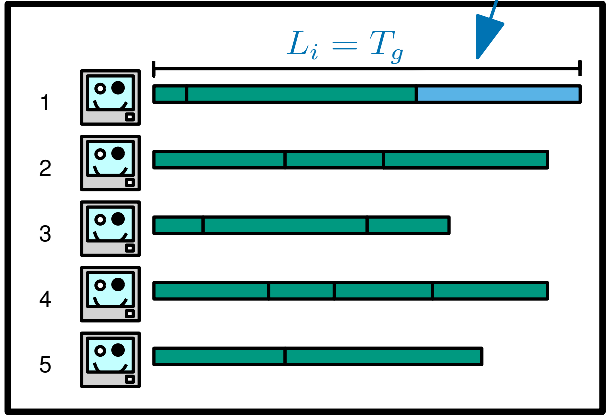

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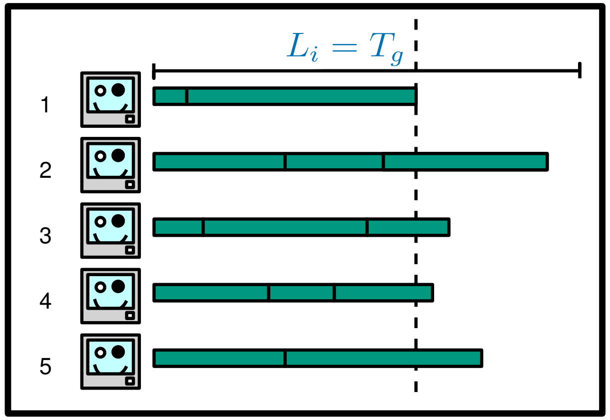

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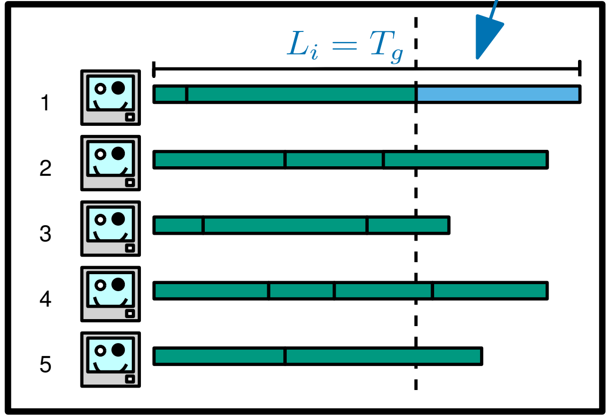

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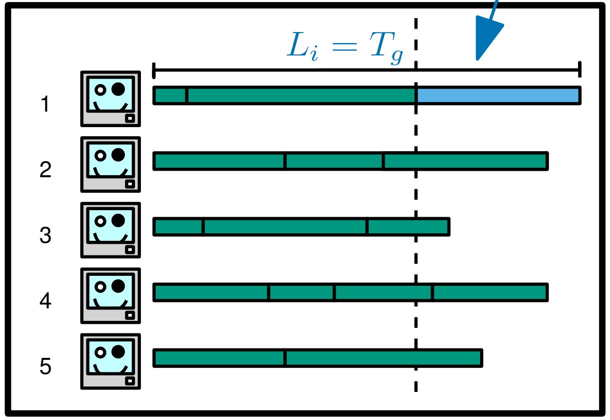

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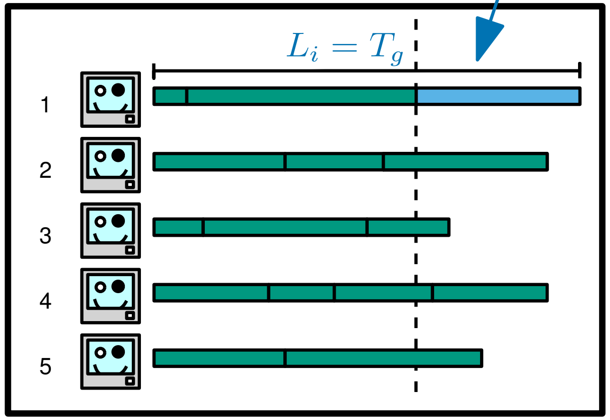

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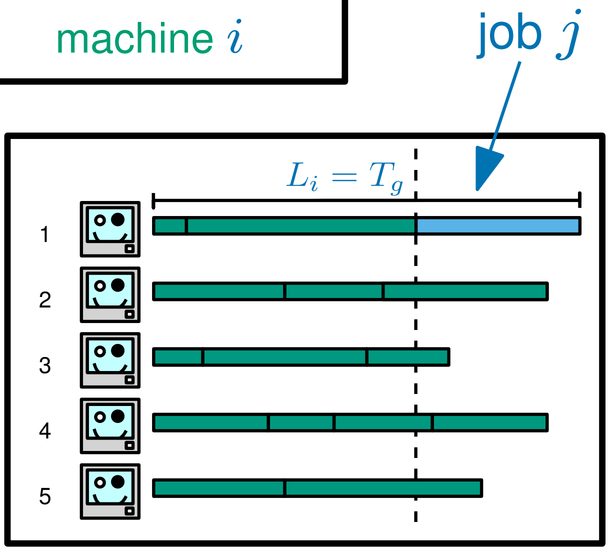

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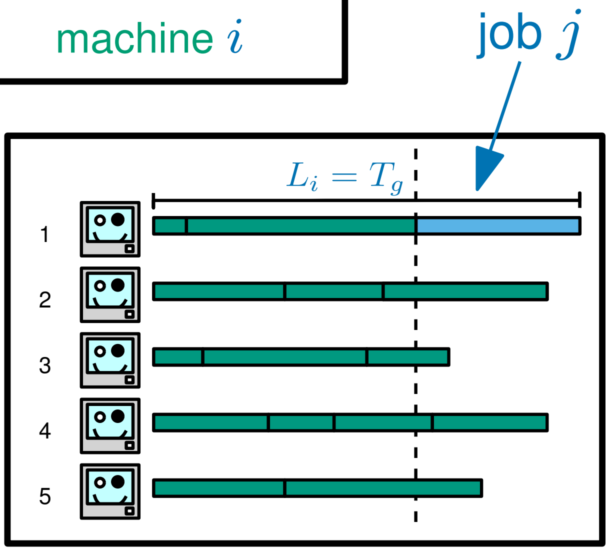

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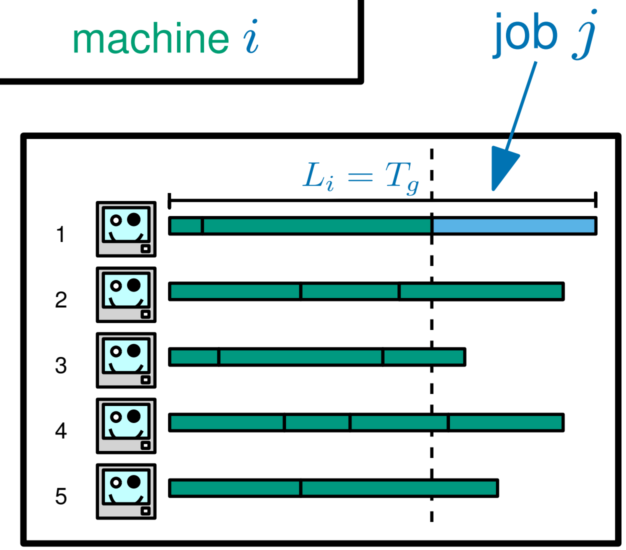

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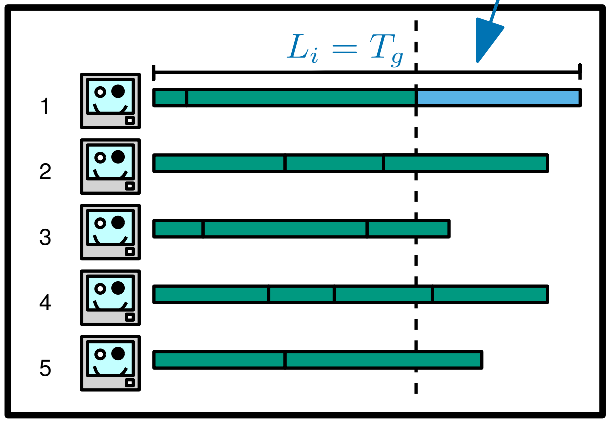

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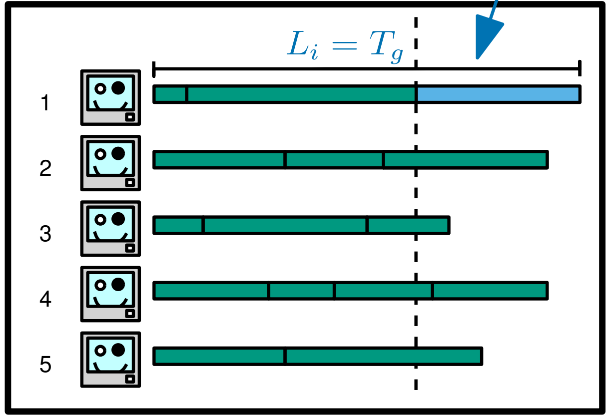

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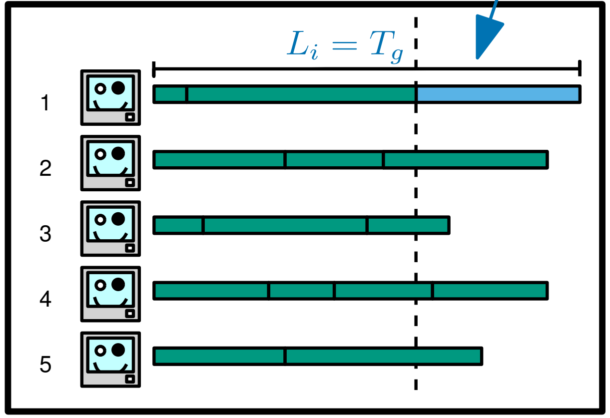

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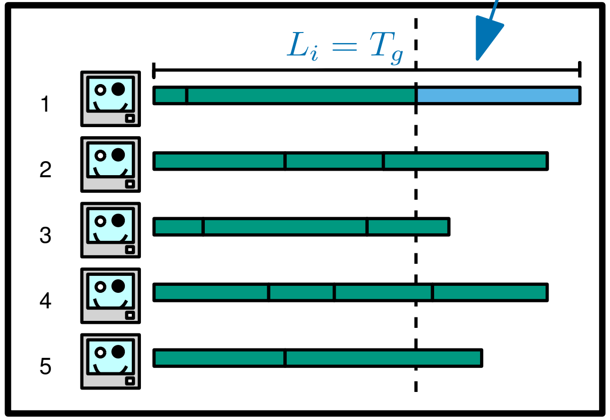

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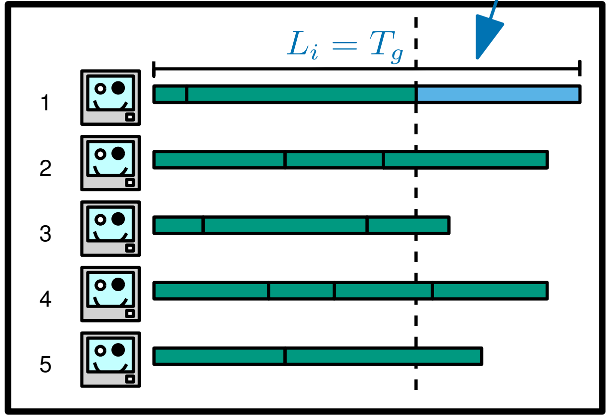

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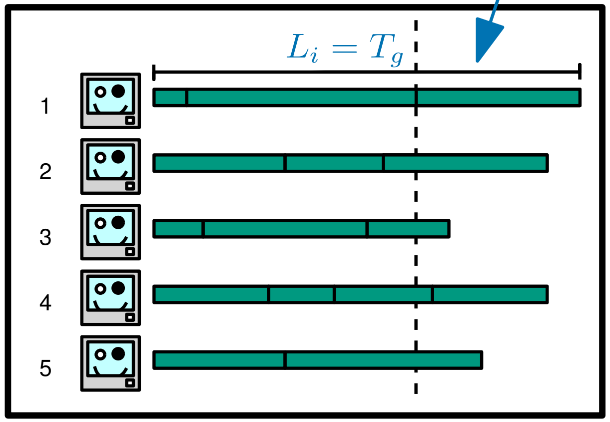

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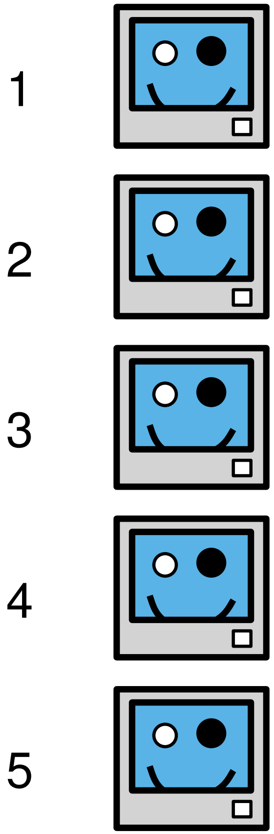
Therefore, $T_g = L_i = (L_i - t_j) + t_j \leqslant \mathrm{Opt} + \mathrm{Opt} = 2\mathrm{Opt}$

Longest Processing Time (LPT)



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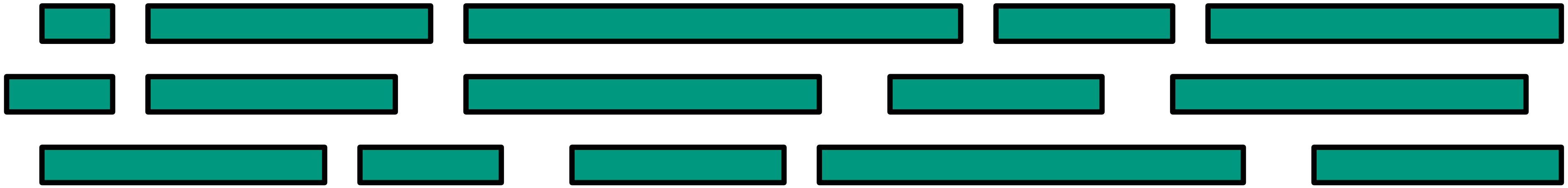

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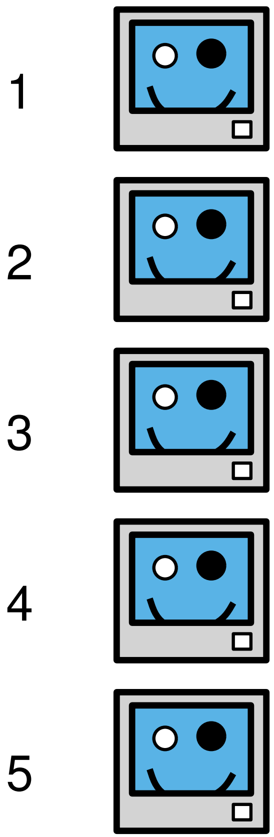

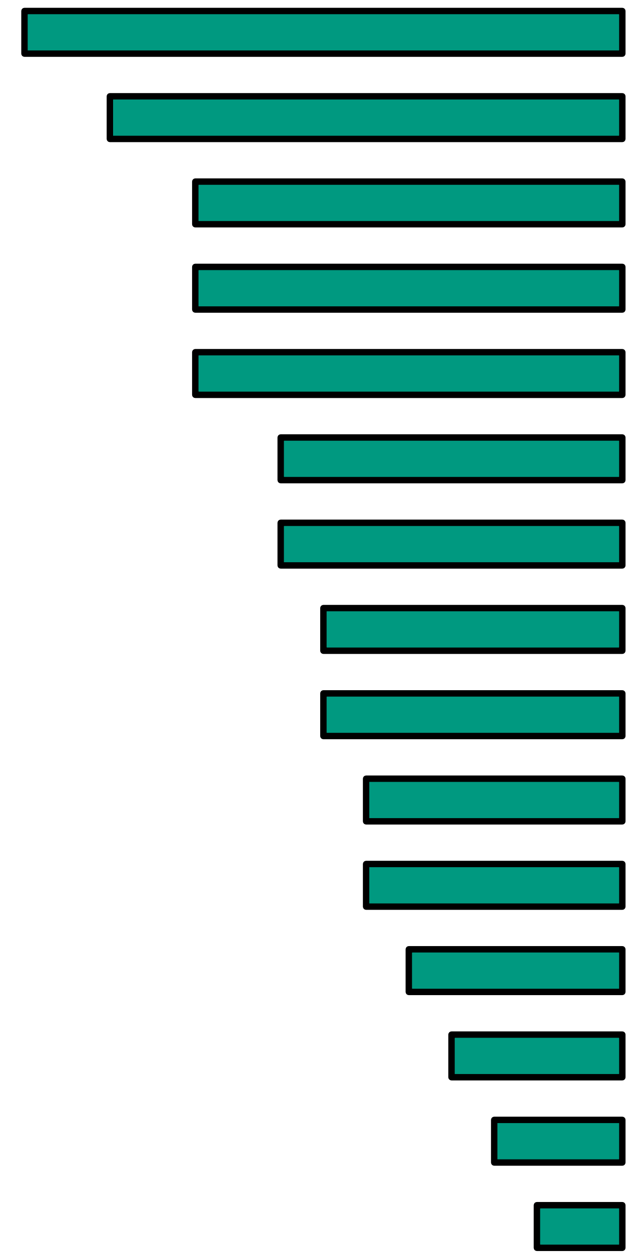

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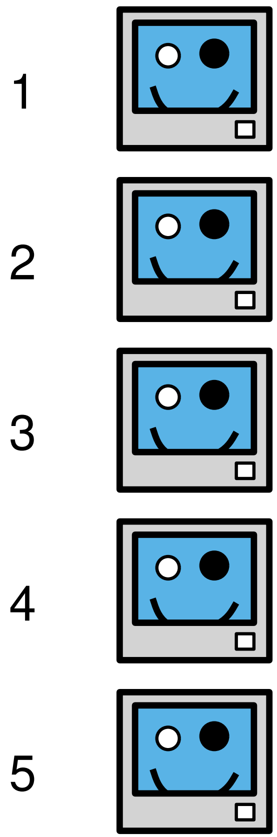

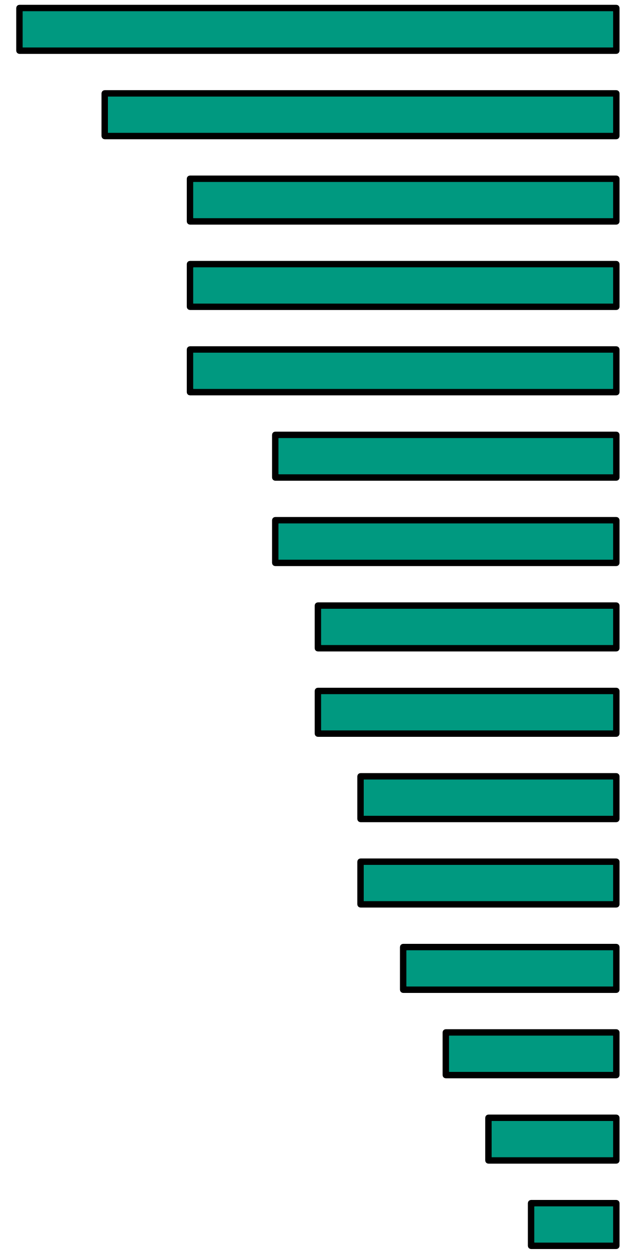

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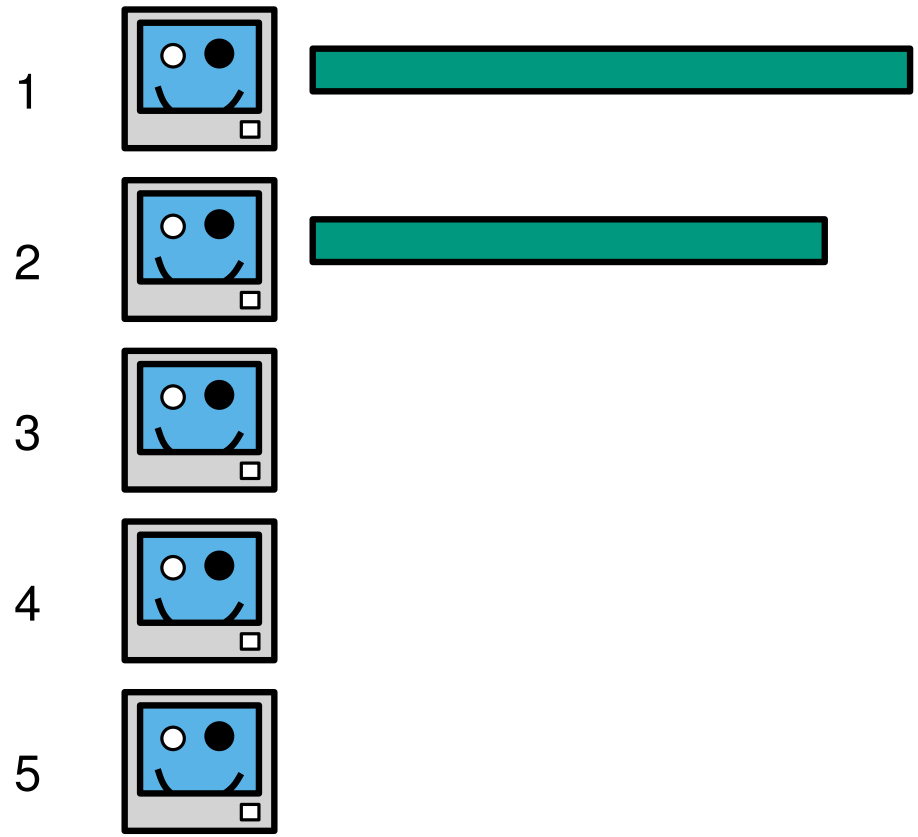

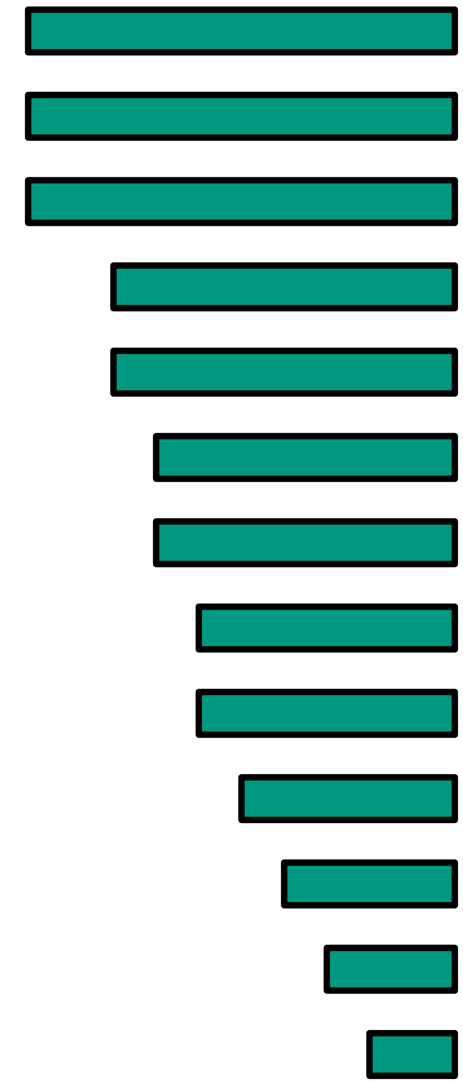

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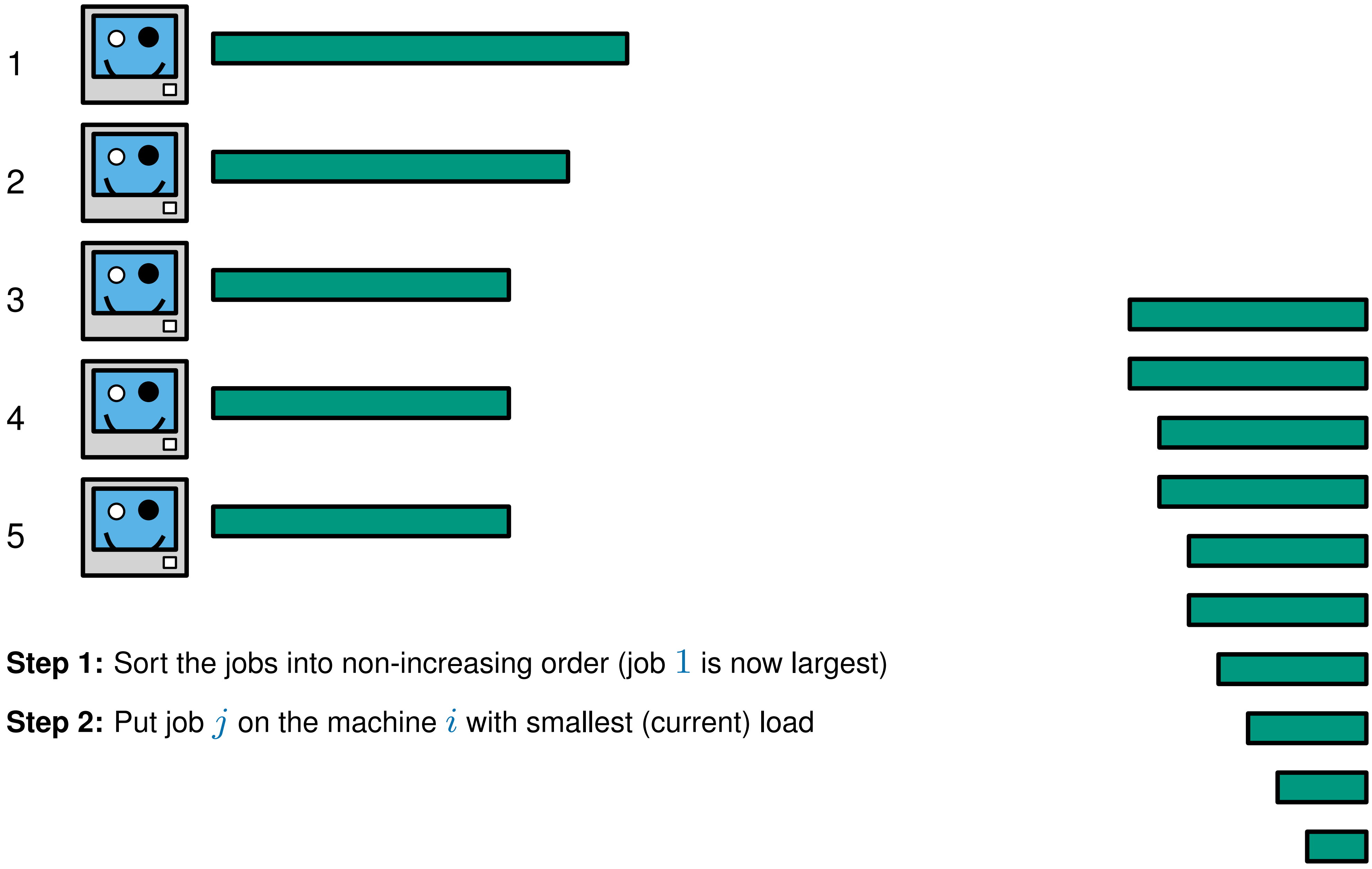

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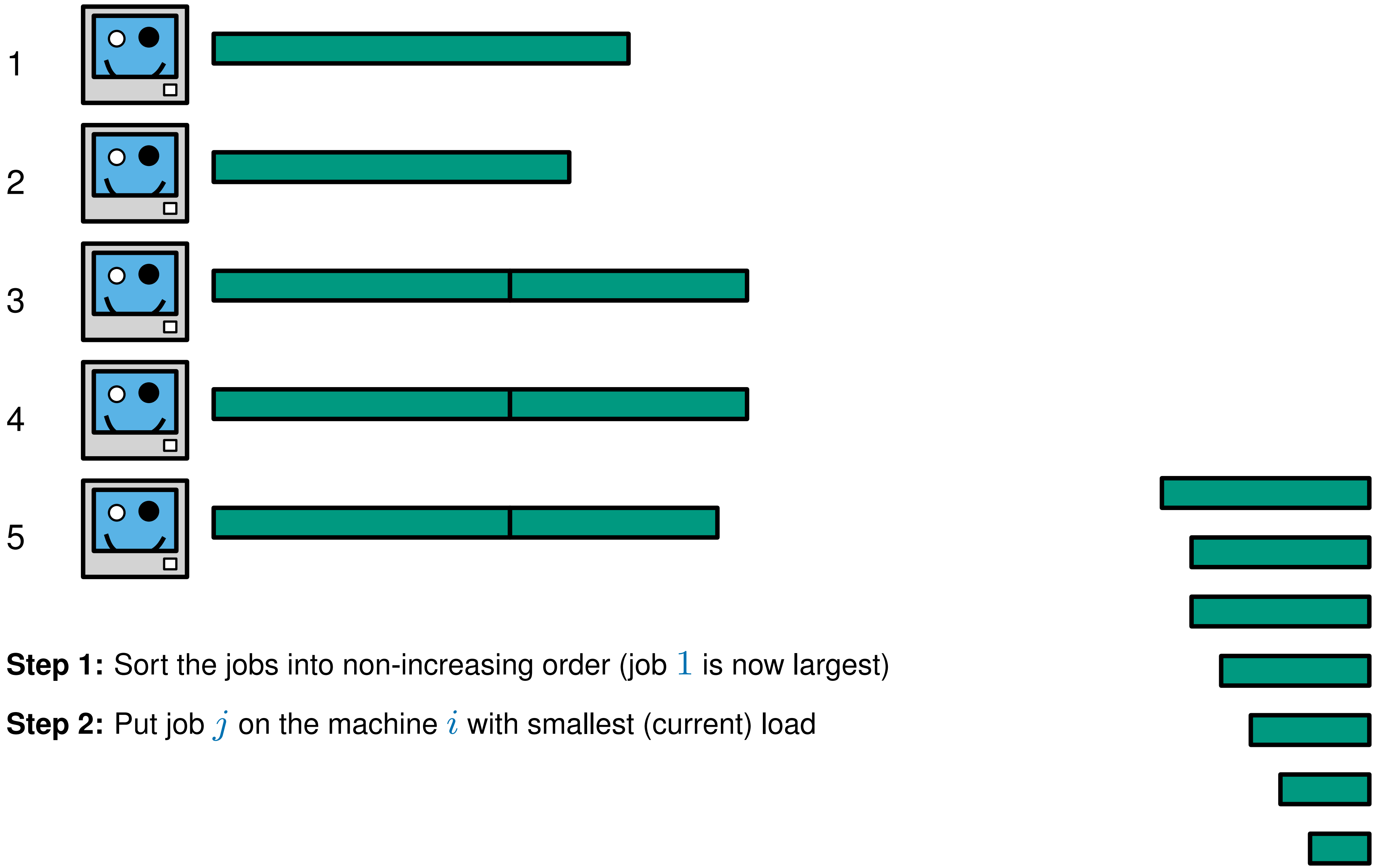

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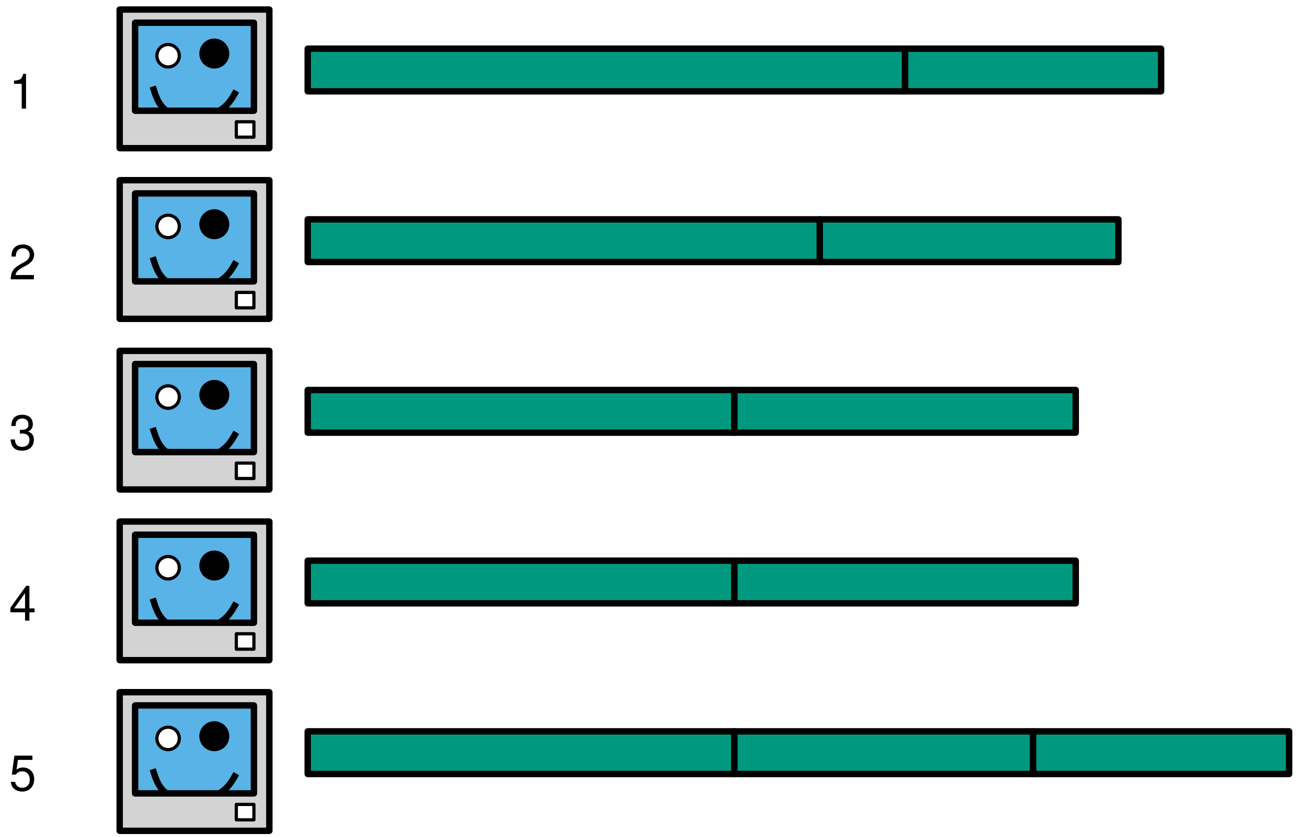

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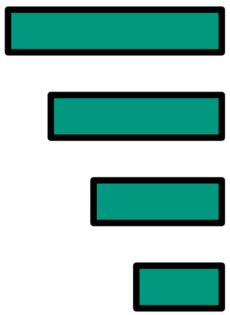

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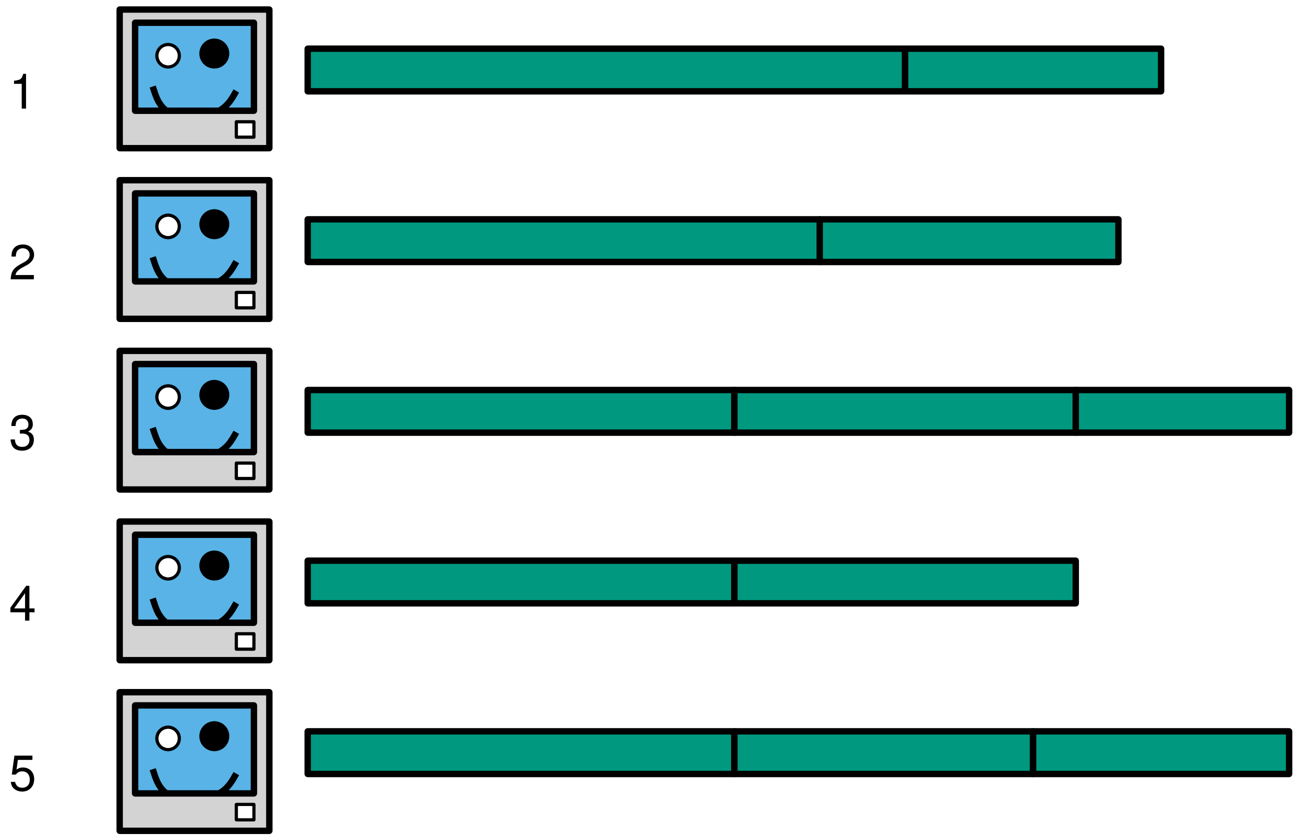

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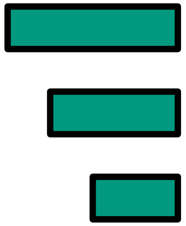

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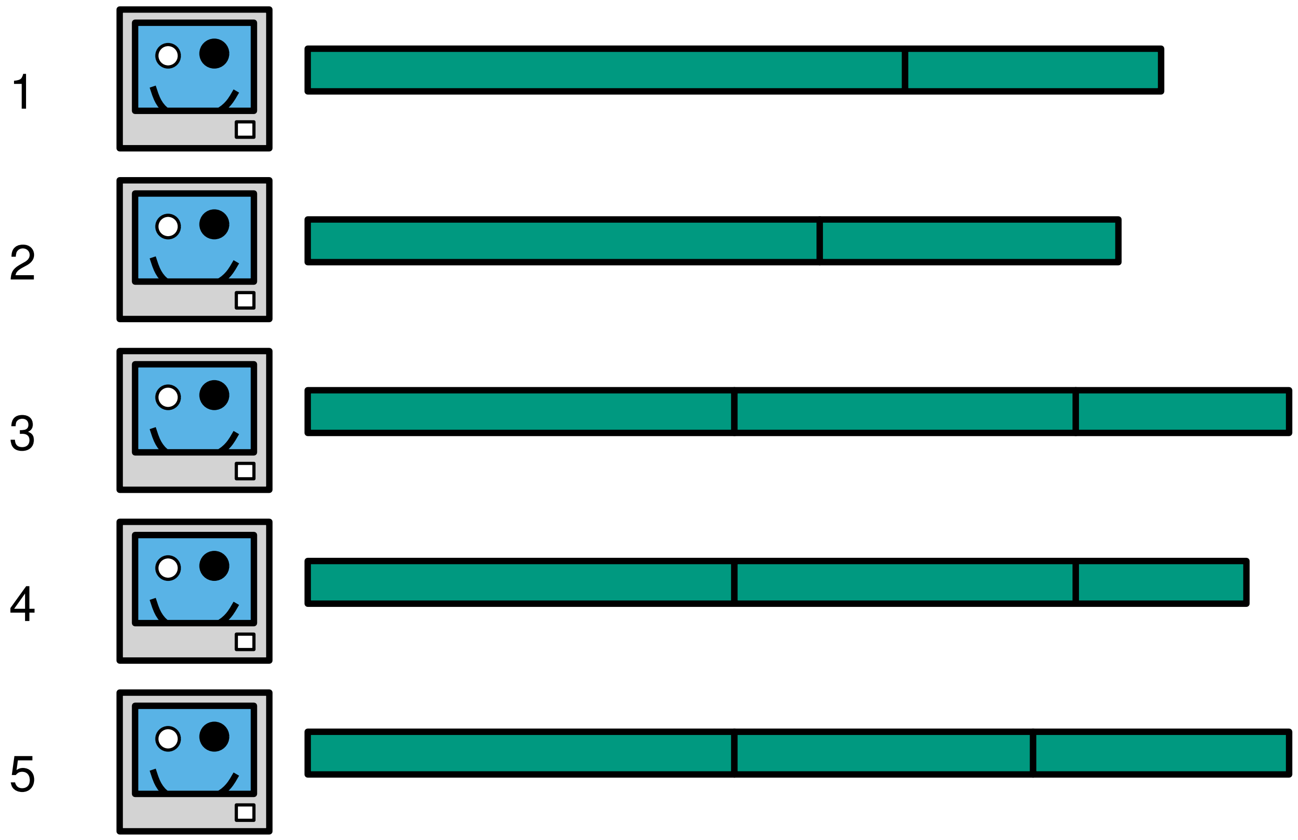

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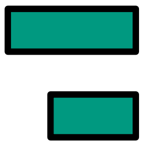

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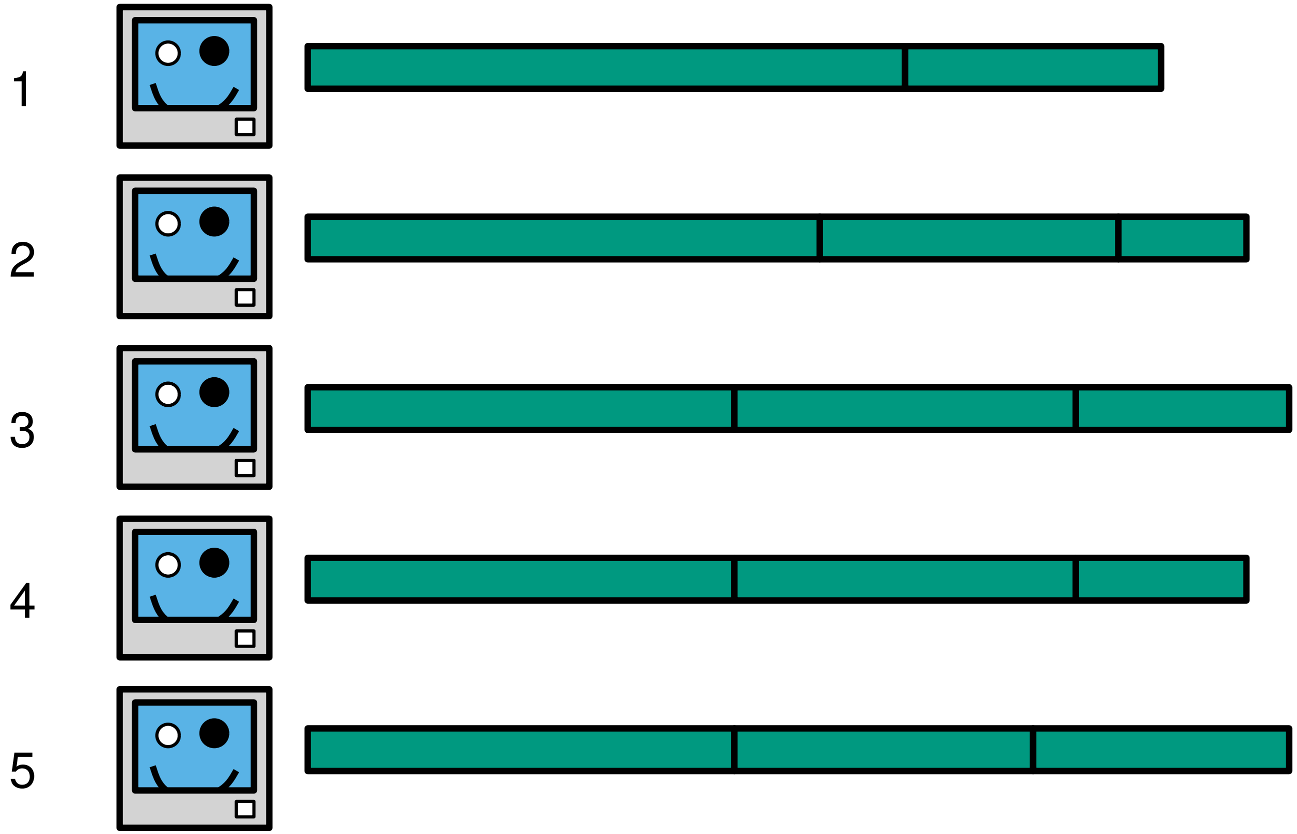

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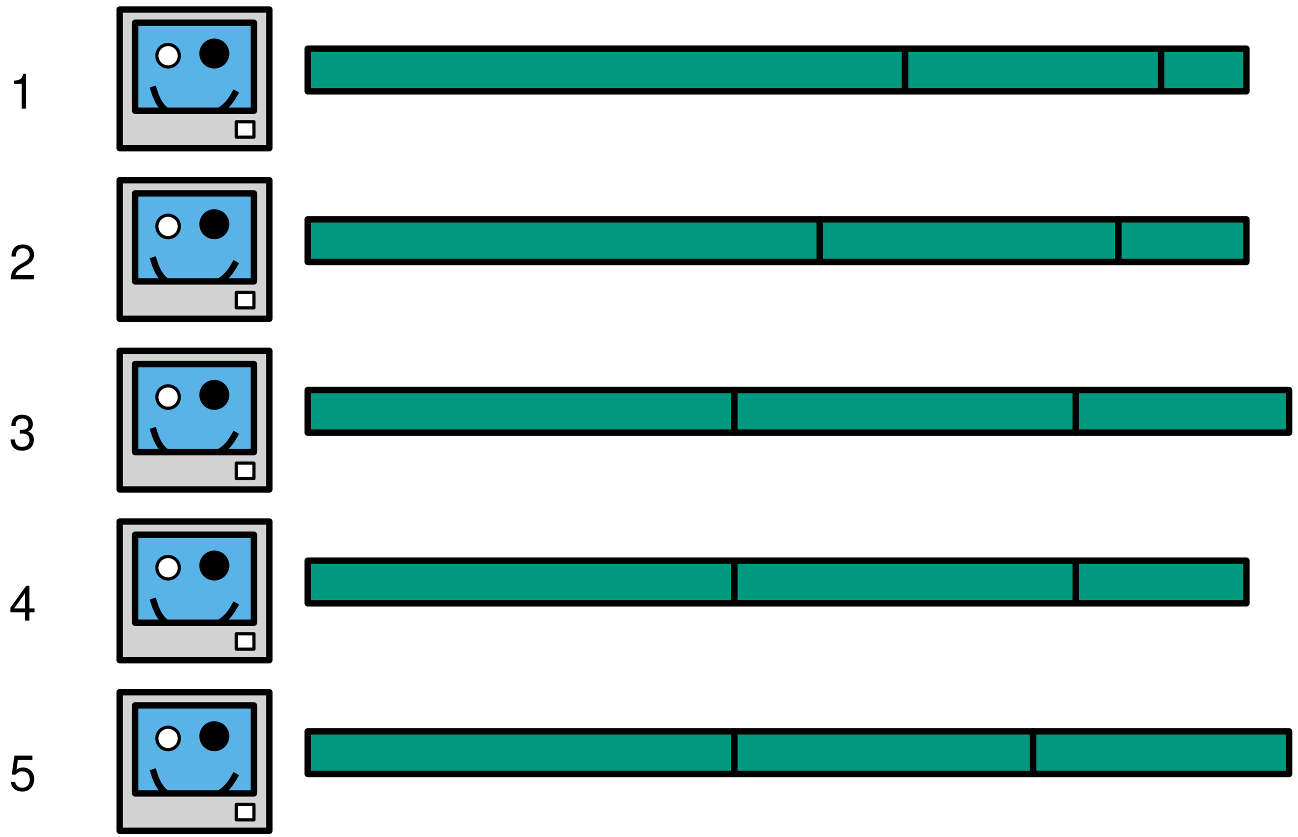

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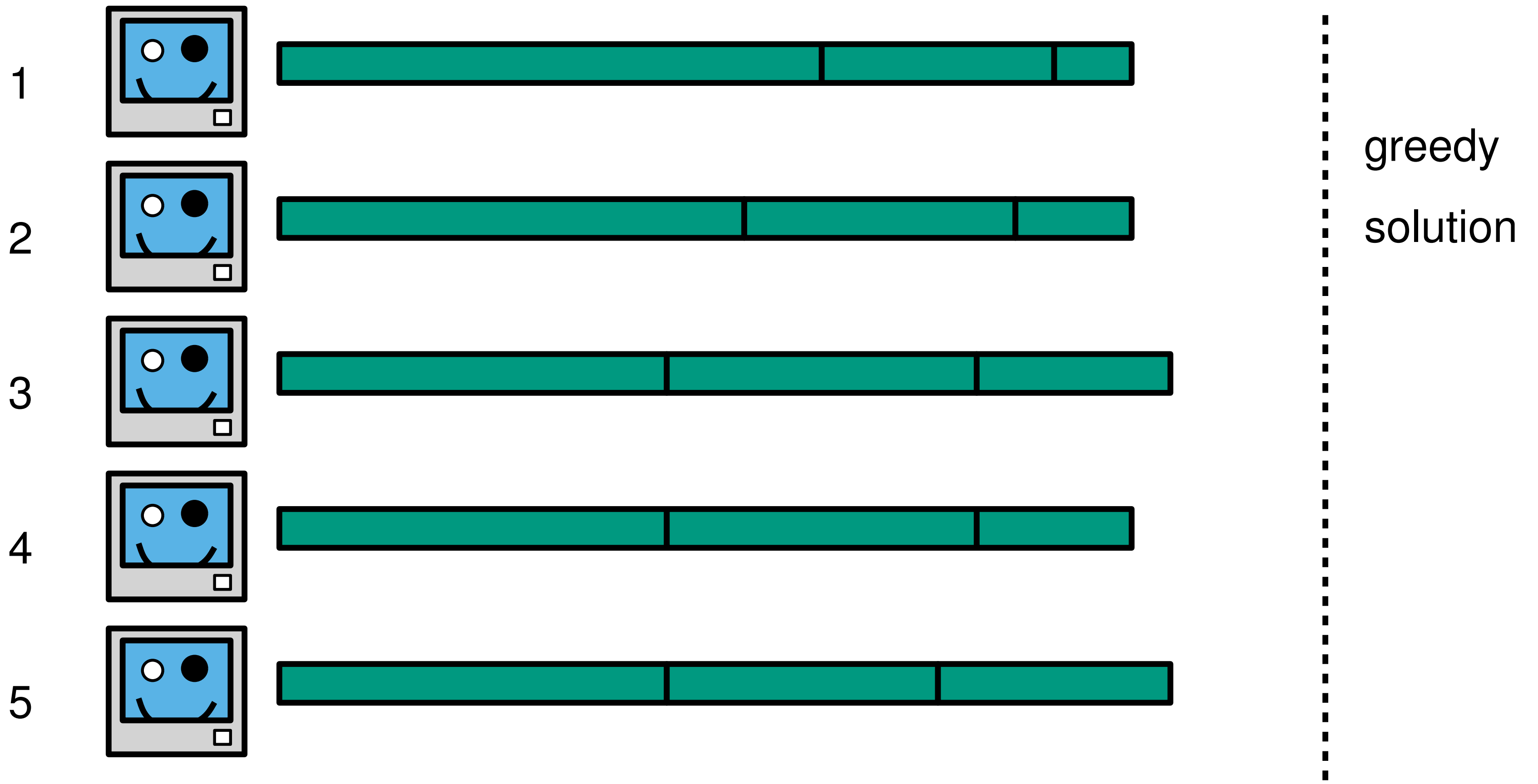

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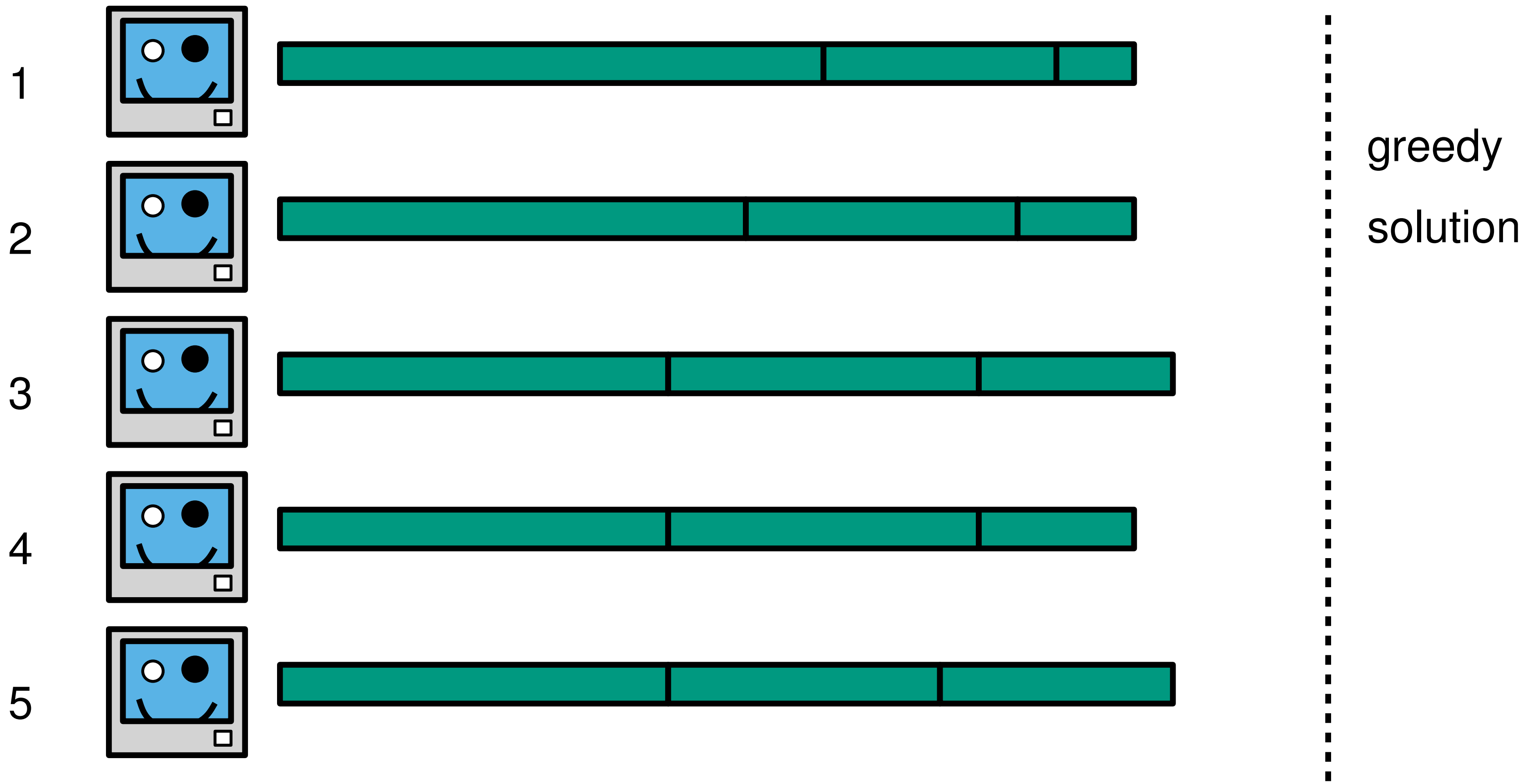

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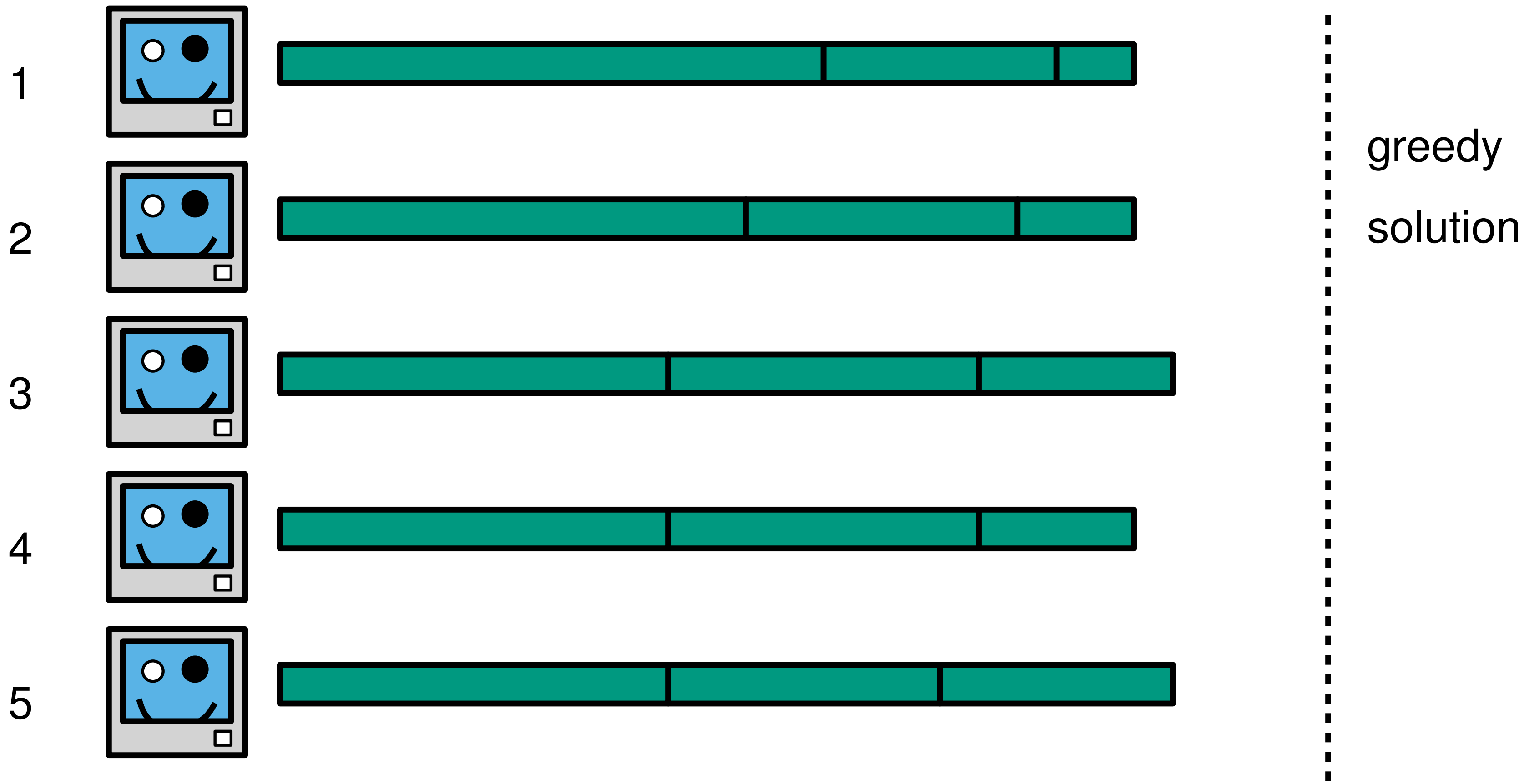

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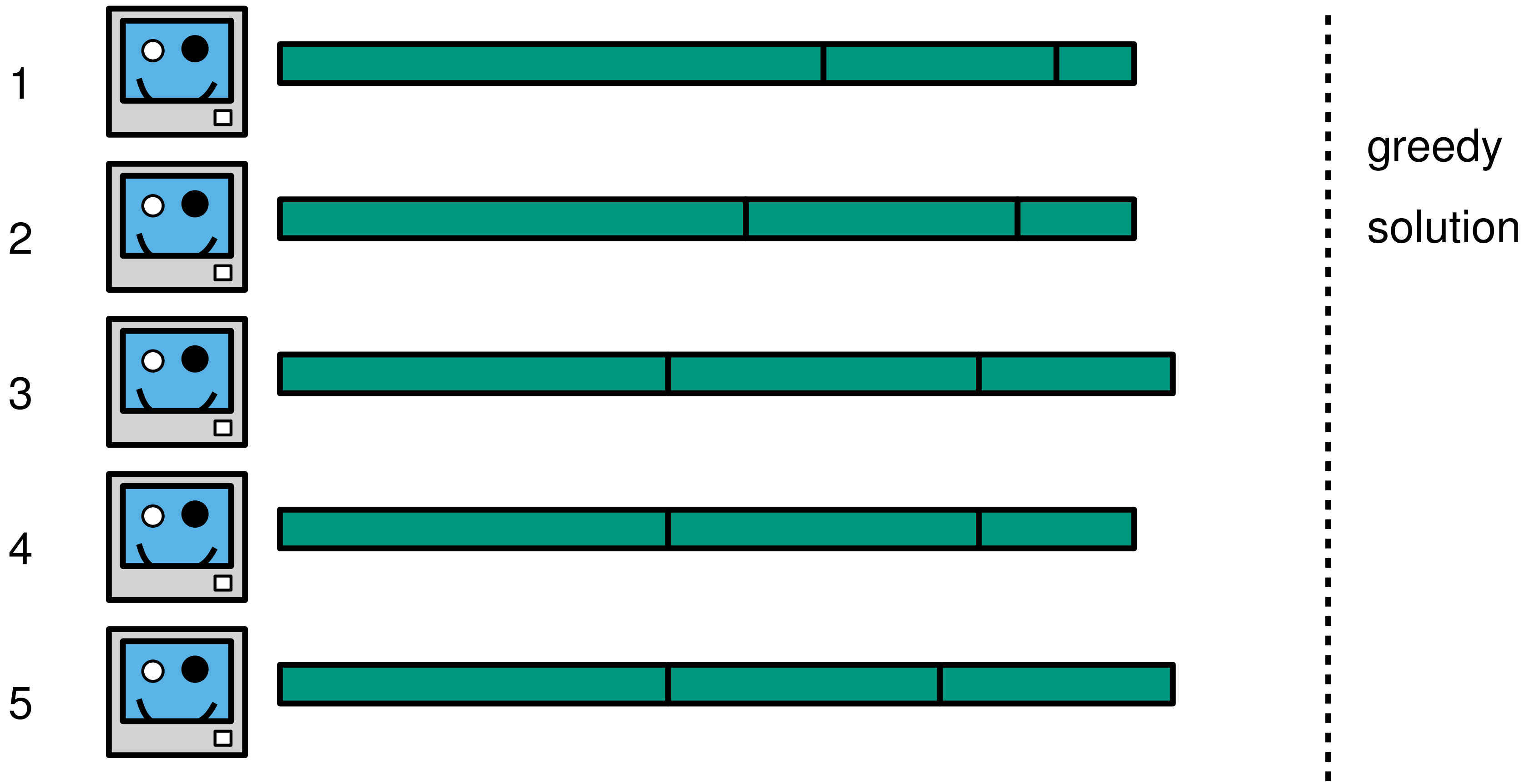

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Proof

- Note that $t_1 \geqslant t_2 \geqslant t_3 \geqslant \ldots t_m \geqslant t_{(m+1)}$

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- Let T_l denote the time taken by the LPT schedule

L_i is the *load* of machine i

Theorem The LPT algorithm is a $3/2$-approximation algorithm

- Before we prove this, we prove another useful fact and a Lemma

m machines

n jobs

Fact If there are at most m jobs $(n \leqslant m)$ then LPT is optimal

If there are at most m jobs then

LPT gives each job its own machine so $\max_i L_i \leqslant \max_j t_j \leqslant \mathrm{Opt}$

Job j takes t_j time units

Lemma If $n > m$ then $\mathrm{Opt} \geqslant 2t_{(m+1)}$ (after sorting)

Proof

- Note that $t_1 \geqslant t_2 \geqslant t_3 \geqslant \ldots t_m \geqslant t_{(m+1)}$
- One of the m machines must be assigned

  (at least) two of these $m+1$ jobs *under any schedule*

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  in particular $\mathrm{Opt} \geqslant 2t_{(m+1)}$

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Proof Consider the machine i with largest load $T_l = L_i$

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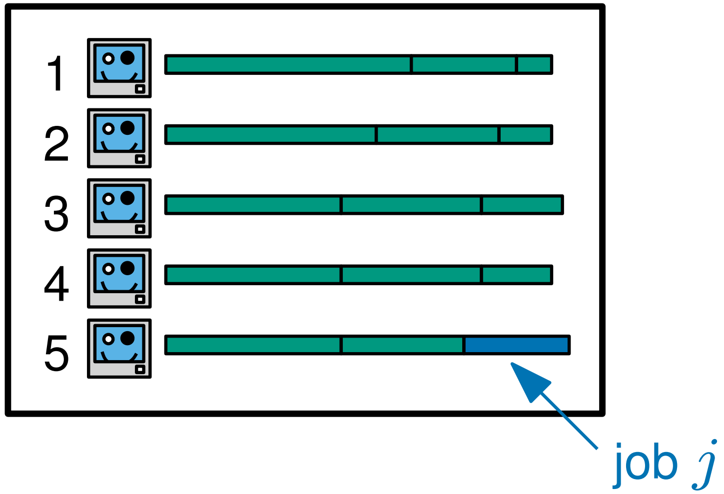

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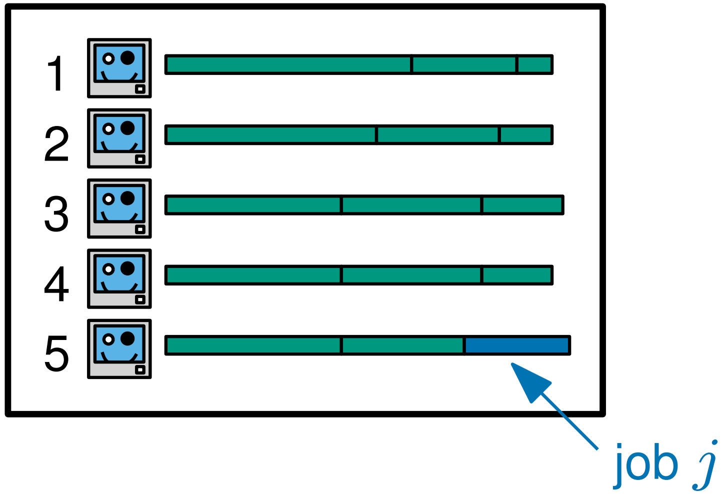

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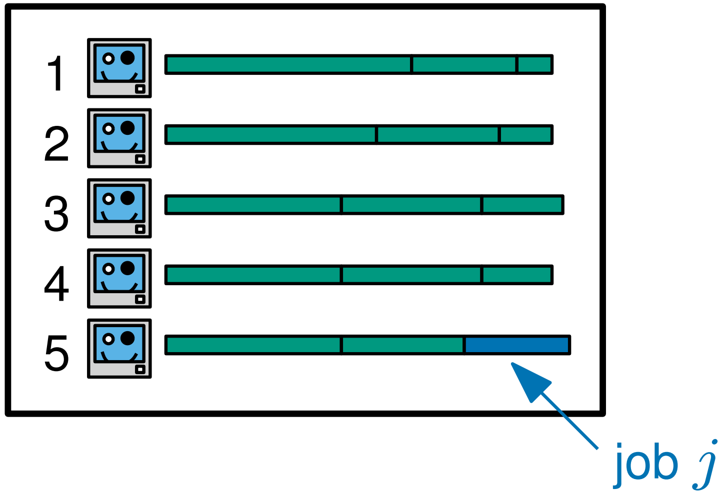

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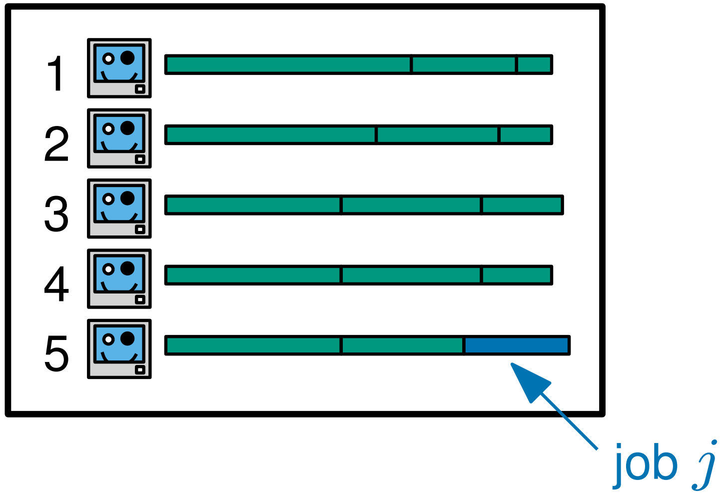

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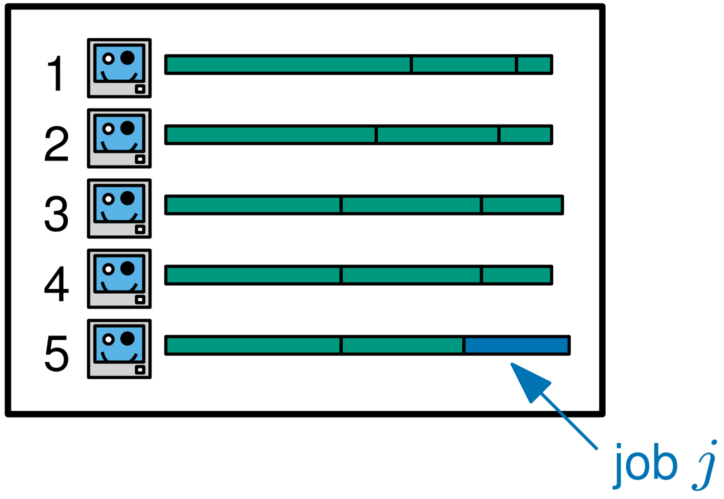
because LPT is optimal in this case



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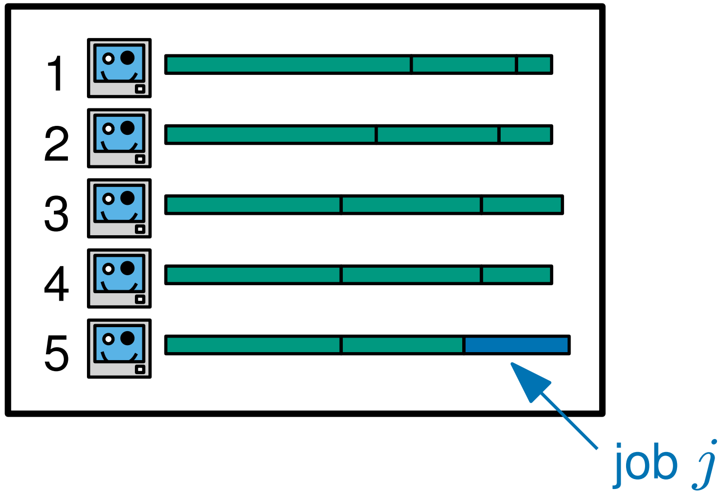

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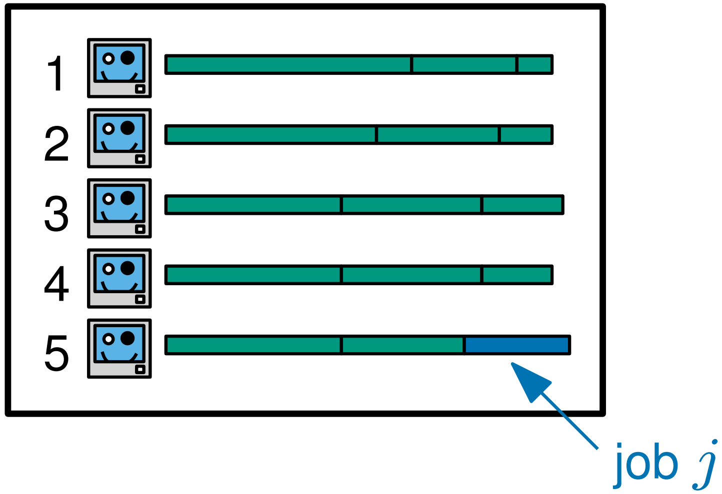

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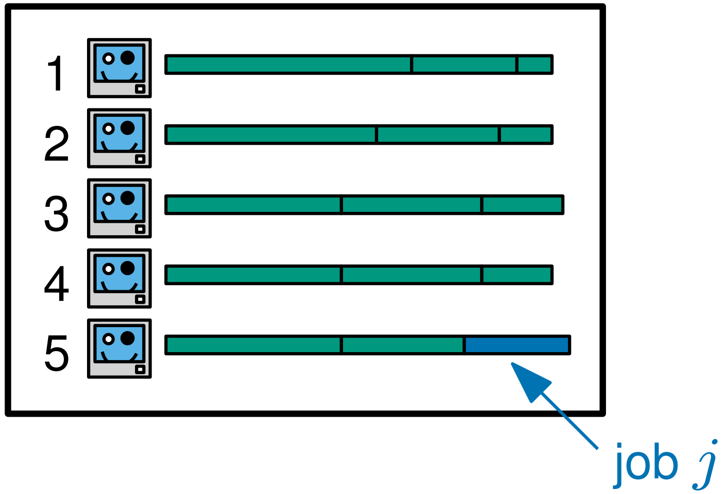
Fact $\mathrm{Opt} \geqslant \max_j t_j$



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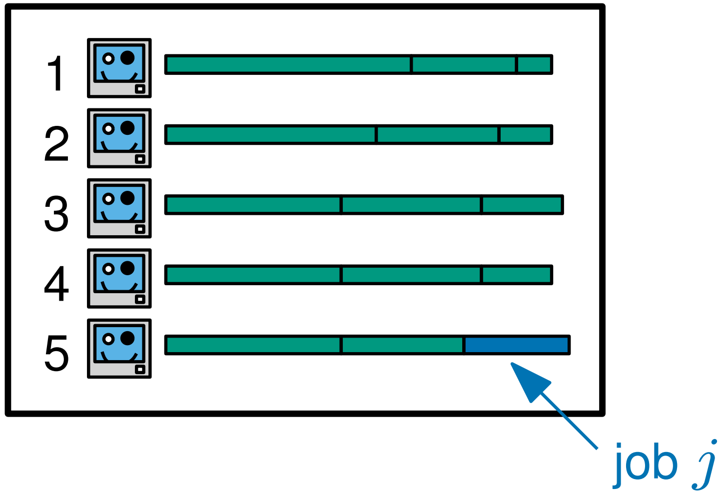

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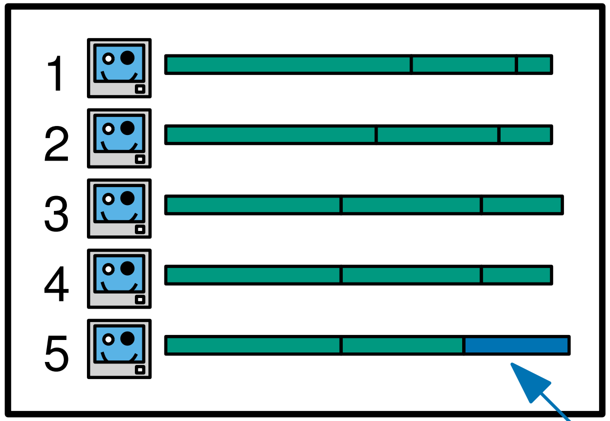

job j

m machines

n jobs

Job j takes t_j time units

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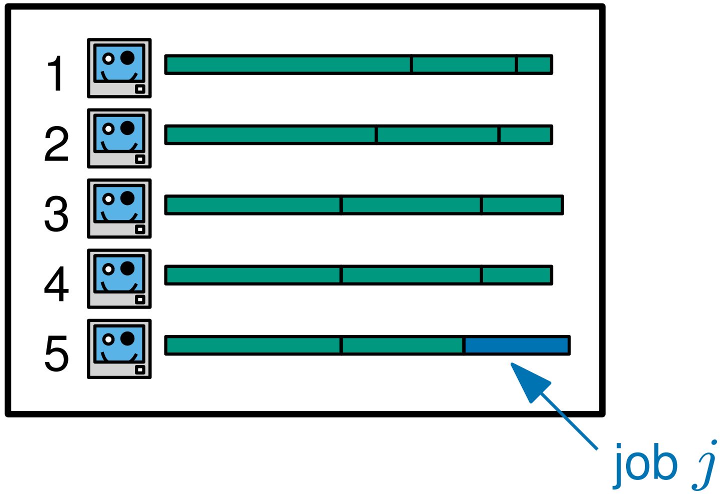
L_i is the *load* of machine i

Theorem The LPT algorithm is a $3/2$-approximation algorithm



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m machines
n jobs

- Therefore machine i was assigned at least two jobs

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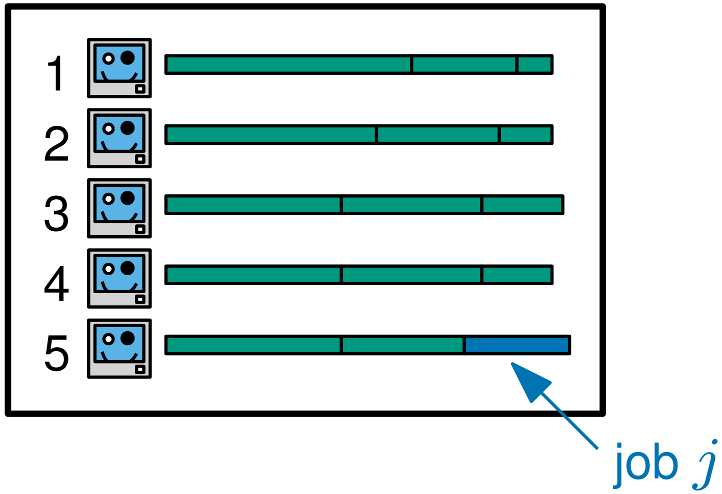

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m machines

n jobs

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  By the algorithm description, we have that $j \geqslant m + 1$

Job j takes t_j time units

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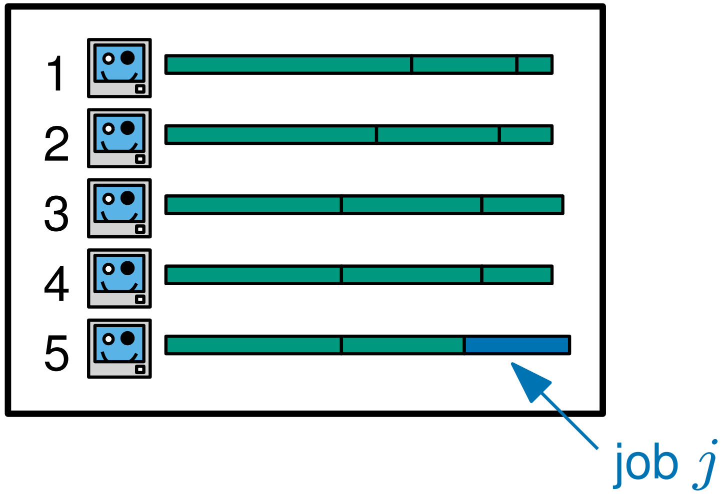

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Job j takes t_j time units

it doesn't assign a second job to any machine until every machine has at least one job

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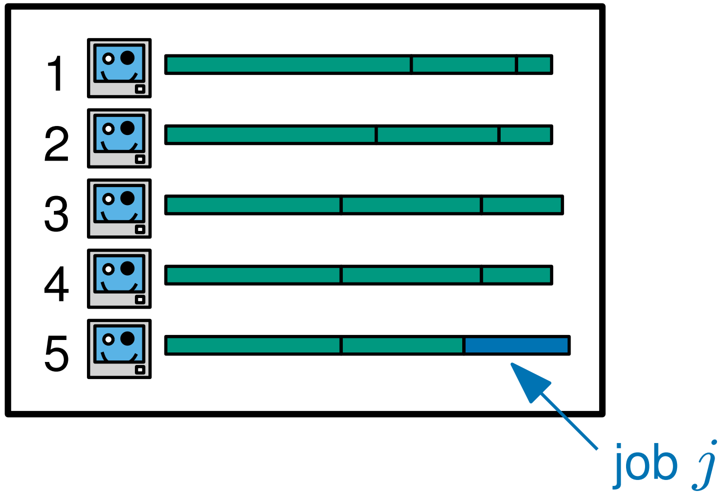

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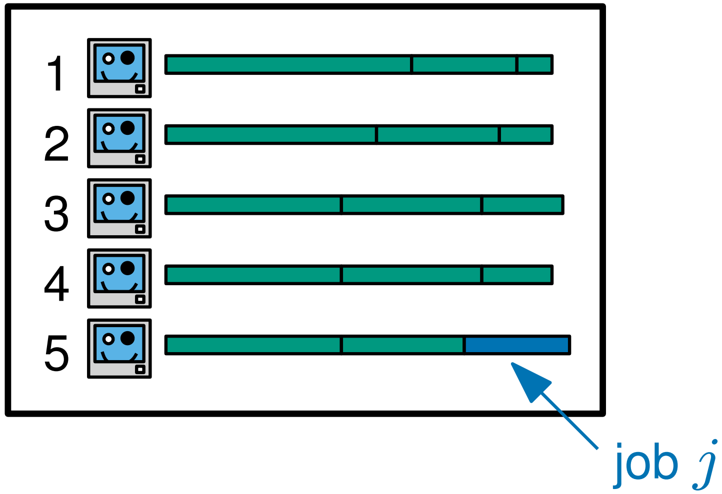

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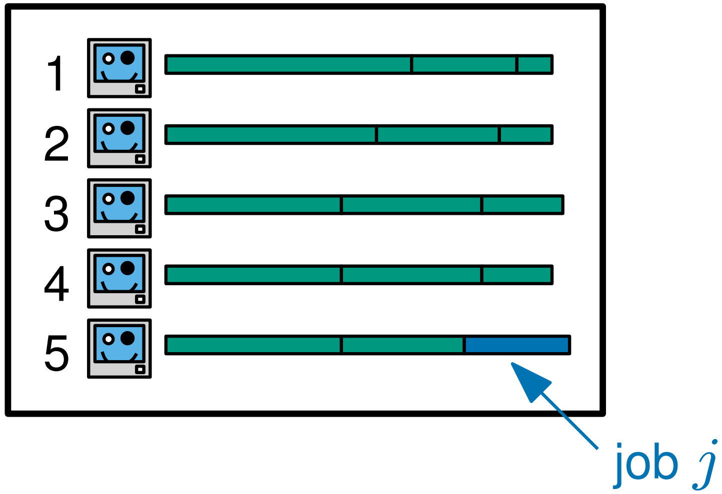

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Job j takes t_j time units

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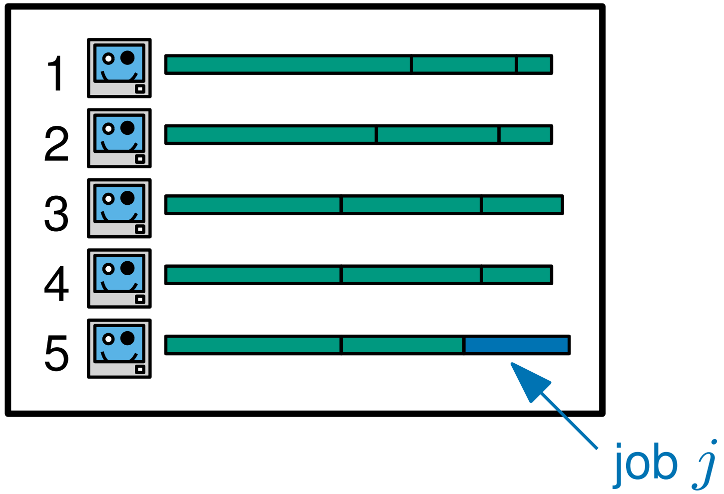

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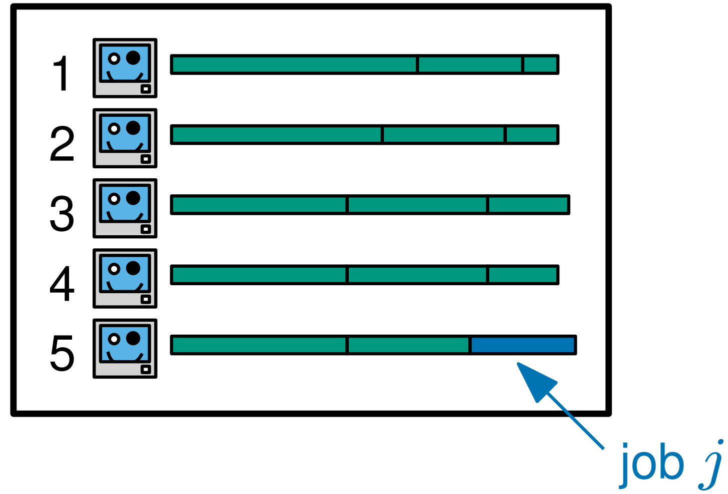

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Job j takes t_j time units

Therefore, $T_l = L_i = (L_i - t_j) + t_j \leqslant \mathrm{Opt} + \mathrm{Opt}/2 = (3/2) \cdot \mathrm{Opt}$

Scheduling conclusions

m machines

n jobs

Theorem The greedy algorithm is a 2-approximation algorithm

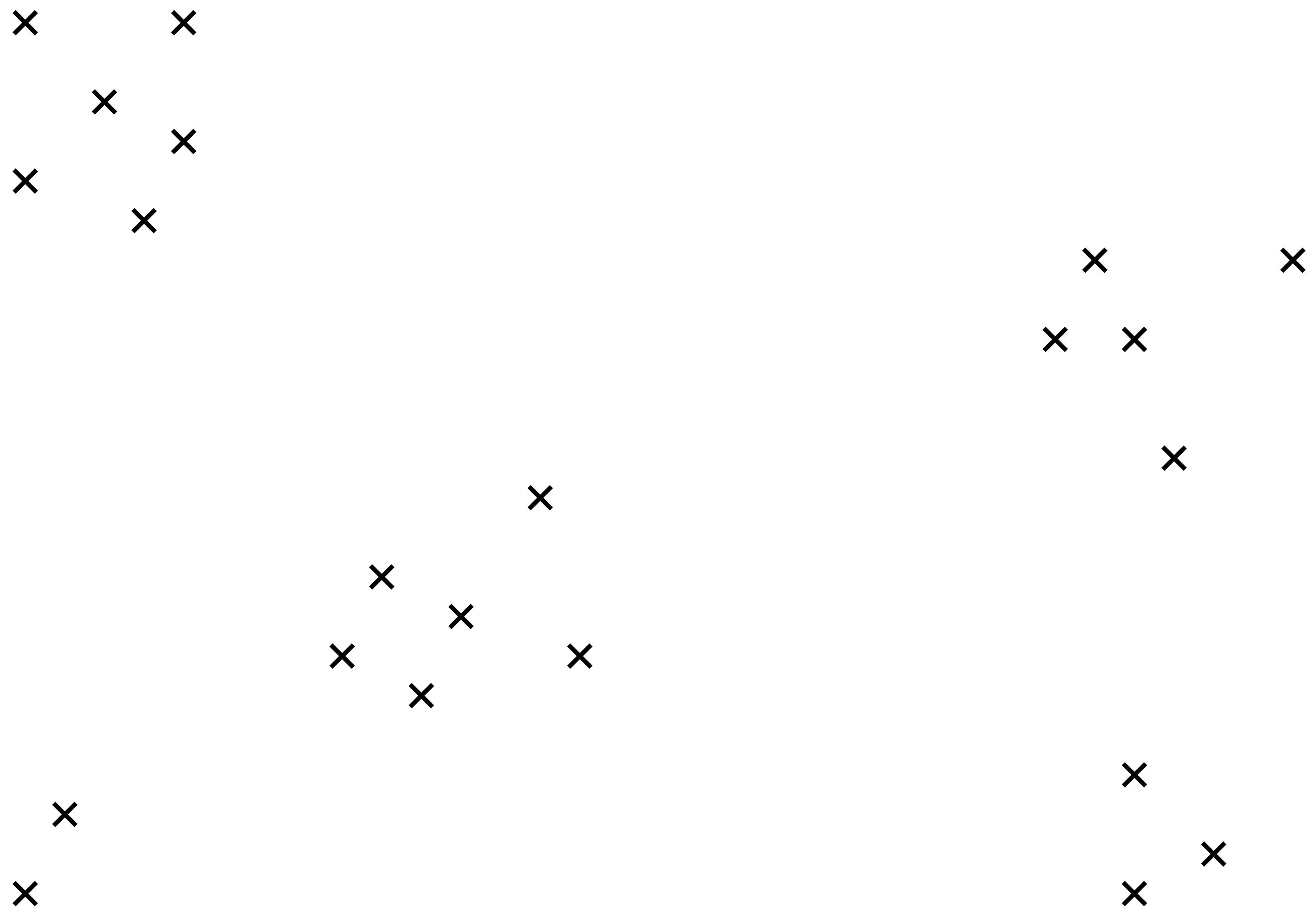
which runs in $O(n \log m)$ time and it's online

Theorem The LPT algorithm is a $3/2$-approximation algorithm

which runs in $O(n \log n)$ time

In fact, LPT is a $4/3$-approximation algorithm (using better analysis)

k-centers



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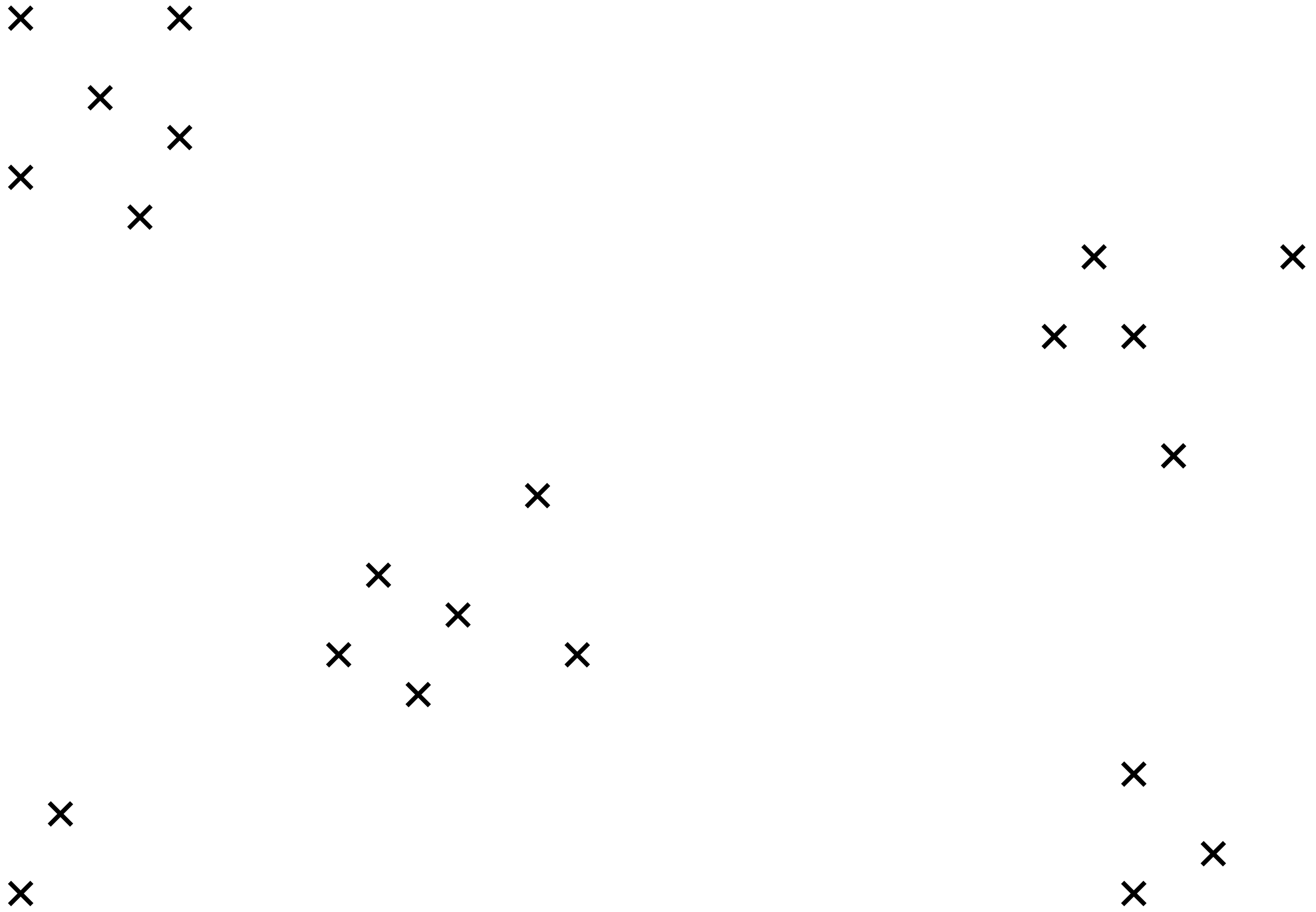
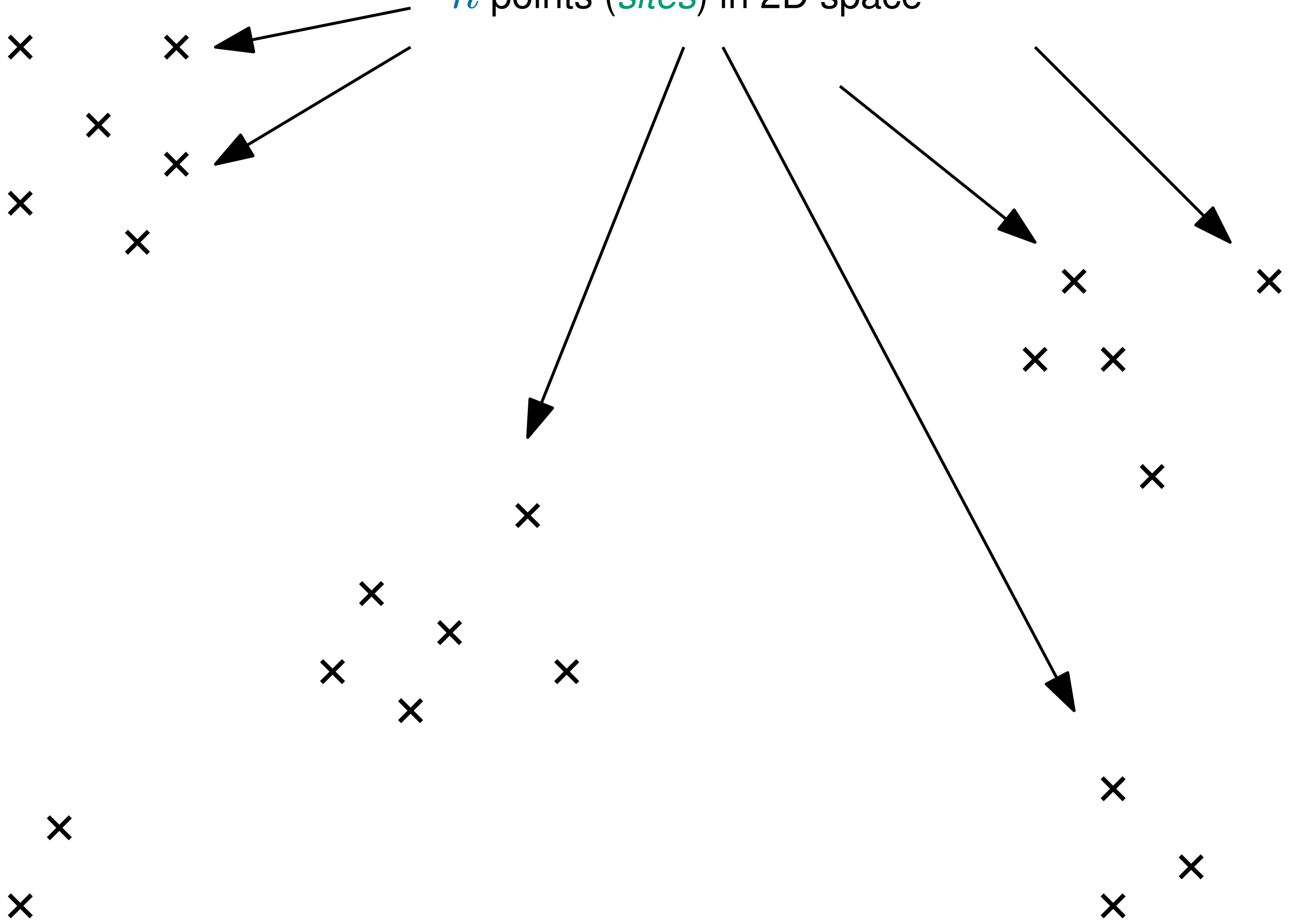


k-centers

n points (*sites*) in 2D space



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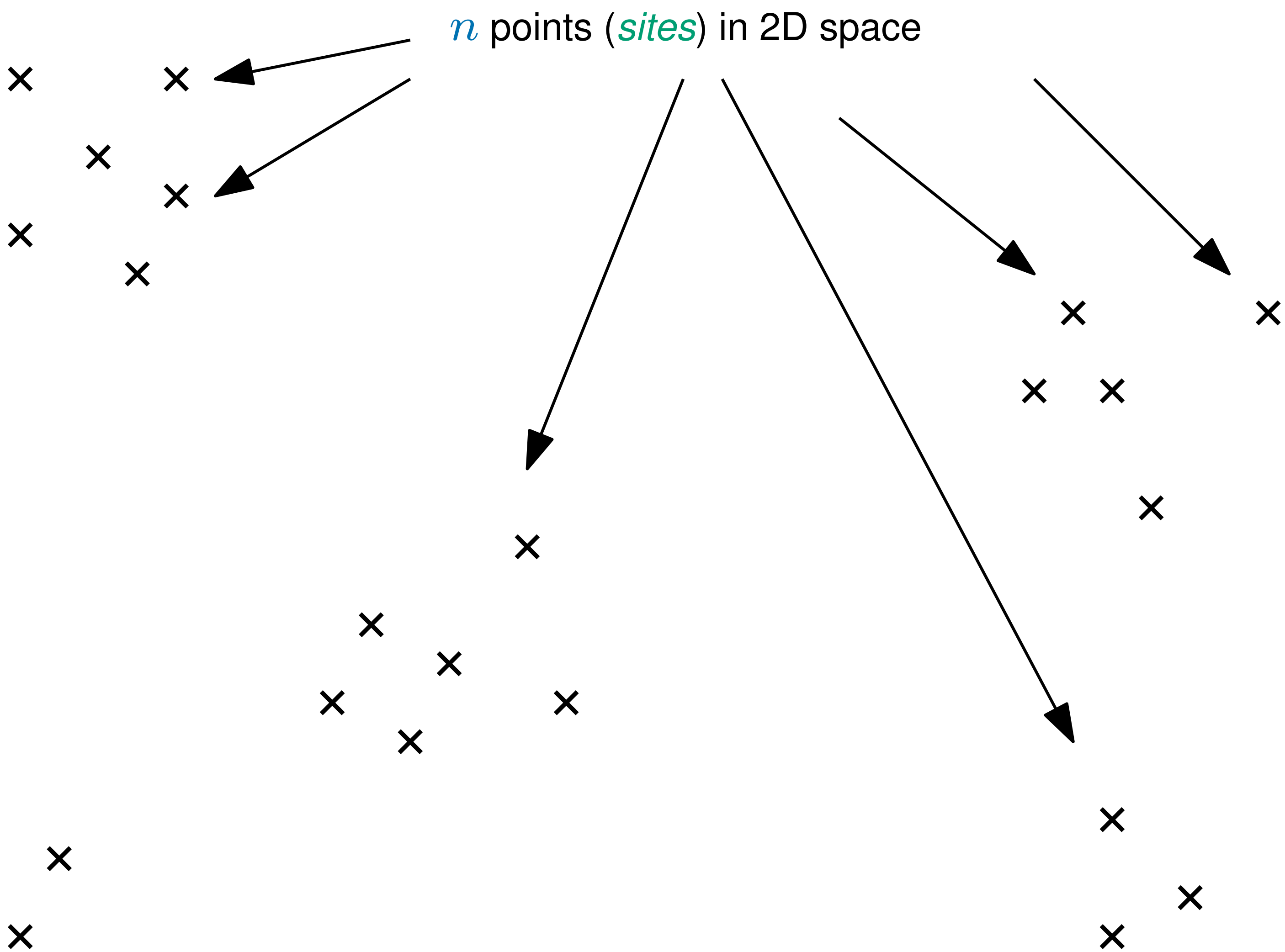
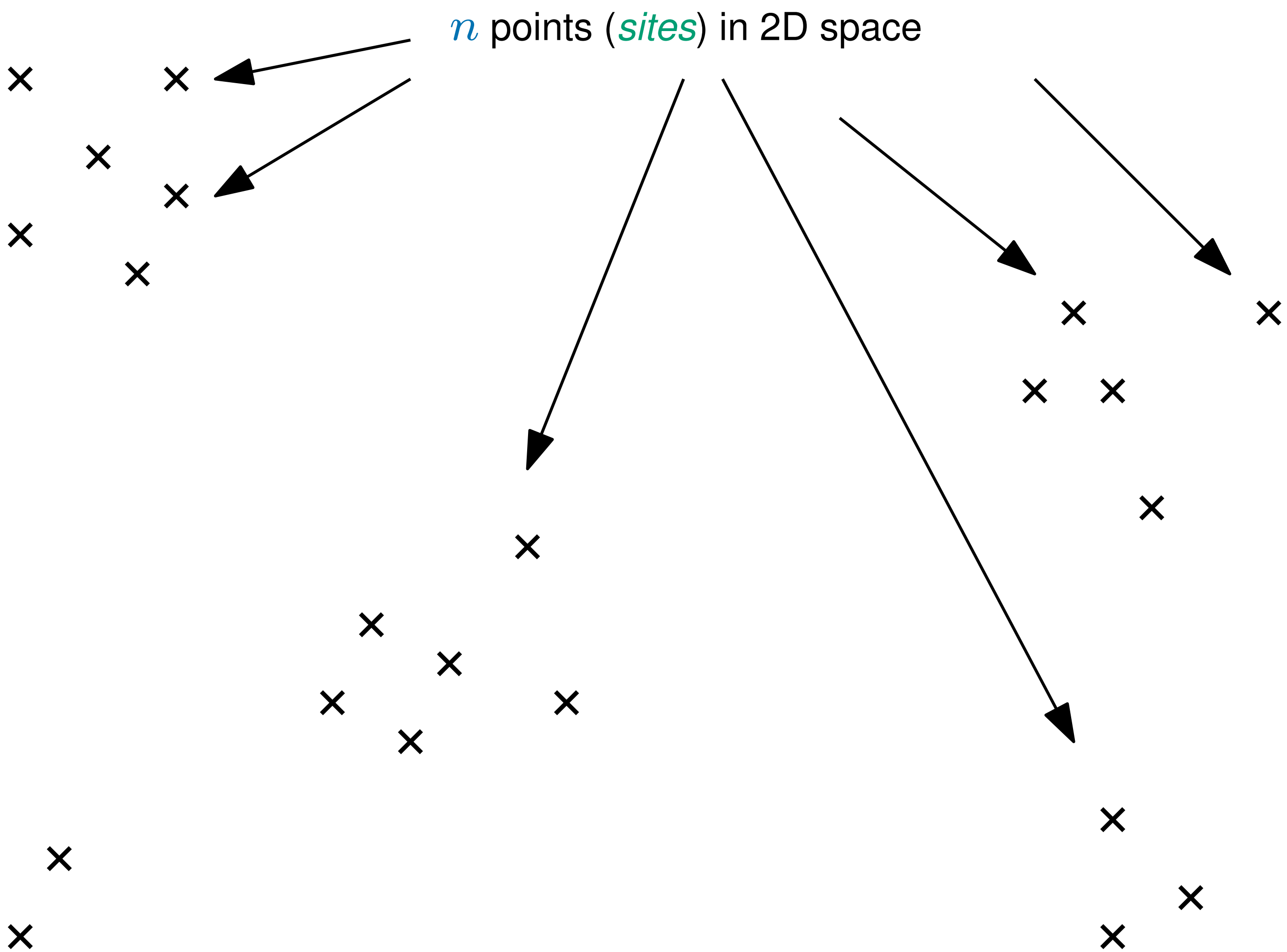


The distance between points s_i, s_j is $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

k-centers



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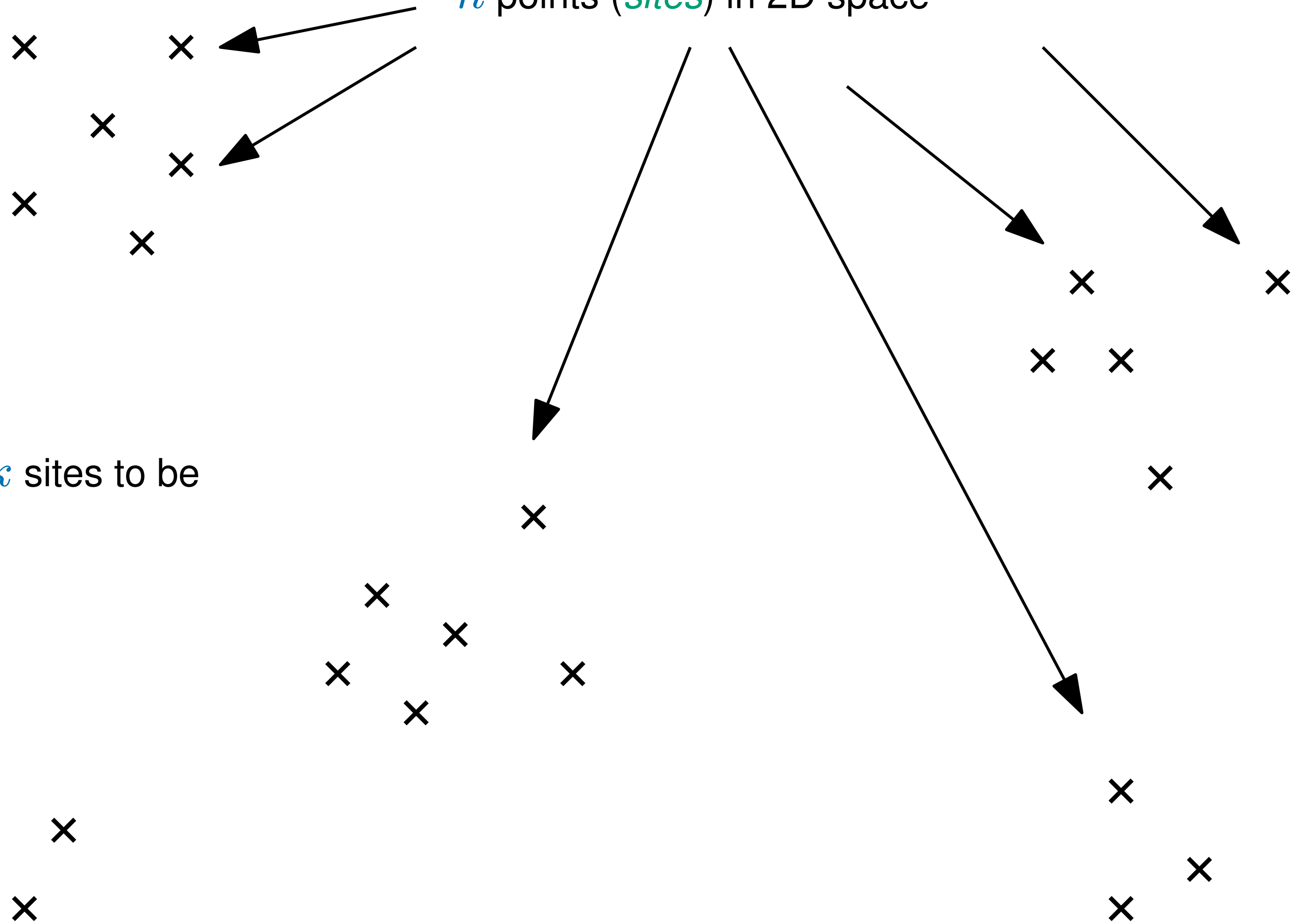
(i.e. ‘normal’ euclidean distance)

k-centers

n points (*sites*) in 2D space



Select k sites to be centers

The distance between points s_i, s_j is $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

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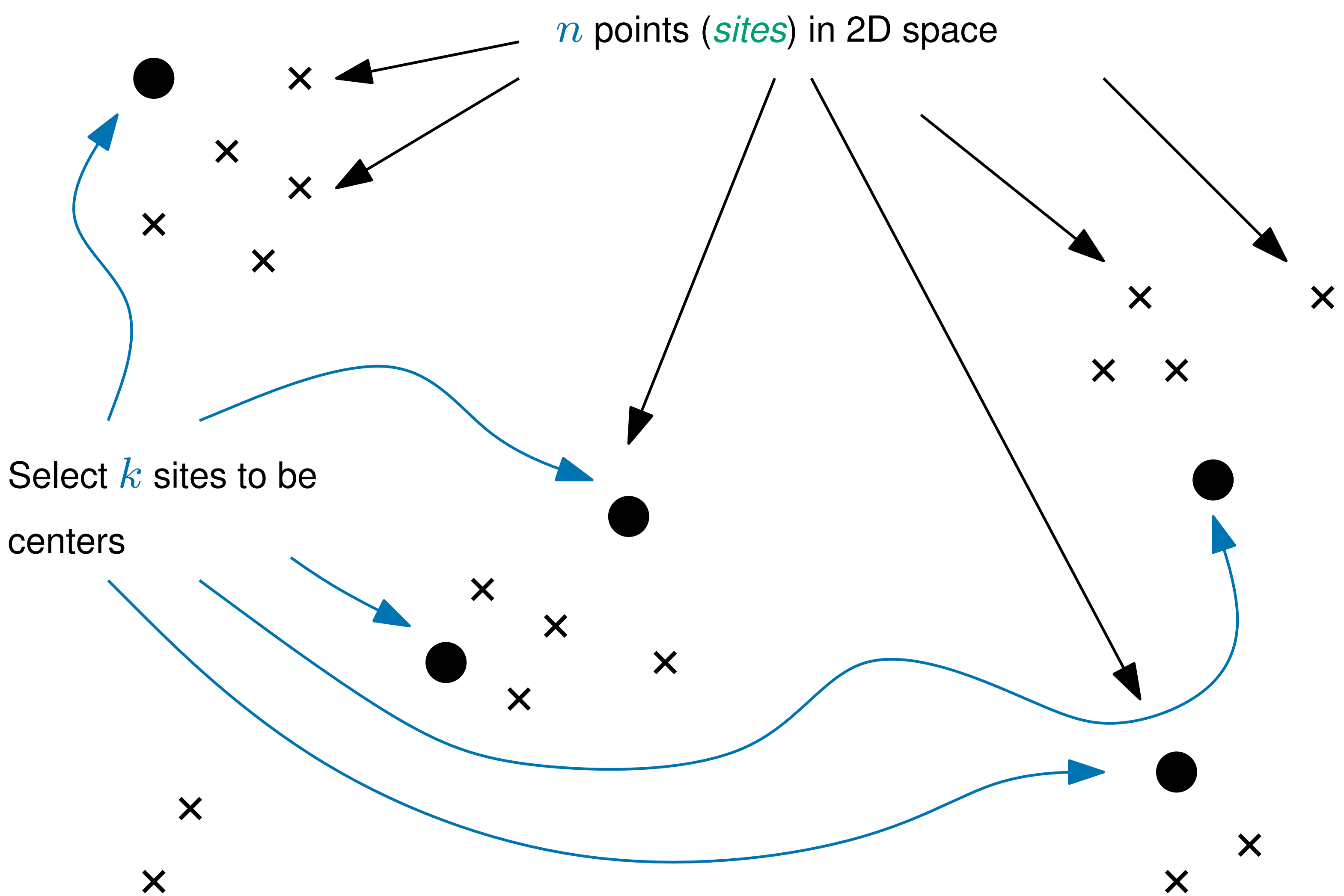

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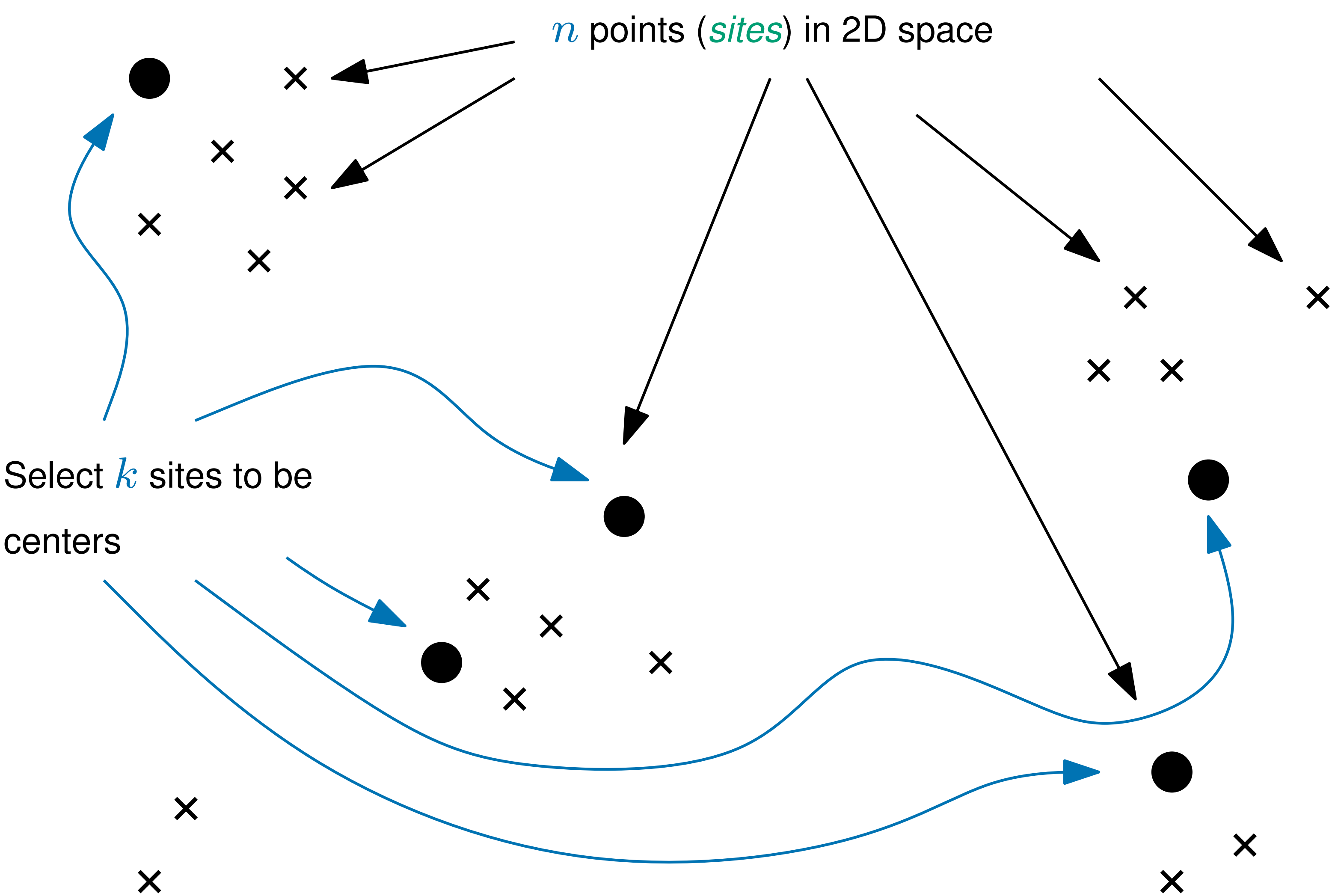

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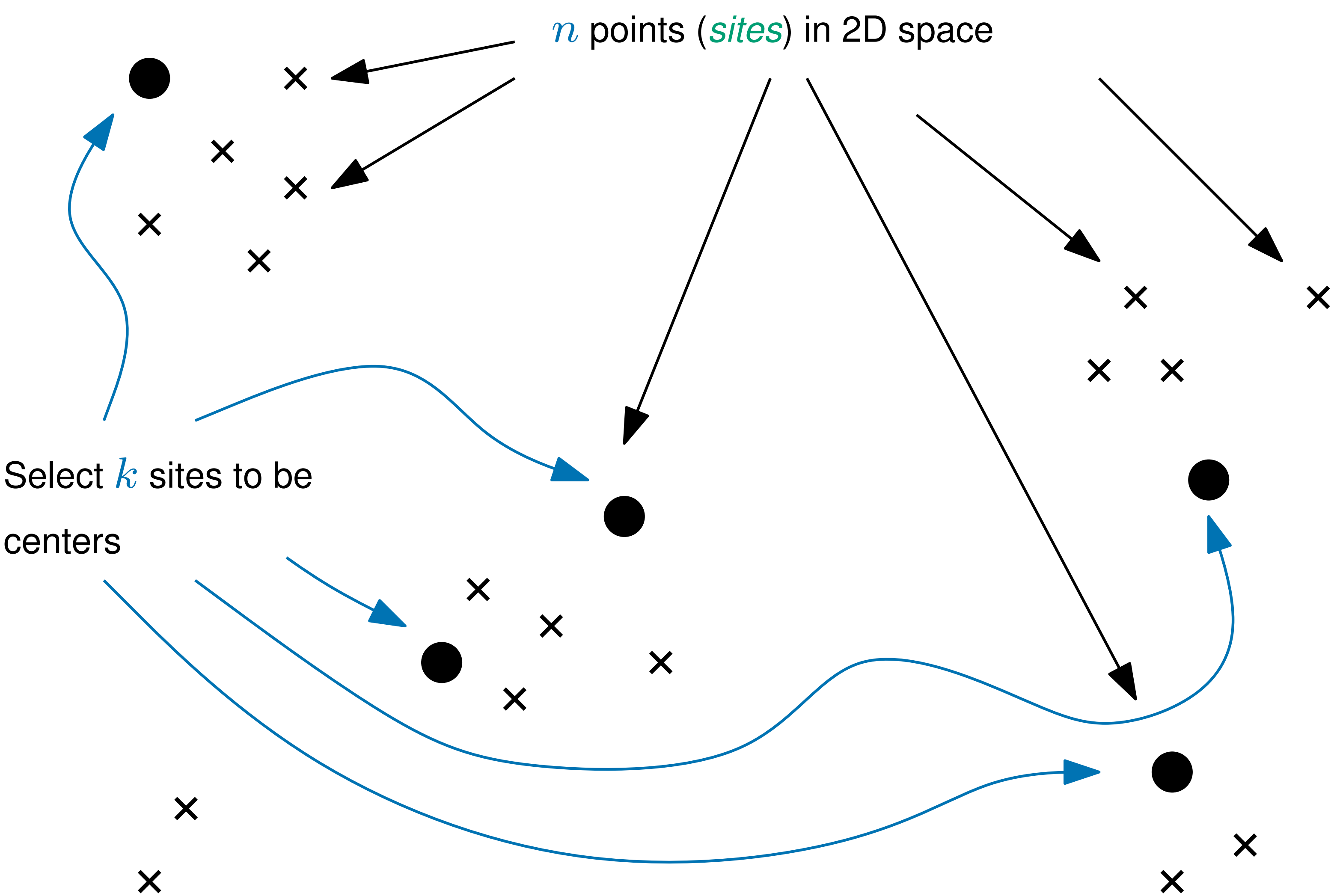

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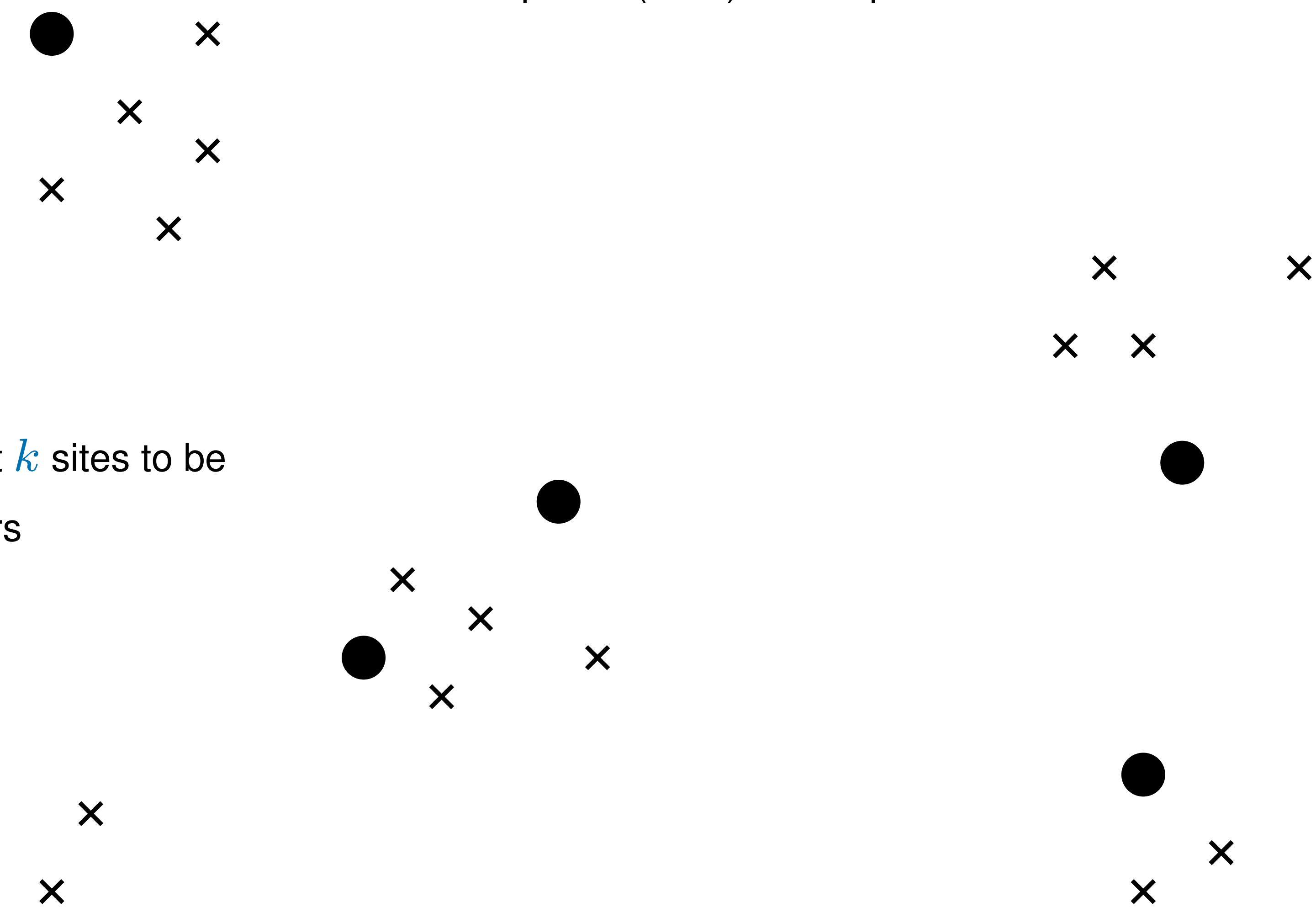
Goal Minimise the largest distance from any site to the closest center

k-centers

n points (*sites*) in 2D space



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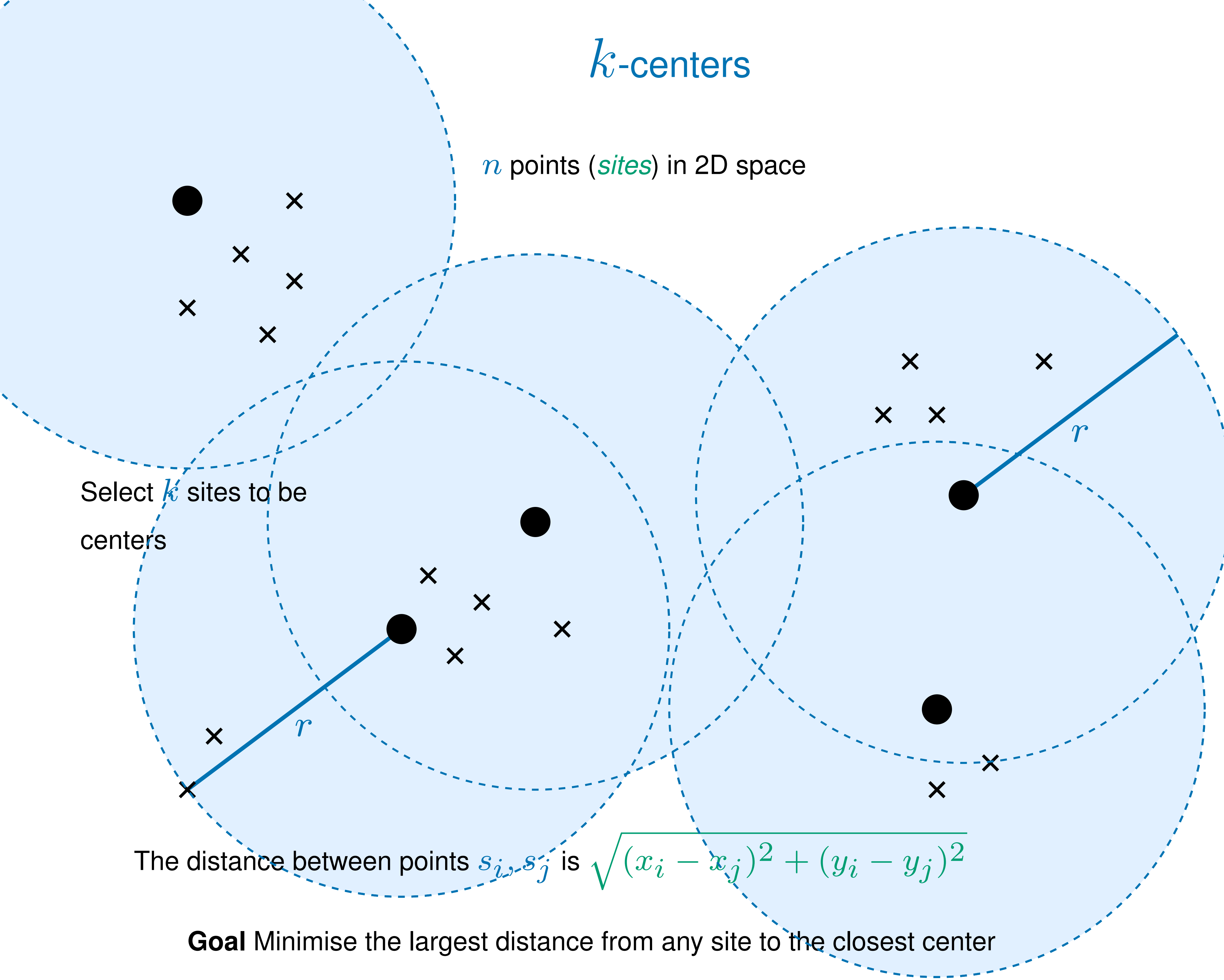

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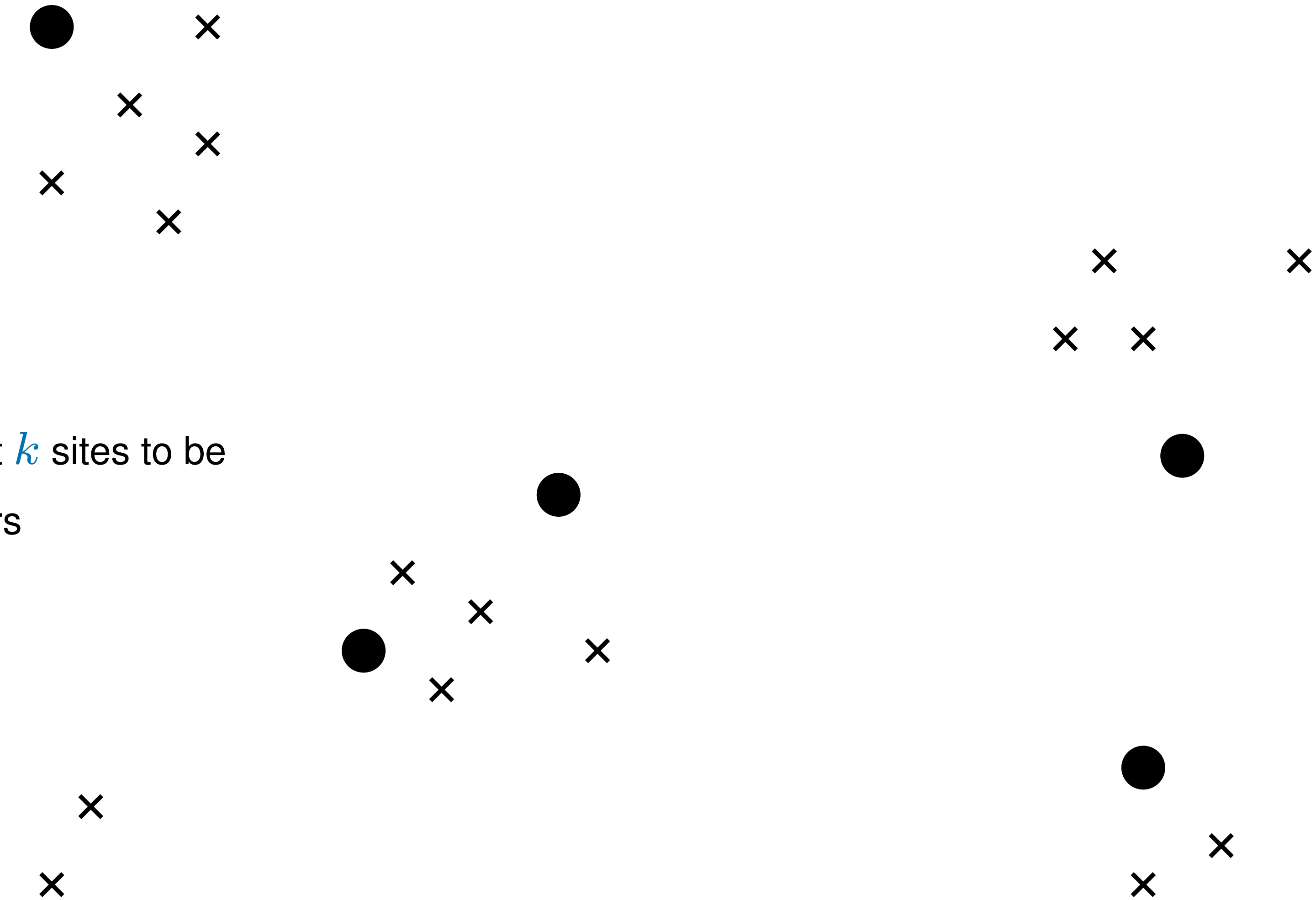
n points (*sites*) in 2D space



Select k sites to be centers

The distance between points s_i, s_j is $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

Goal Minimise the largest distance from any site to the closest center

k-centers

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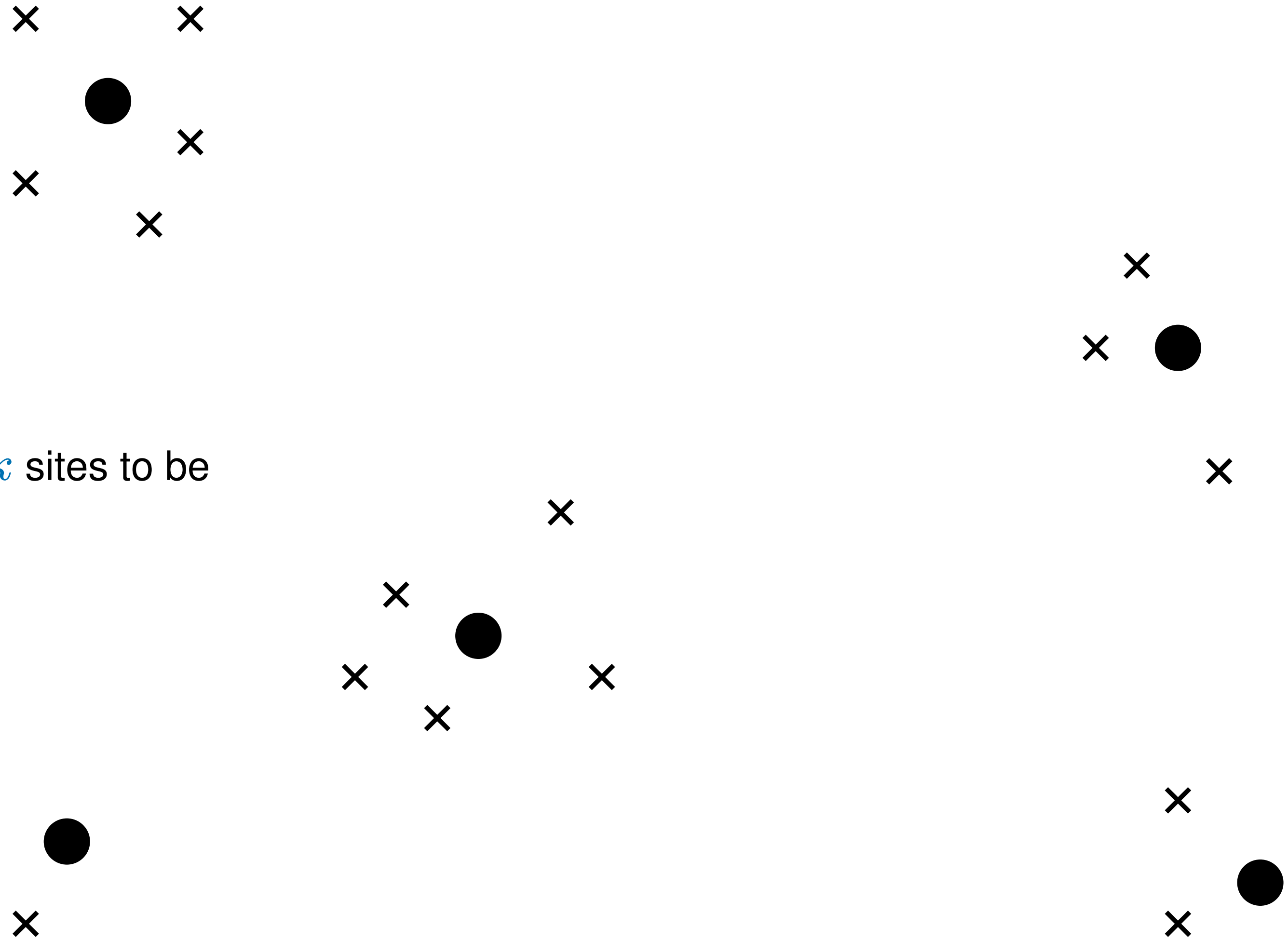

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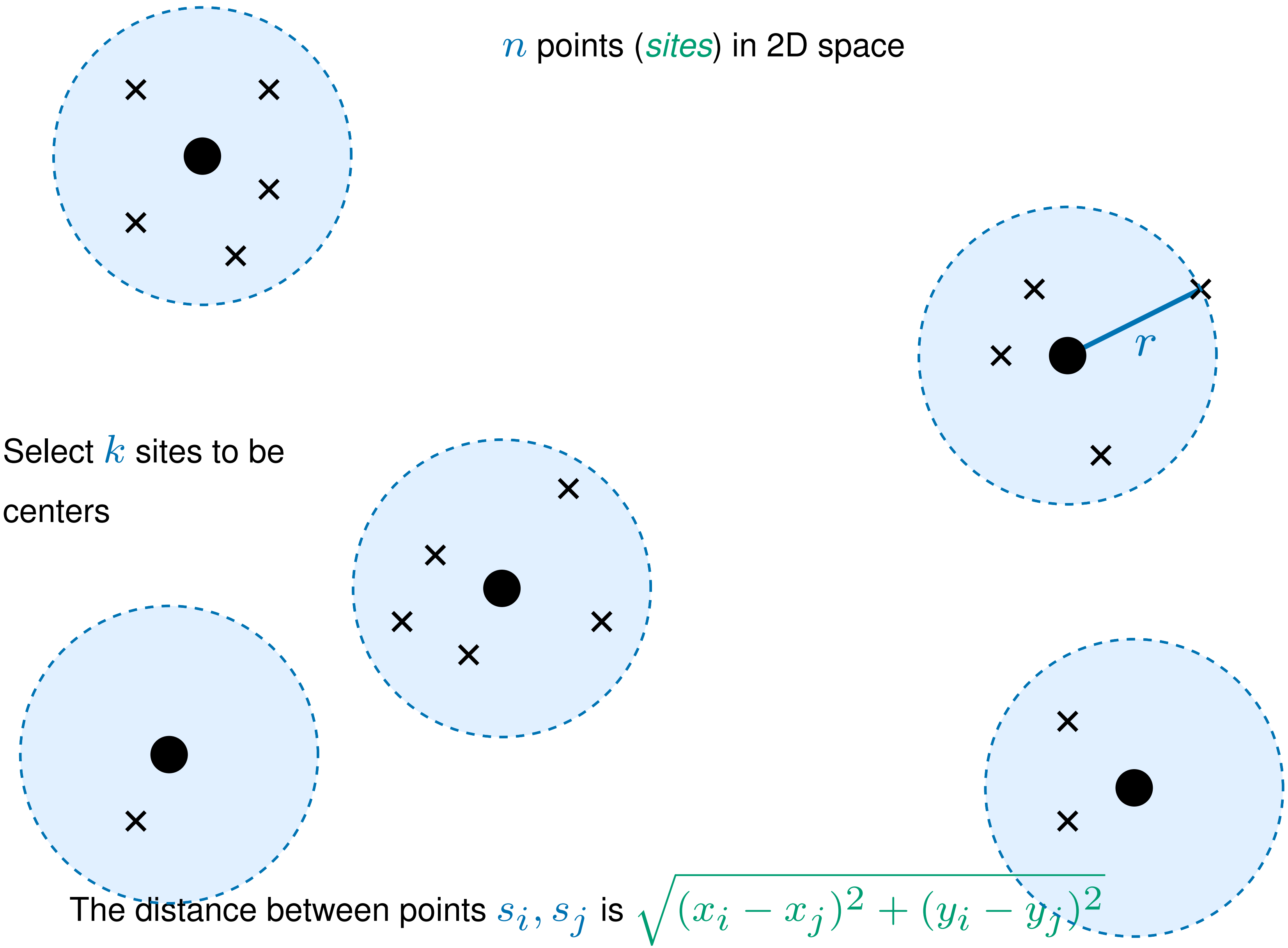

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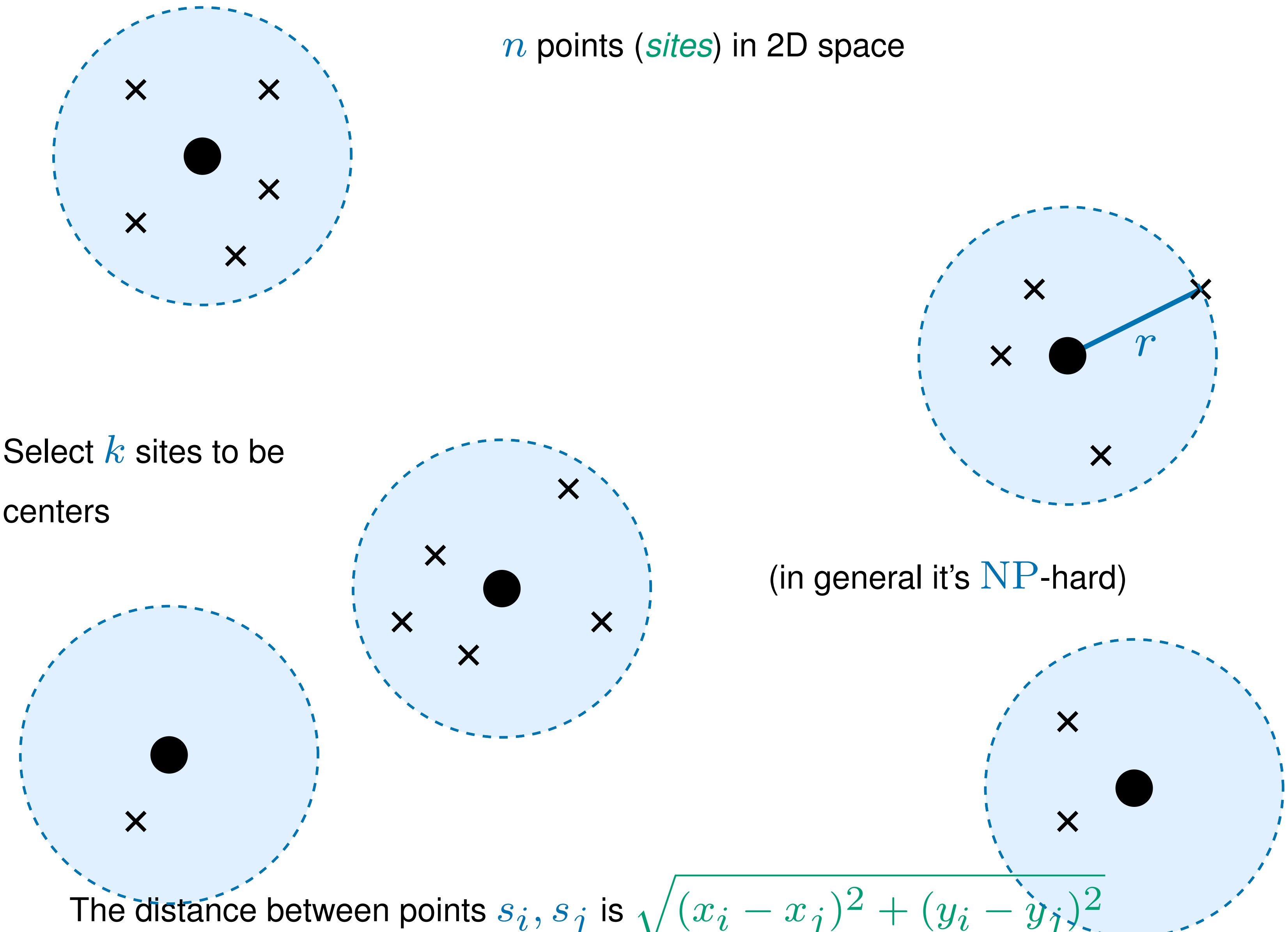

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(in general it's NP-hard)

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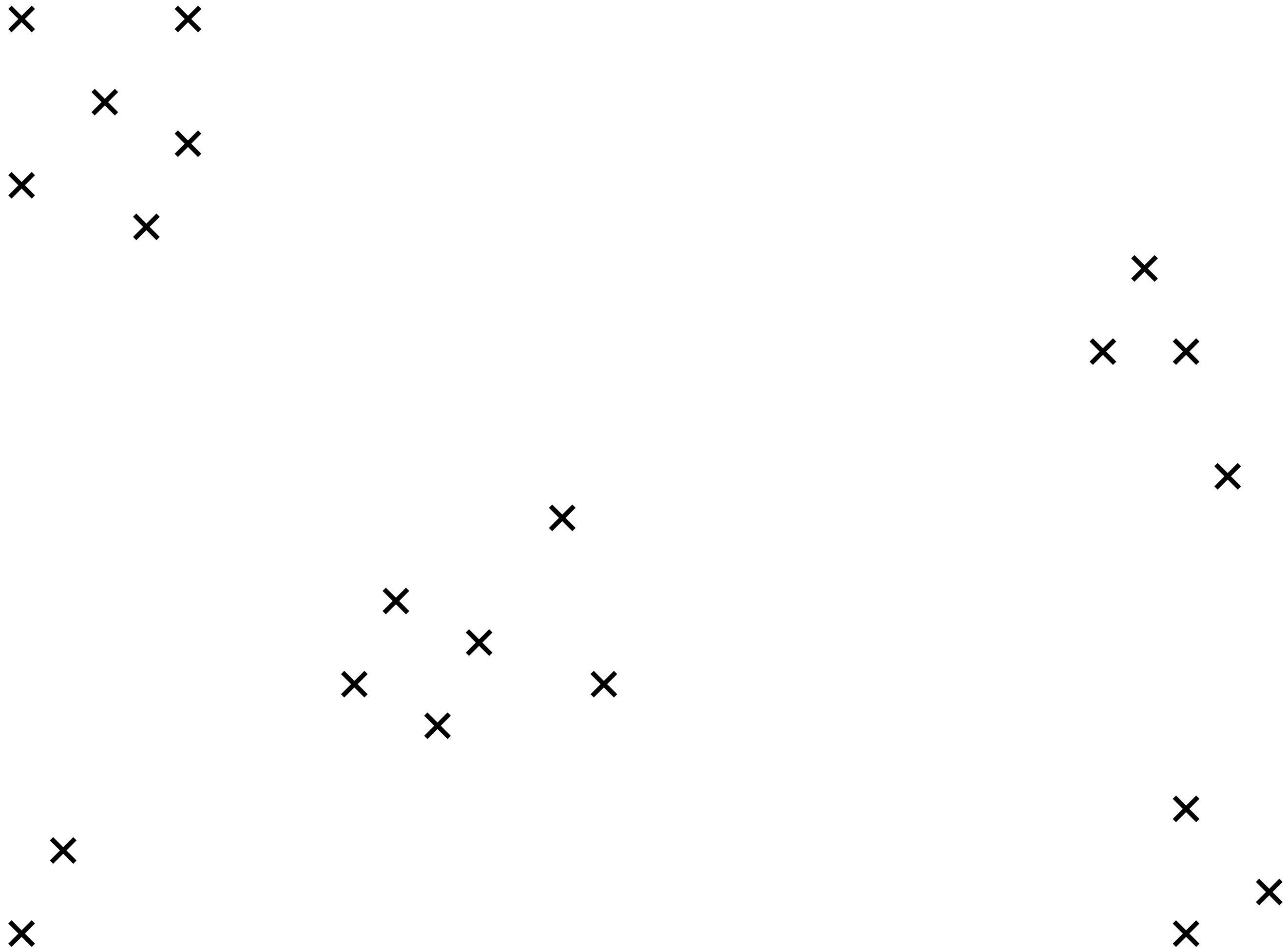
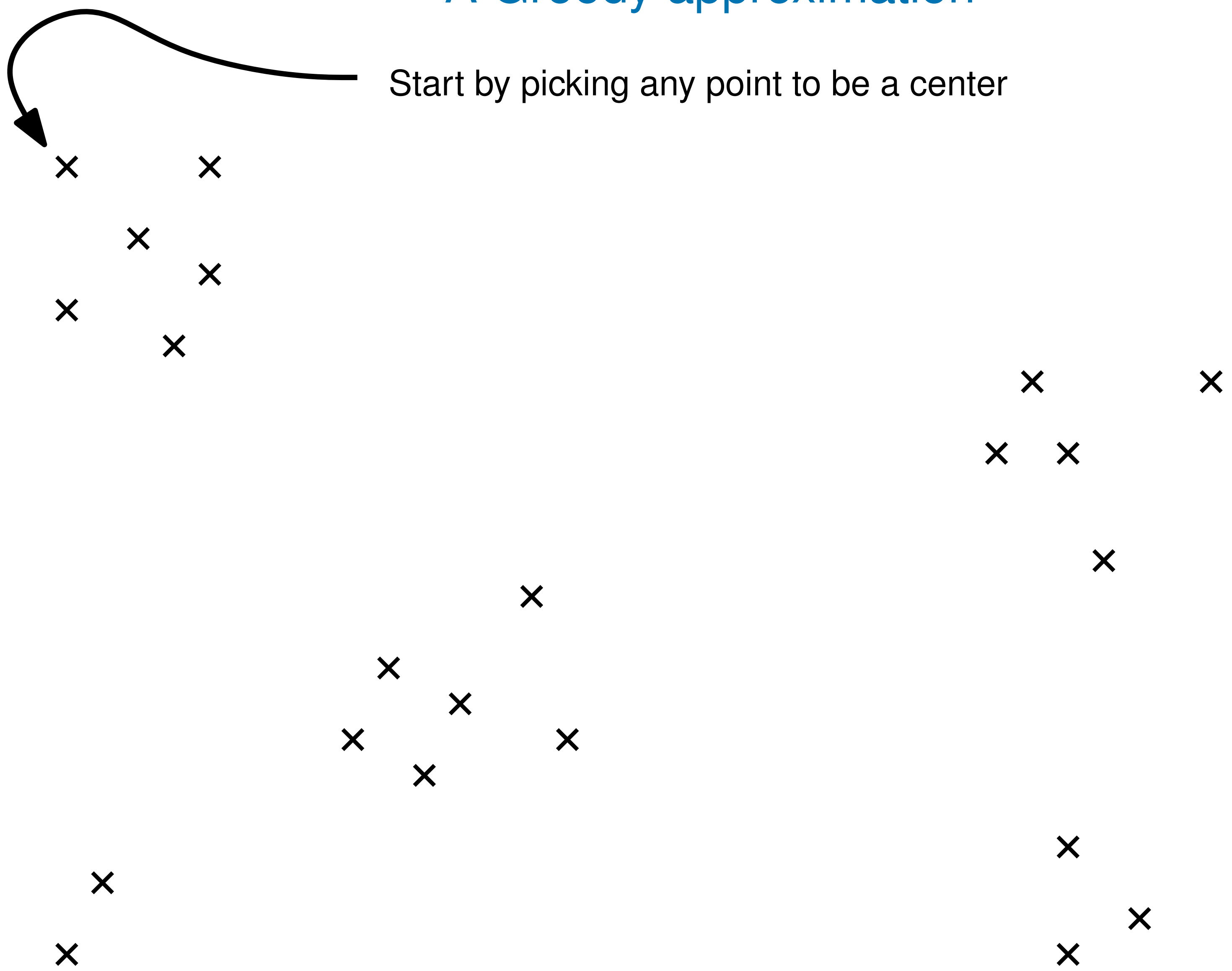
A Greedy approximation



A Greedy approximation

Start by picking any point to be a center



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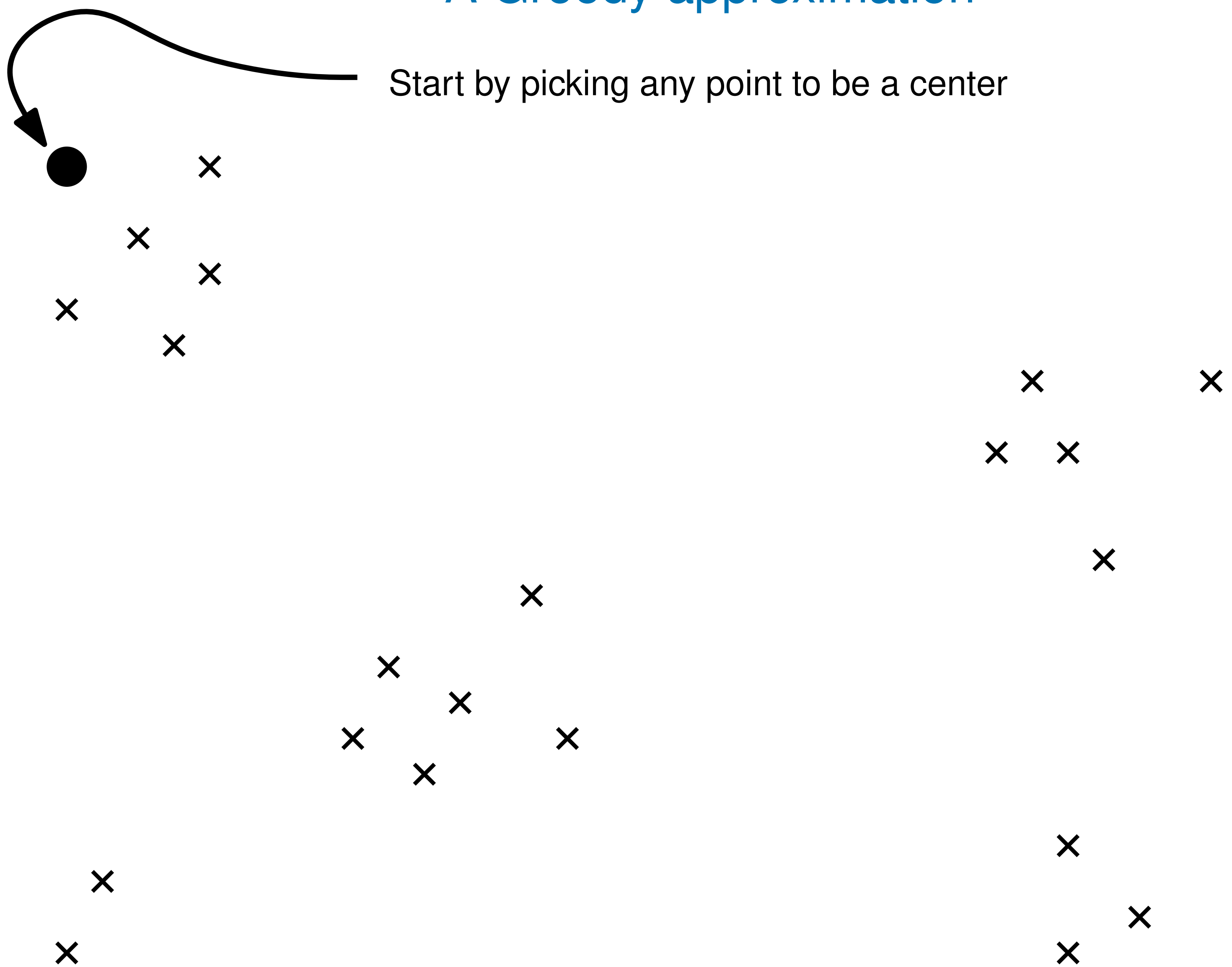

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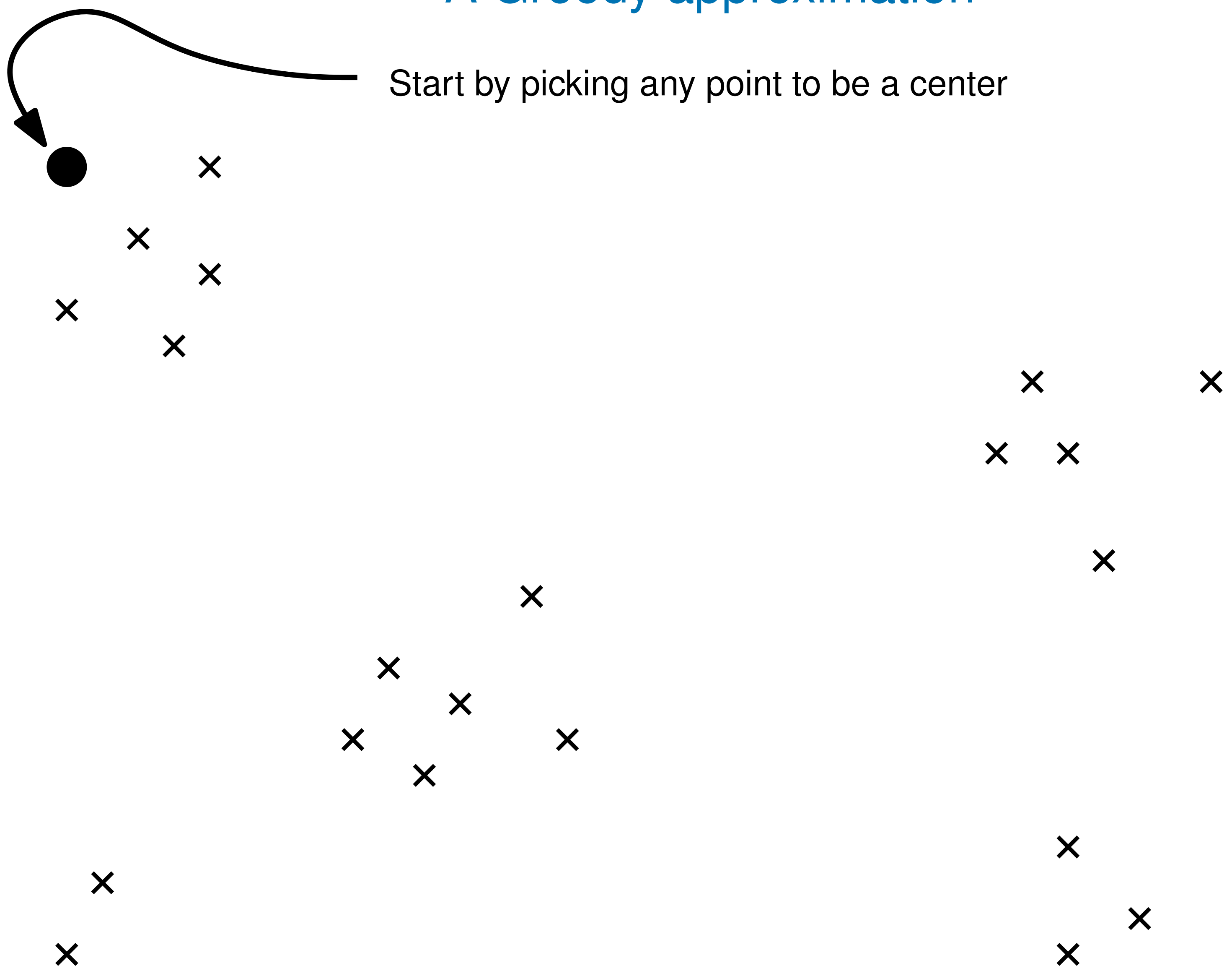

Repeatedly pick the site which is furthest from any existing center

A Greedy approximation

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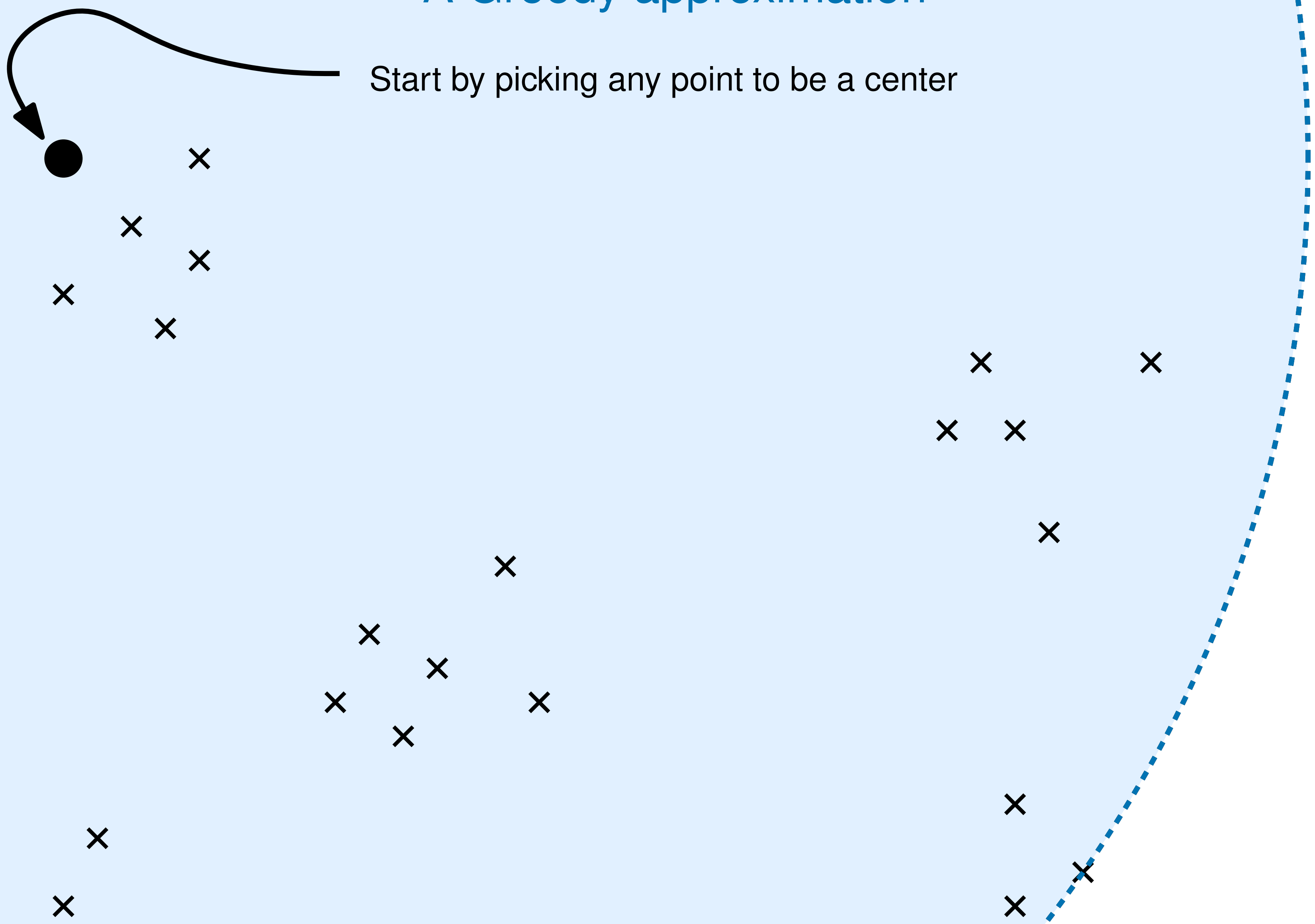

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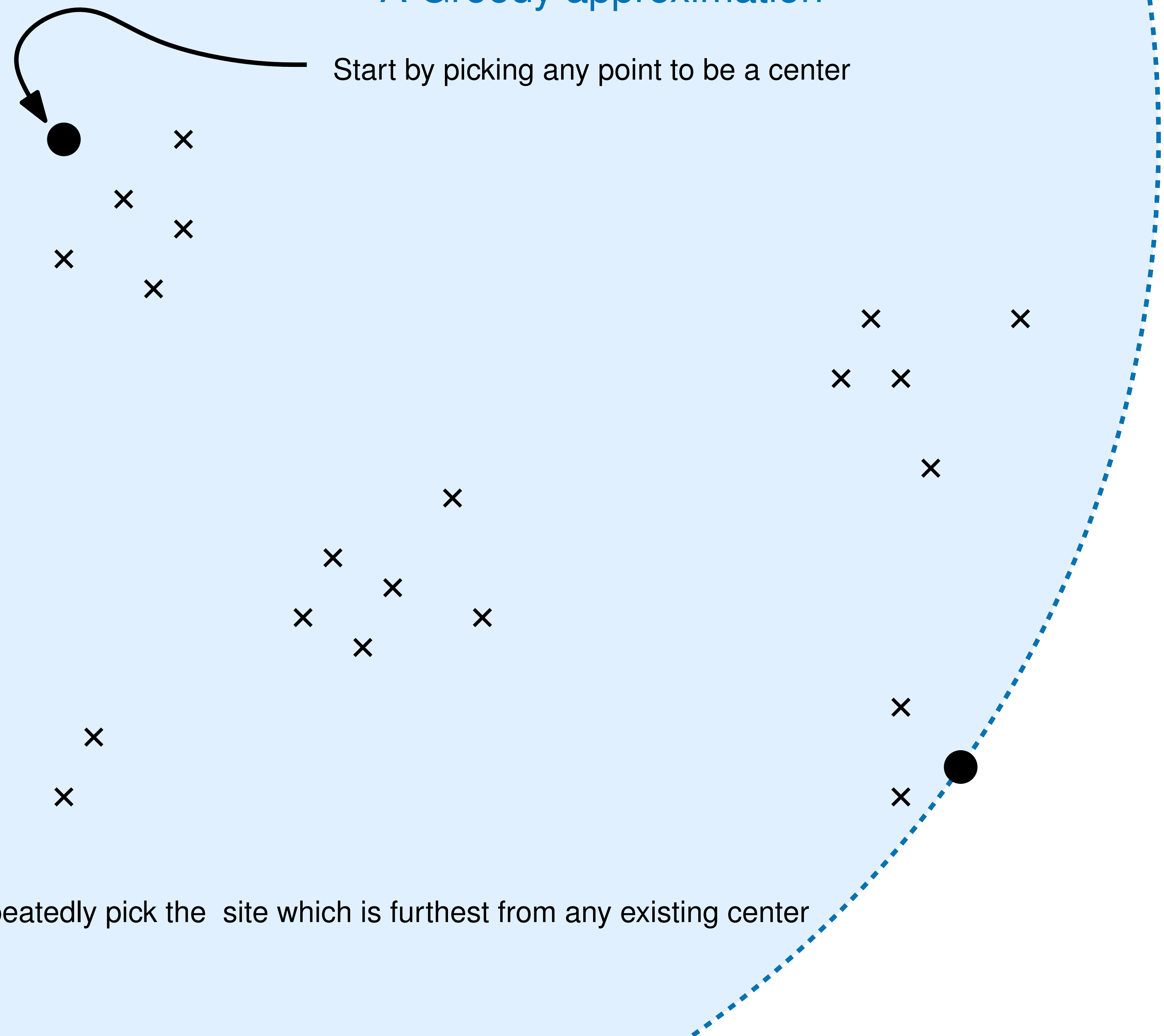

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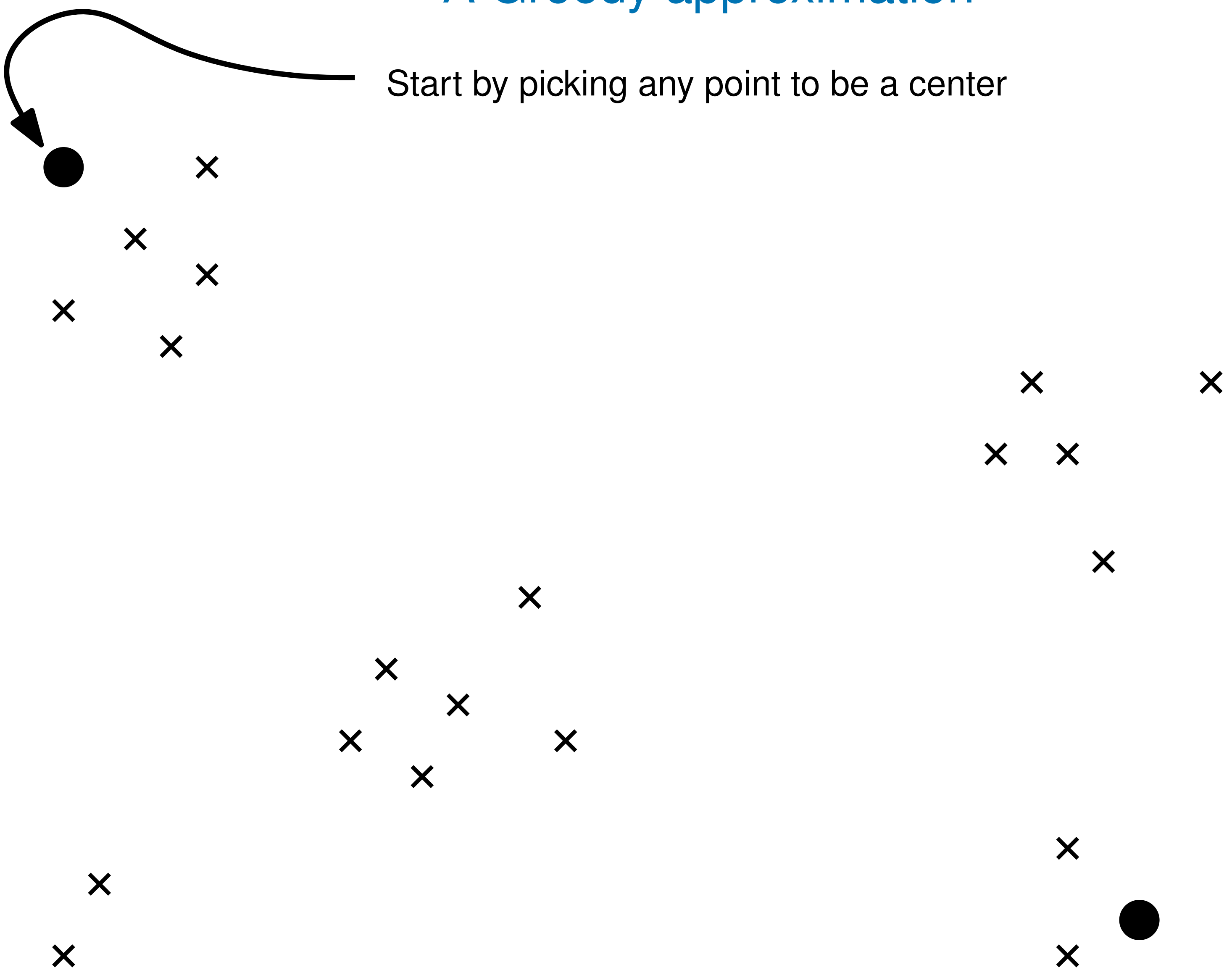

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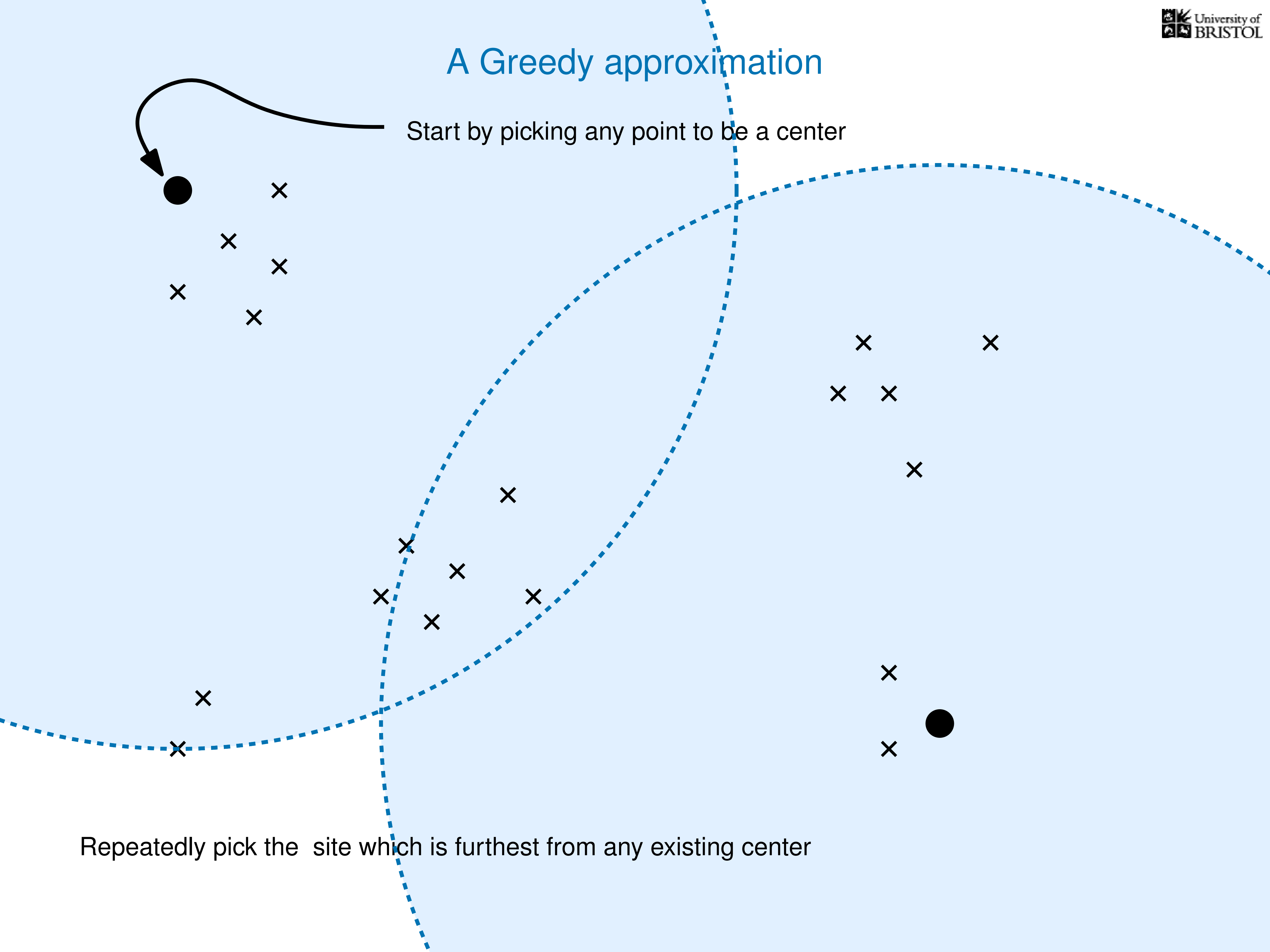

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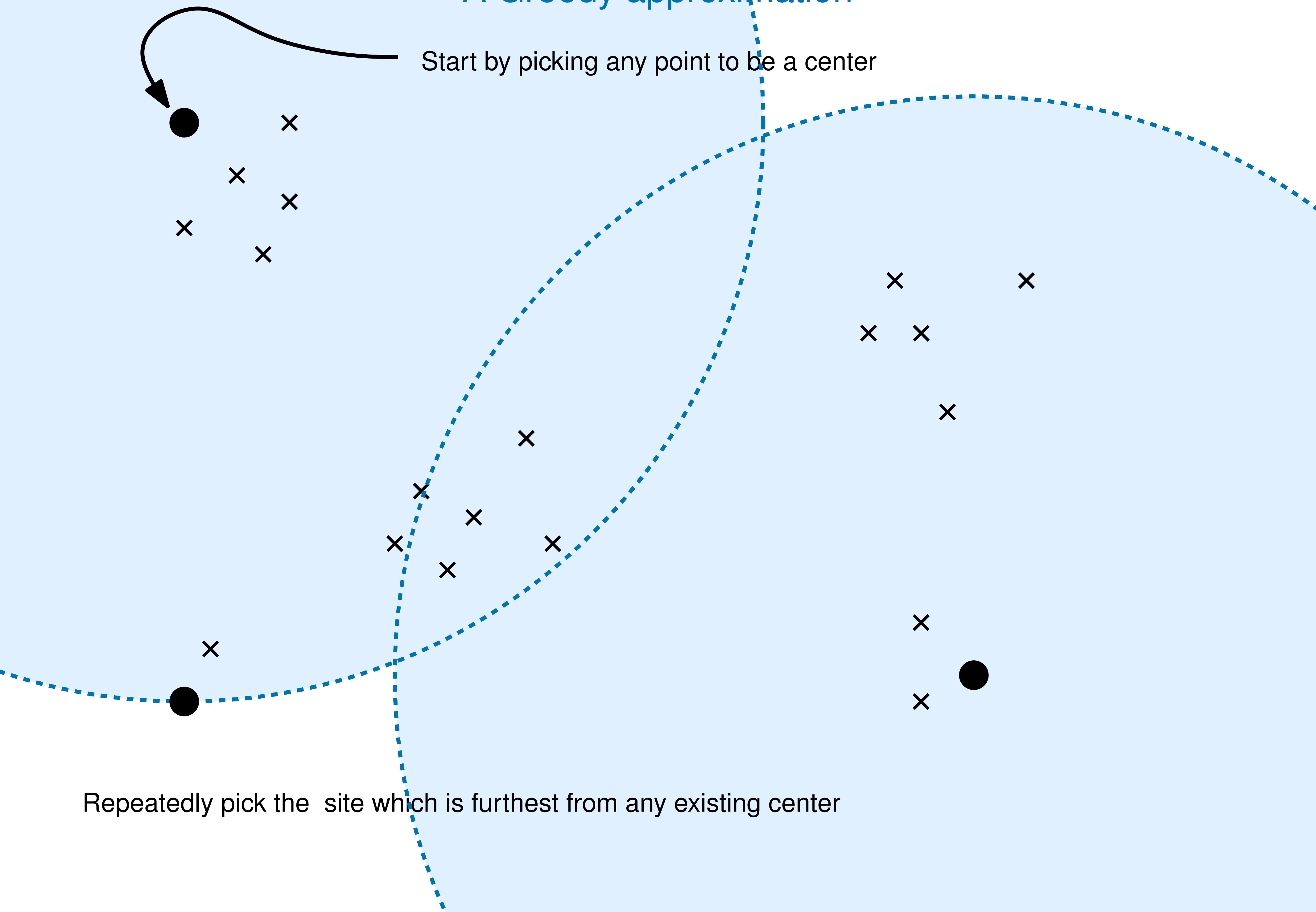

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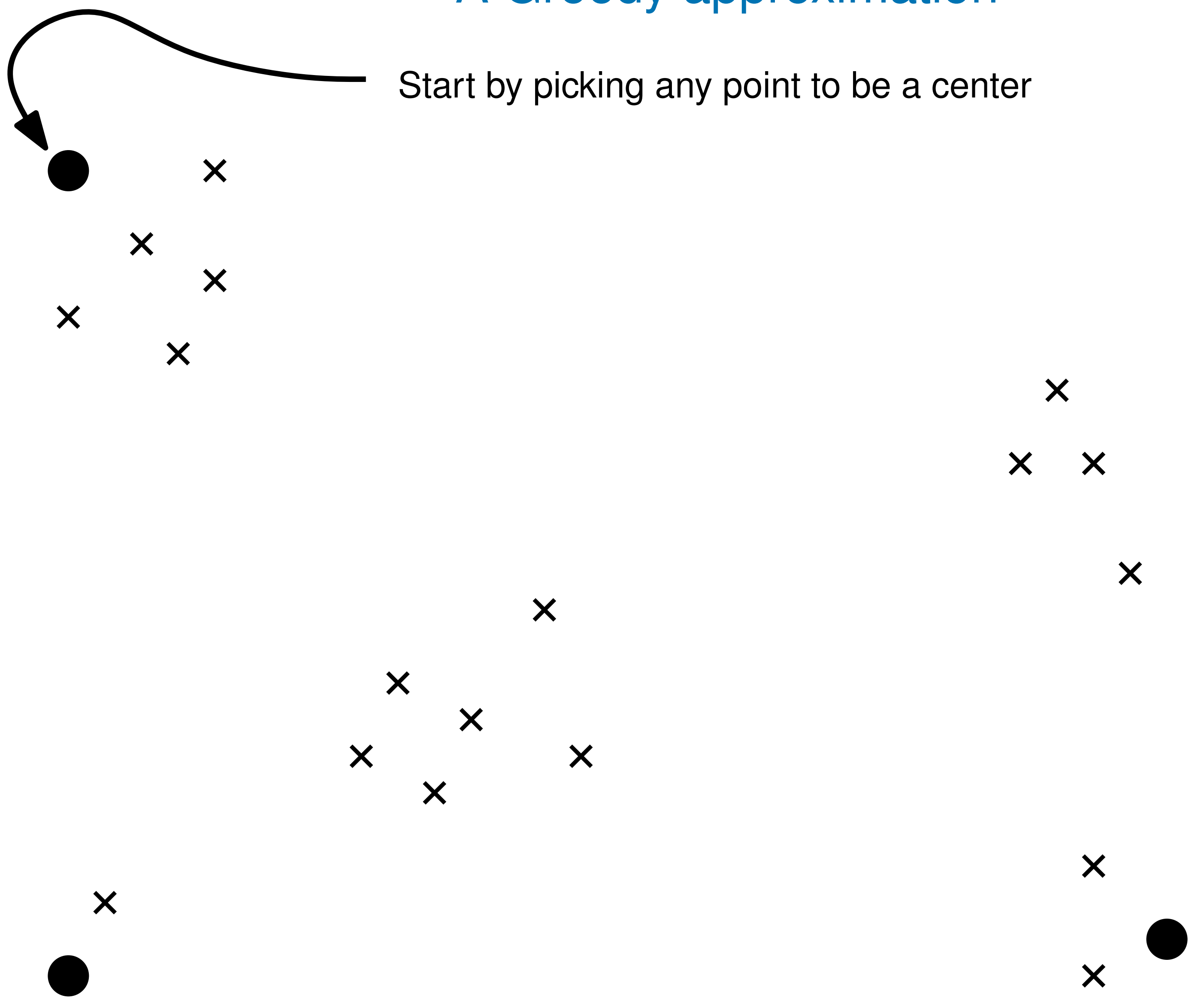

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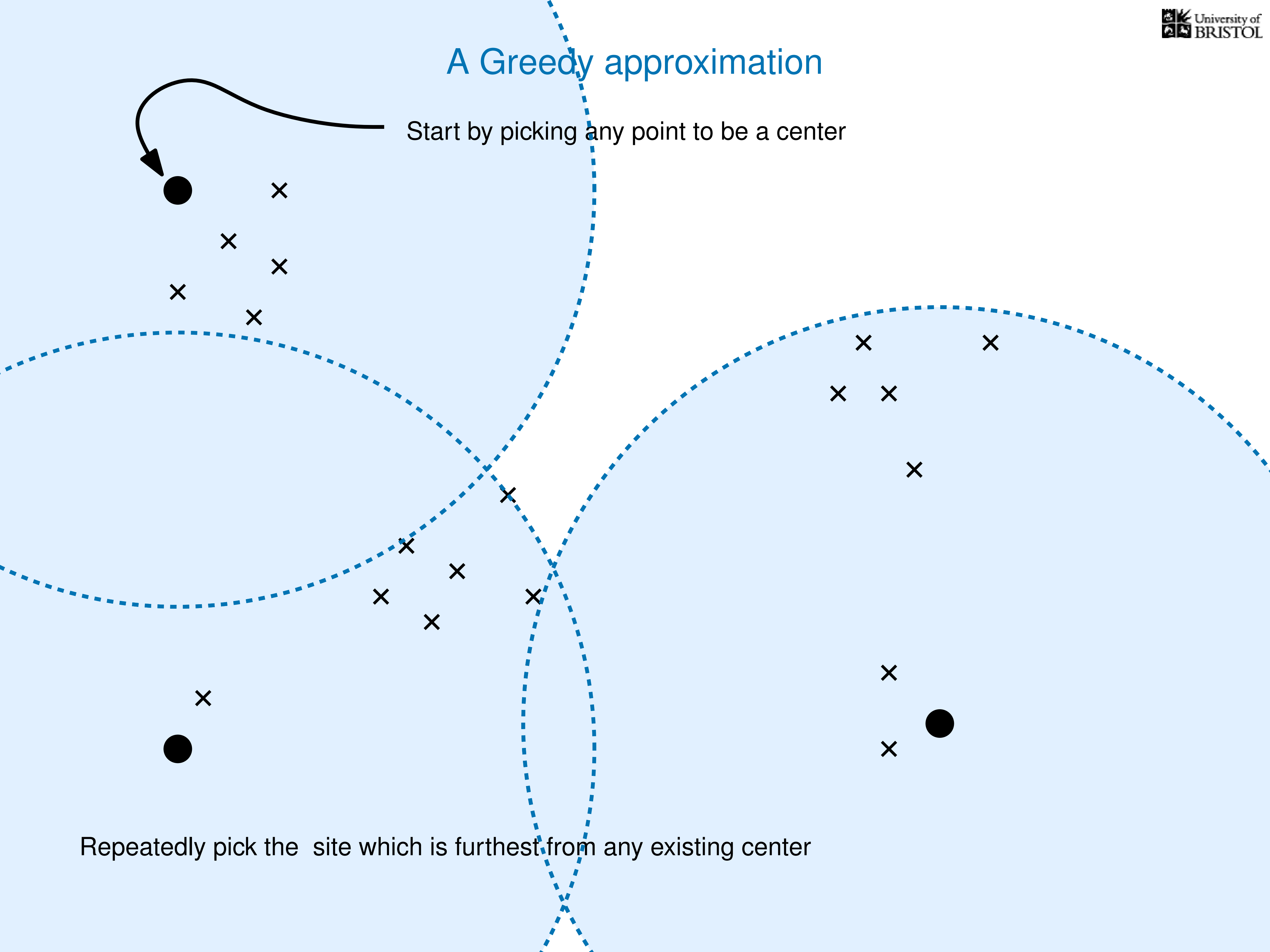

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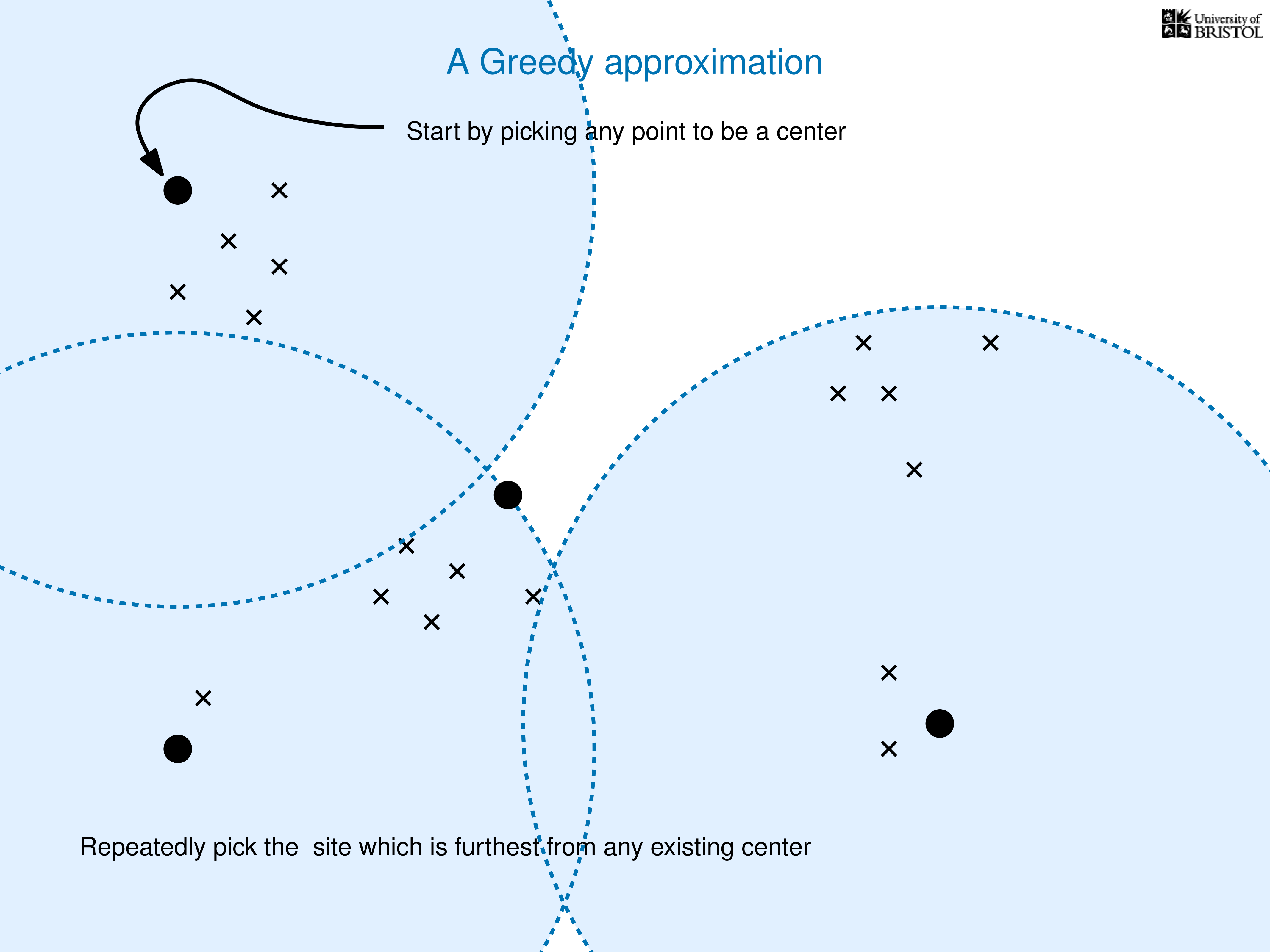

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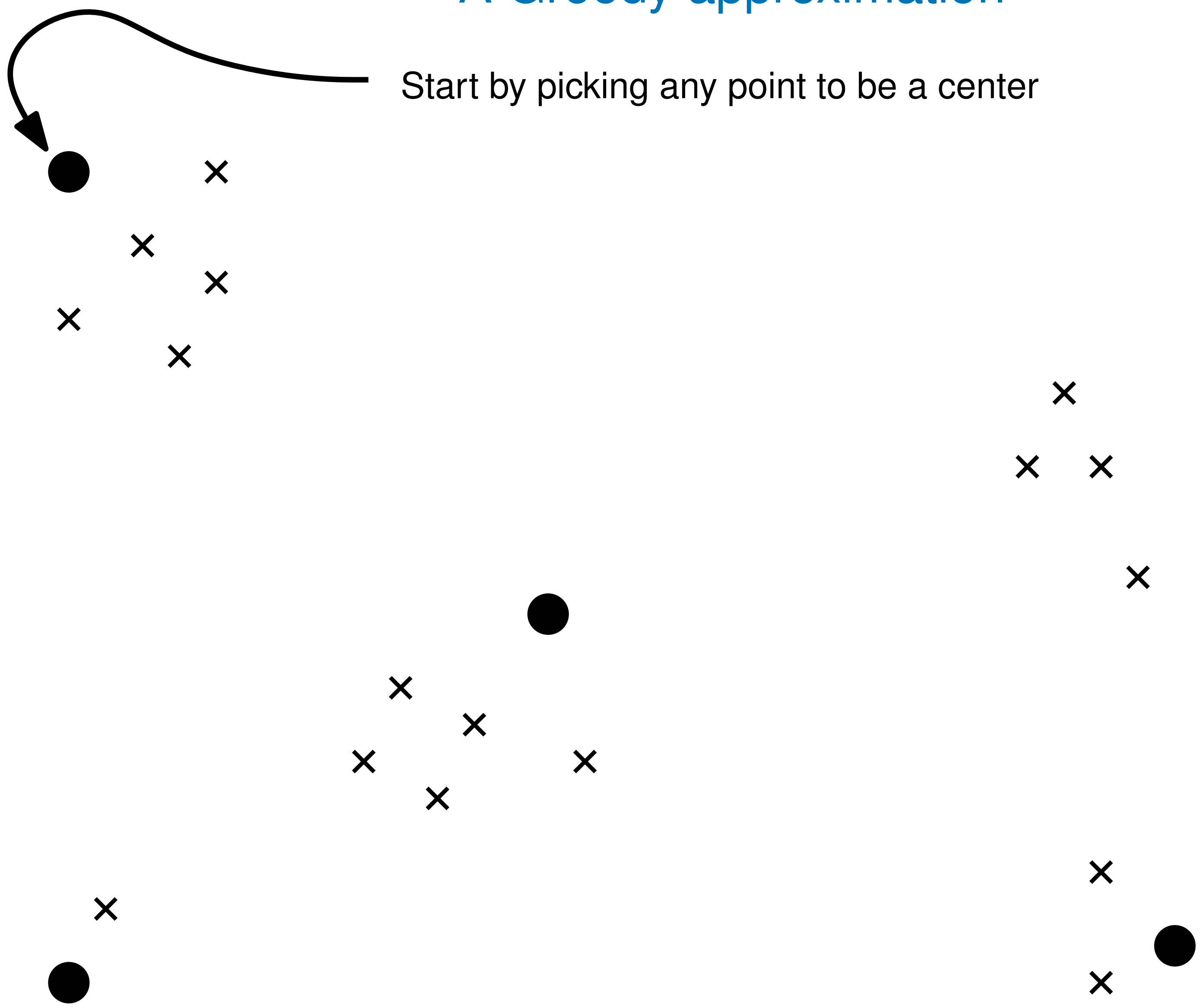

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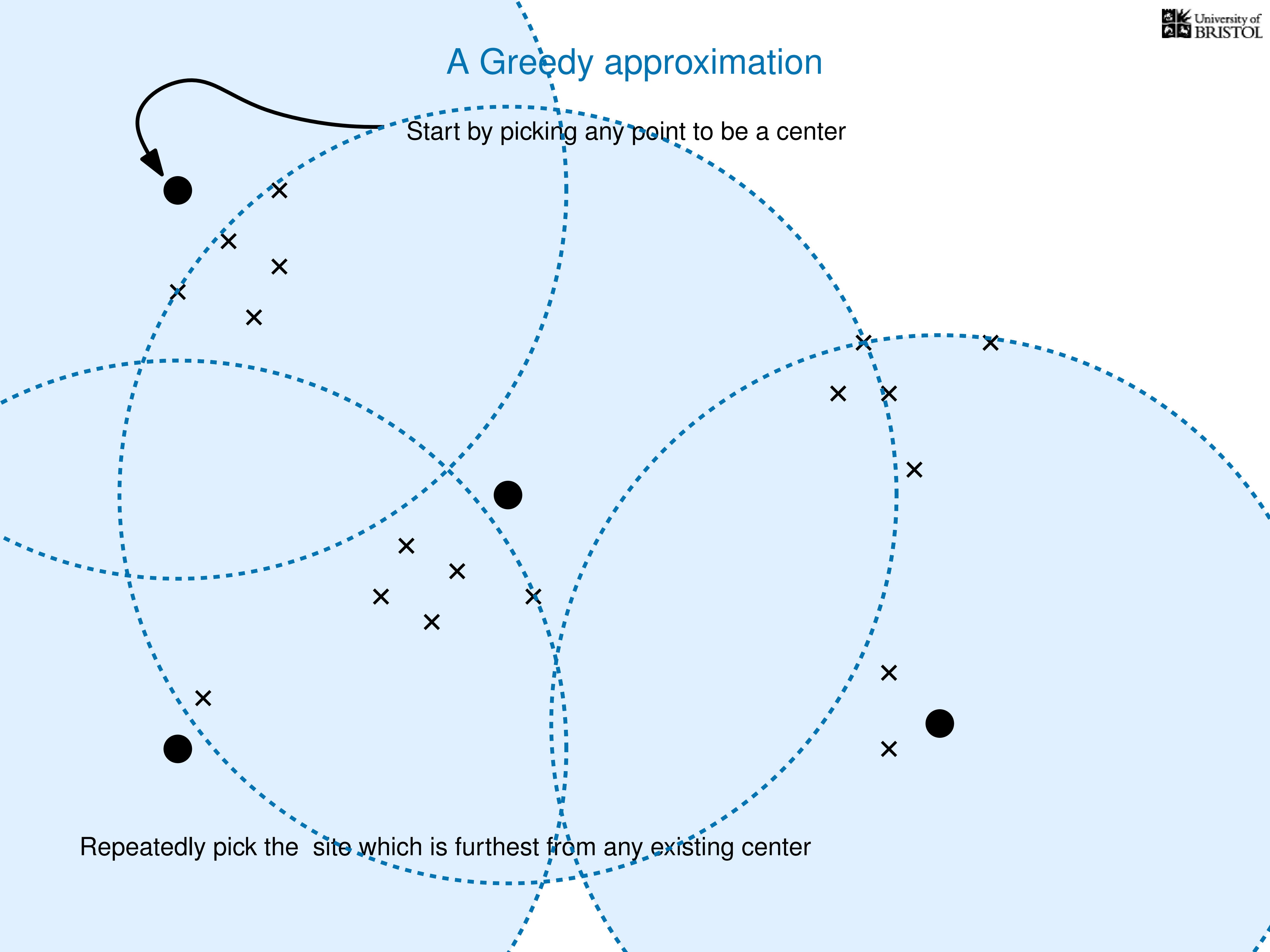

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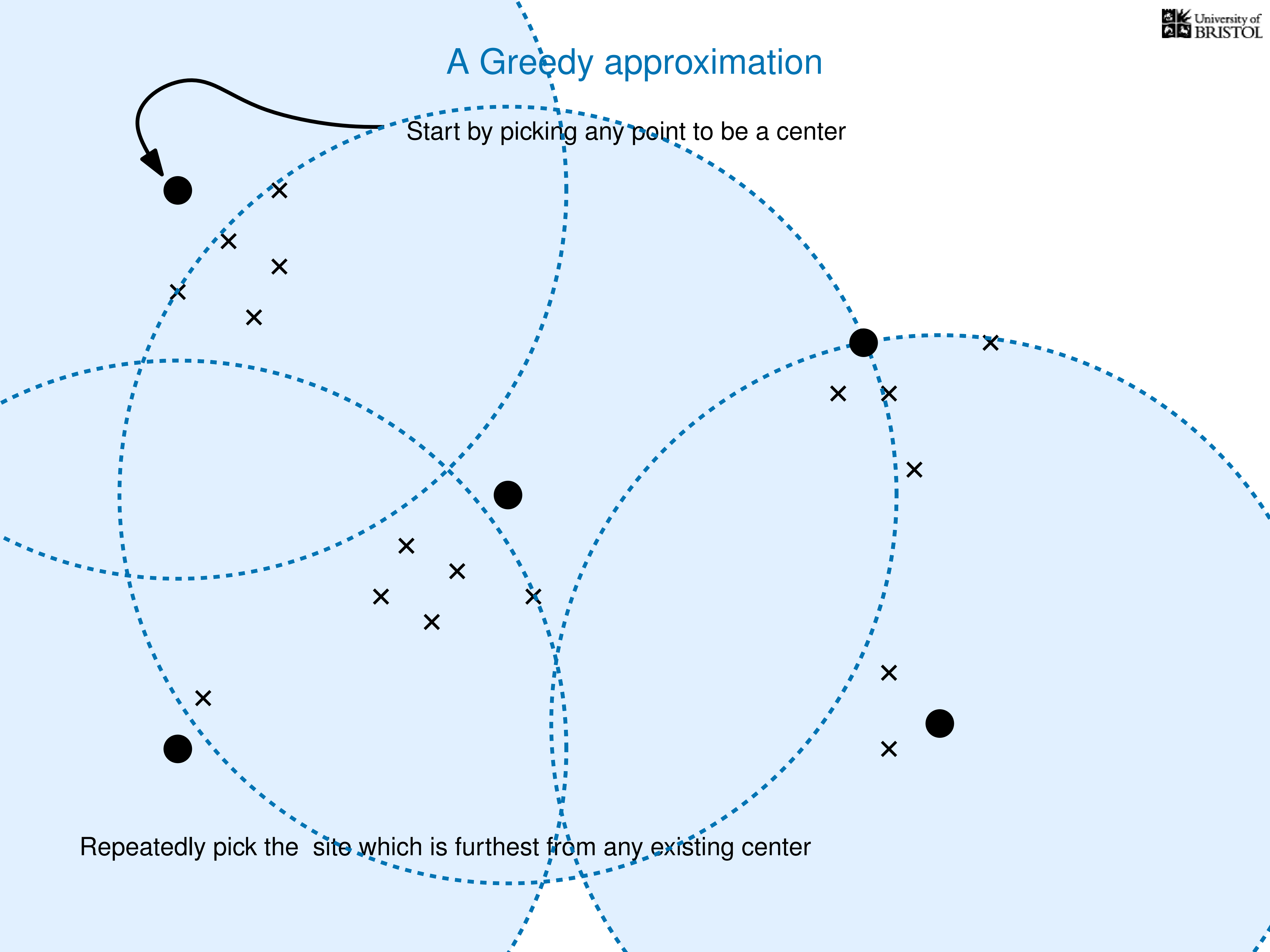

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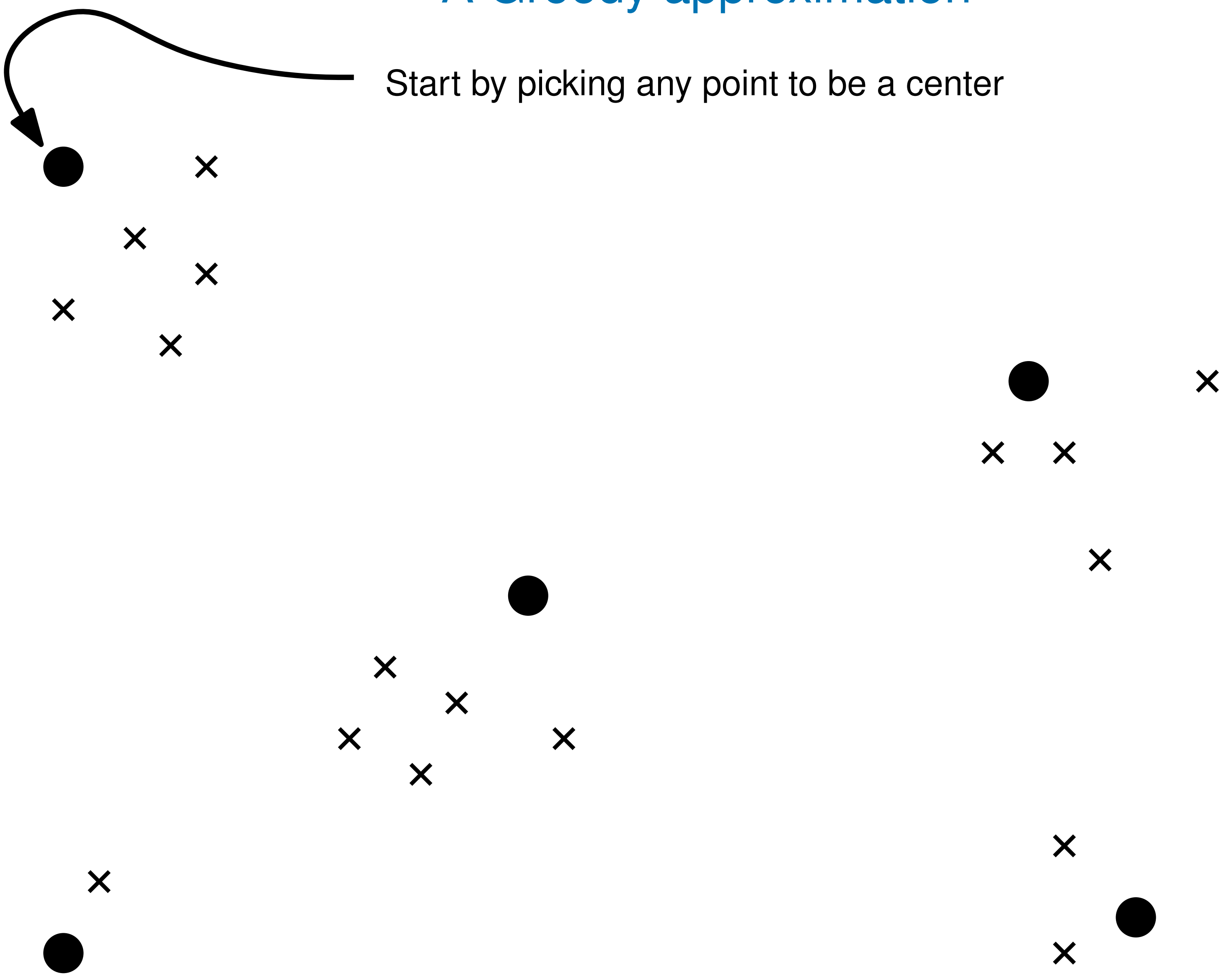

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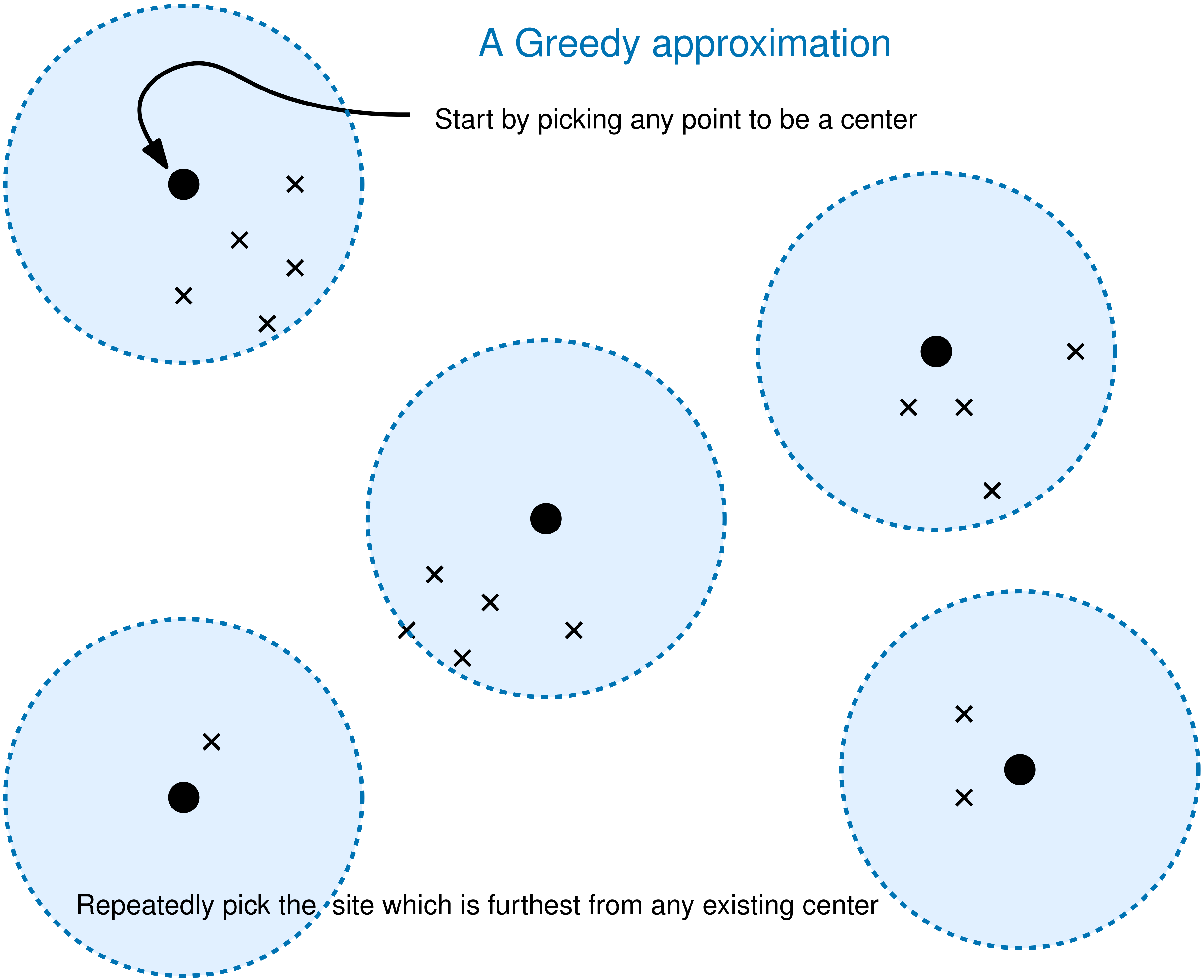

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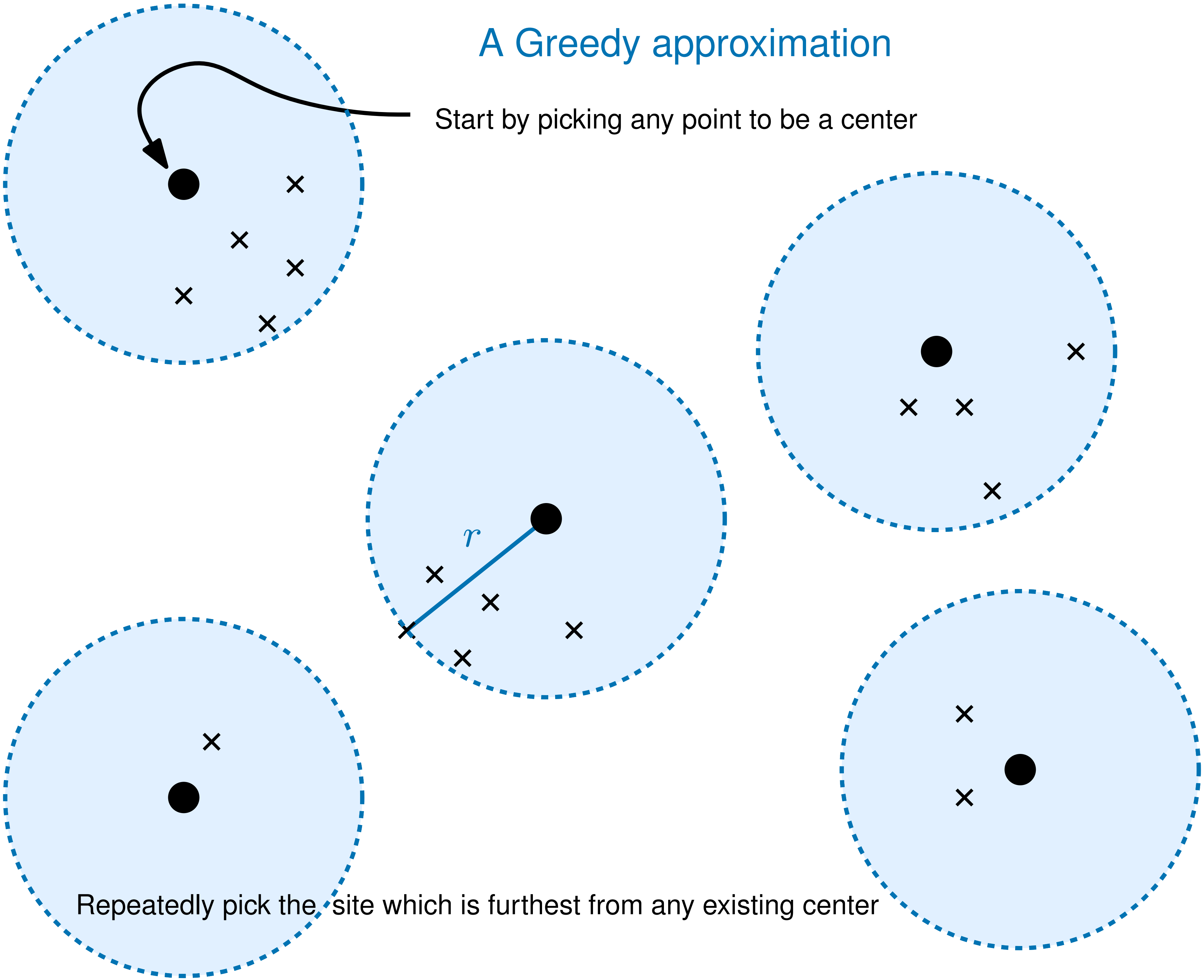

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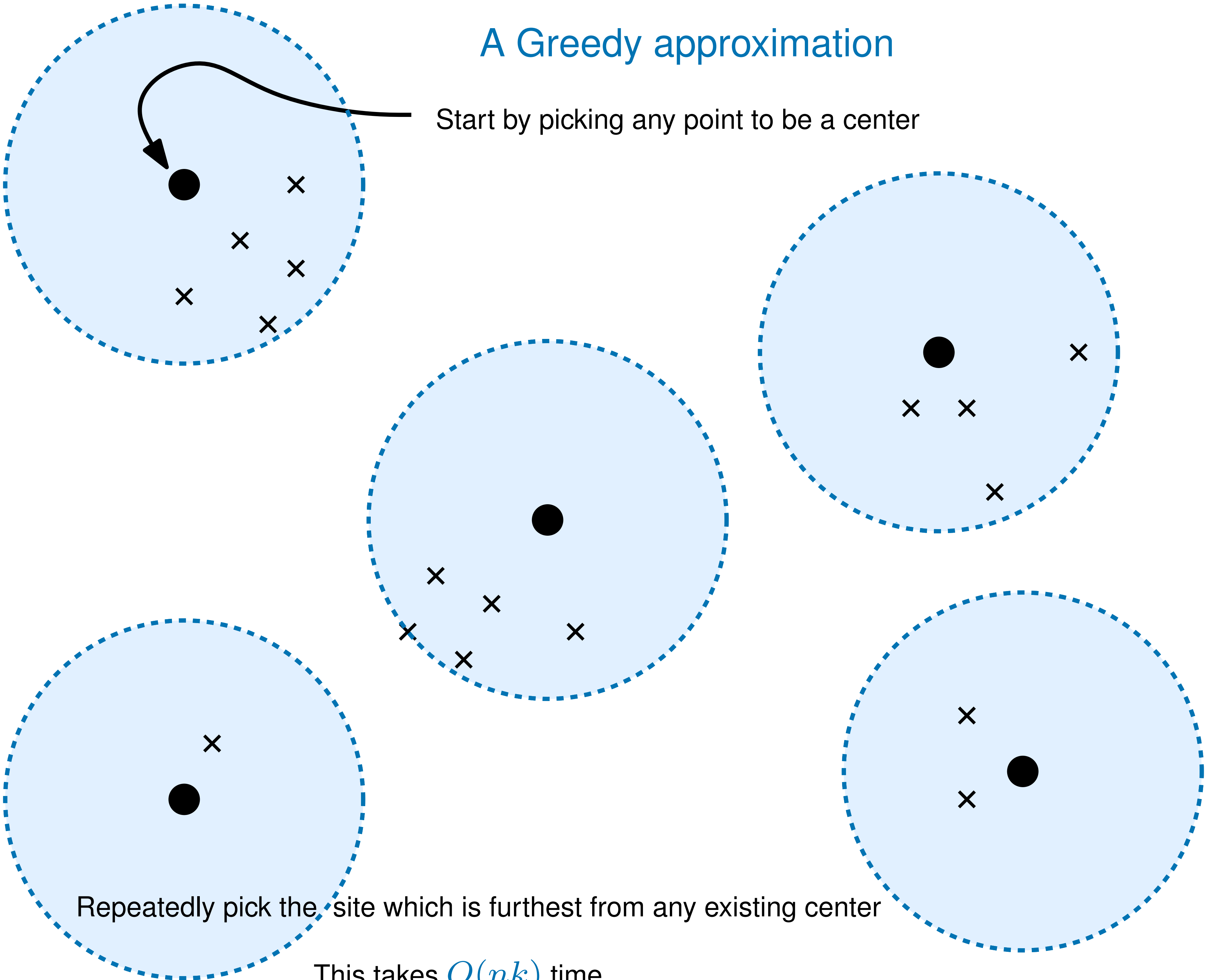

Start by picking any point to be a center

Repeatedly pick the site which is furthest from any existing center

This takes $O(nk)$ time

A Greedy approximation

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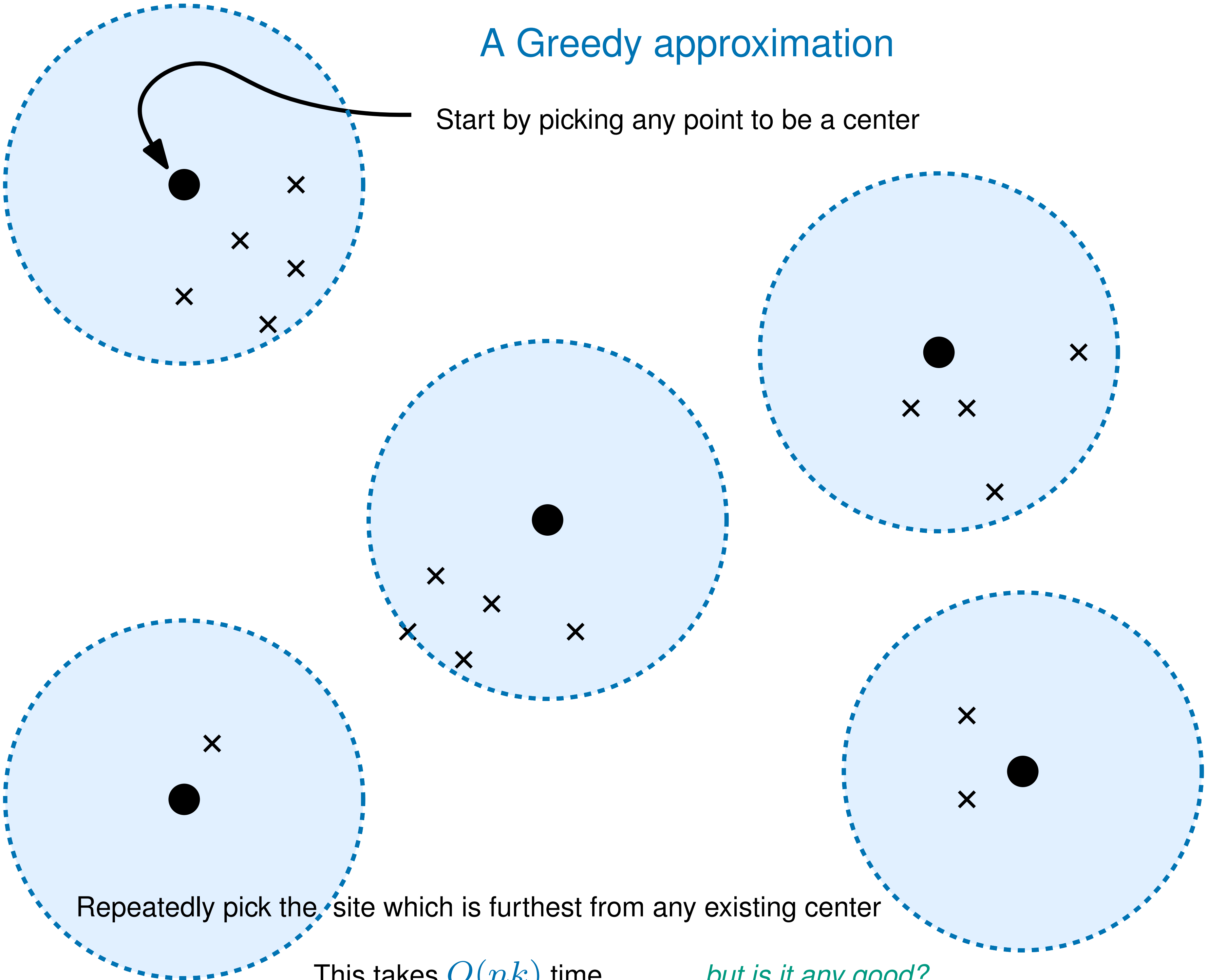

Repeatedly pick the site which is furthest from any existing center

This takes $O(nk)$ time *but is it any good?*

The Greedy approximation

Theorem The Greedy algorithm for k-center is a 2-approximation algorithm

Proof

Let C_g (resp. C_{Opt}) denote the set of centers selected by Greedy (resp. Optimal)

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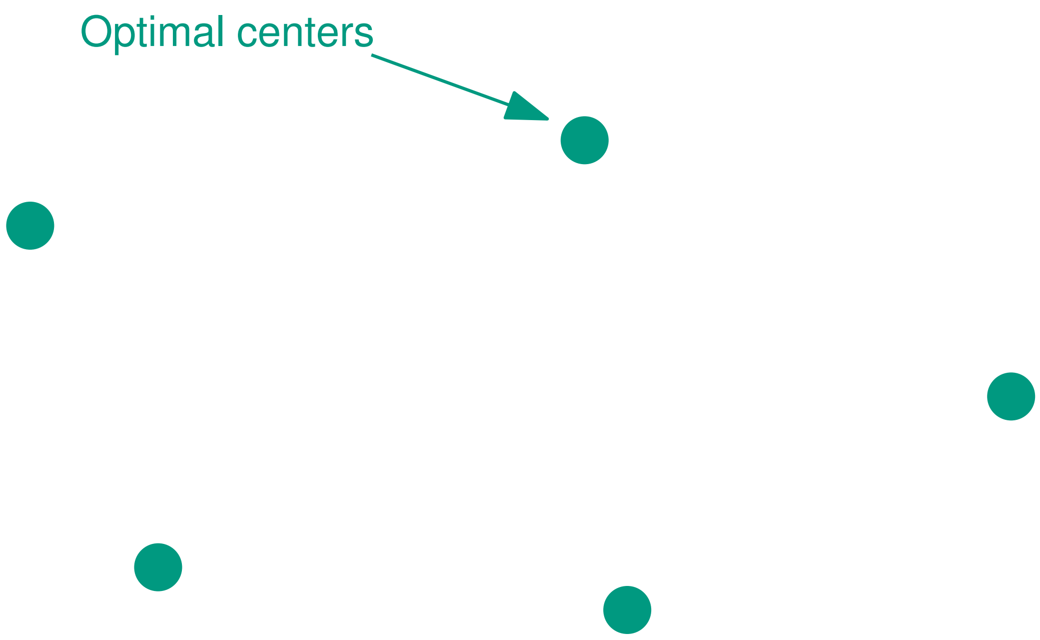

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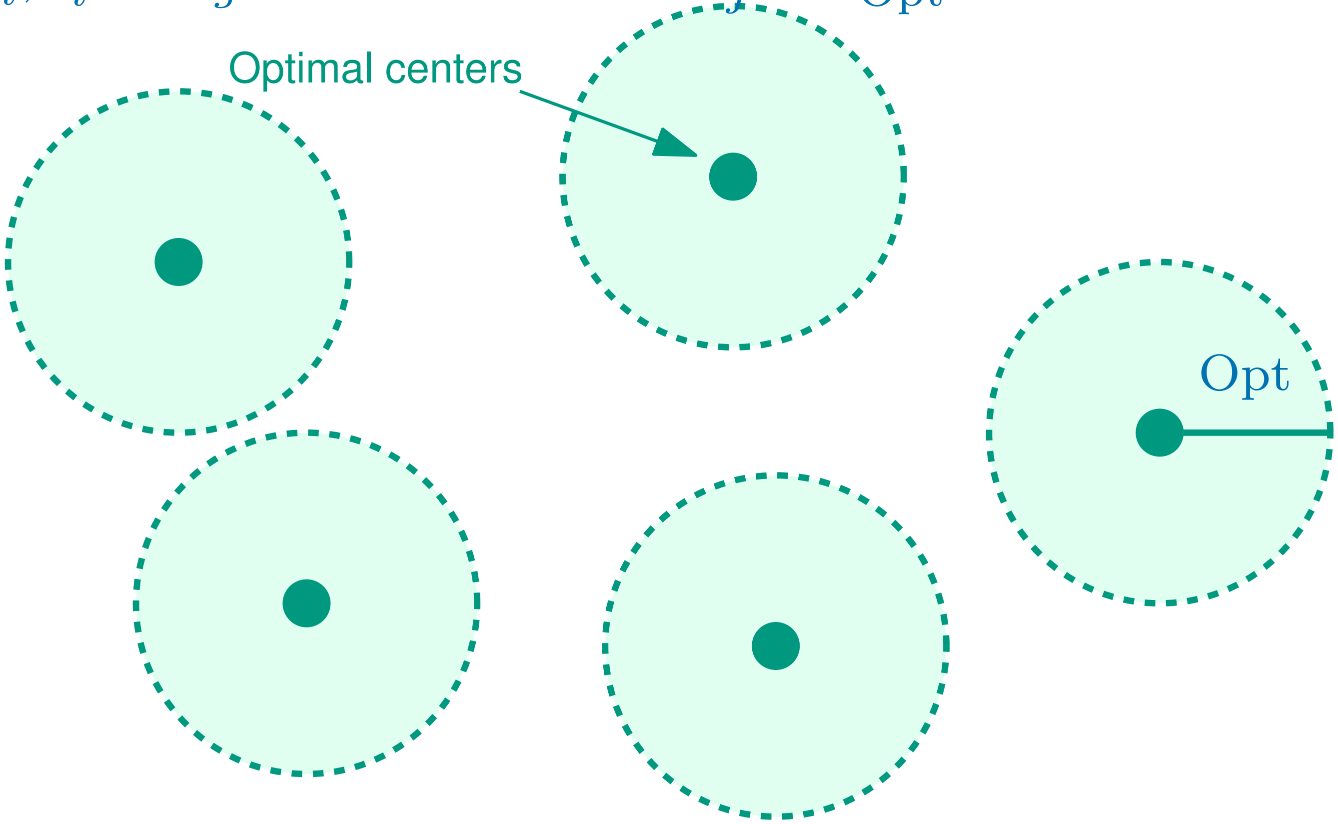

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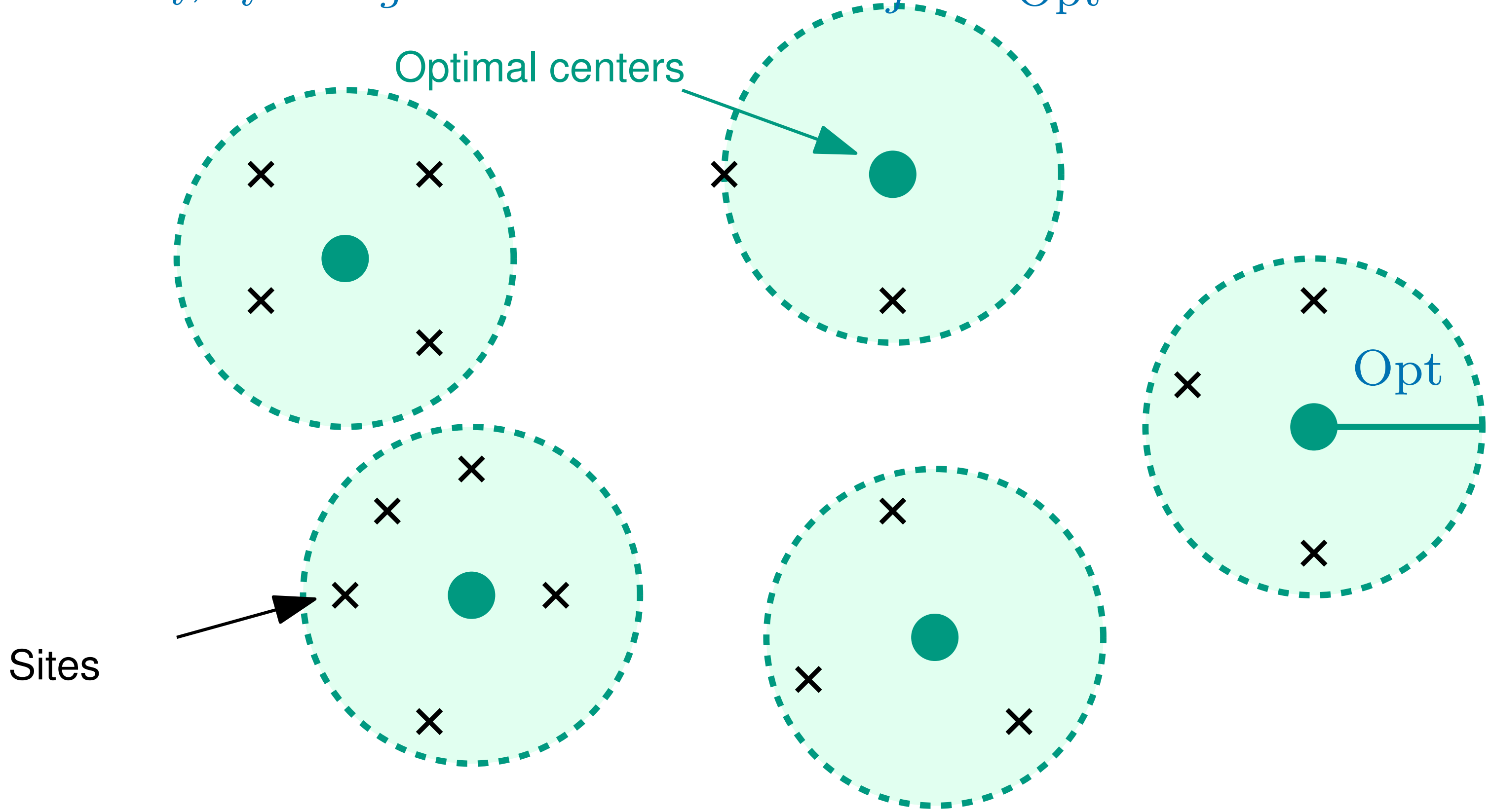

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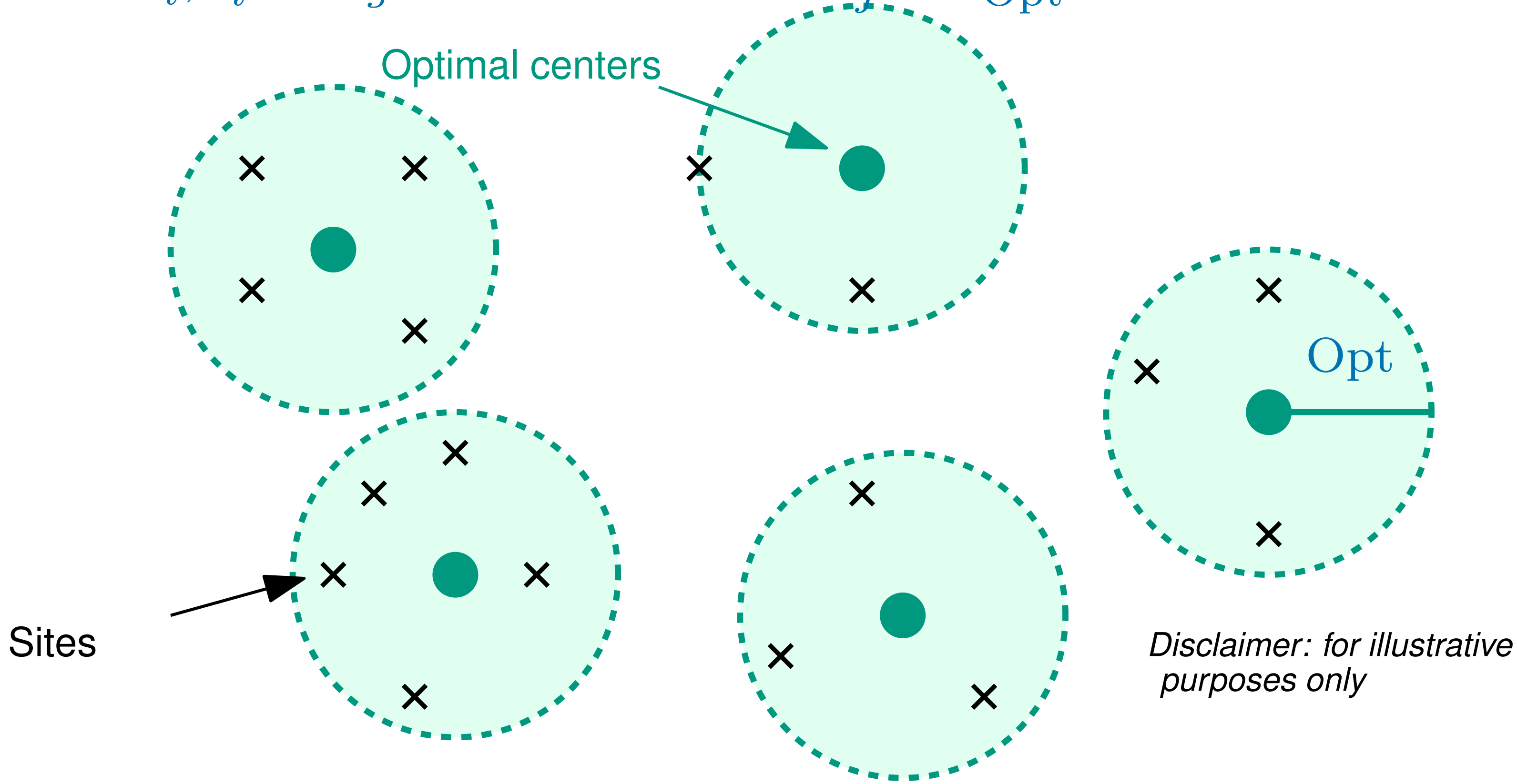

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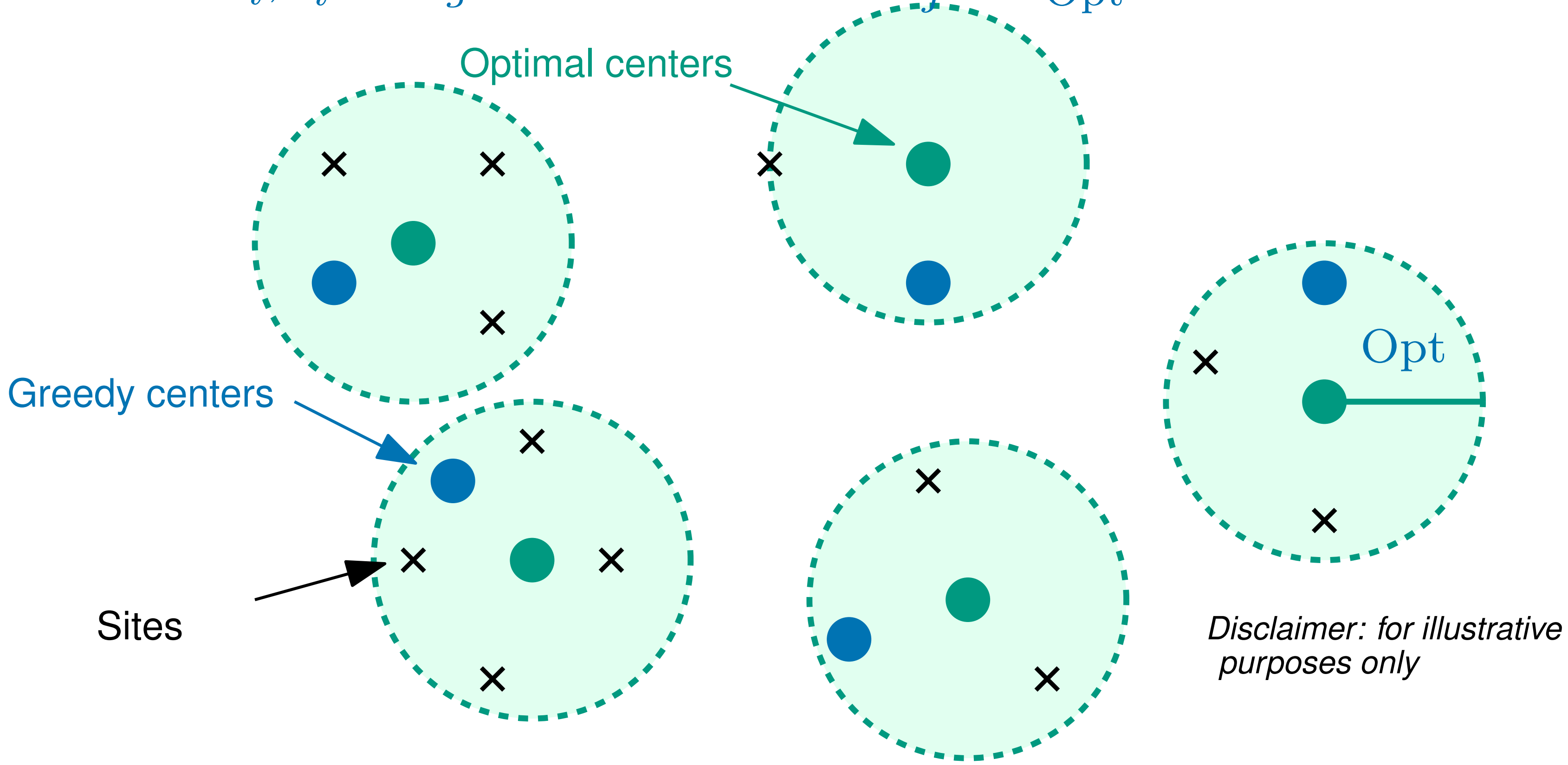

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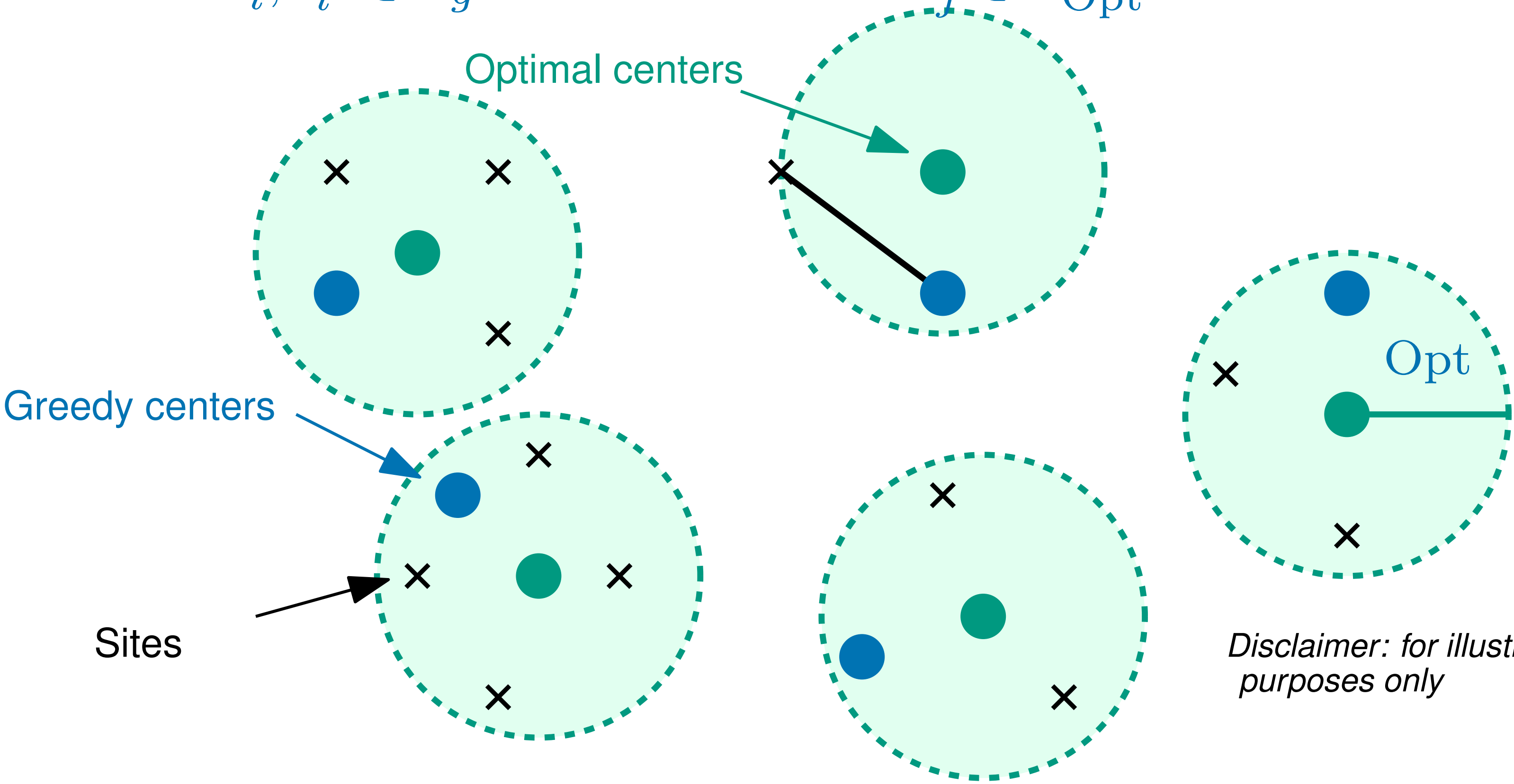

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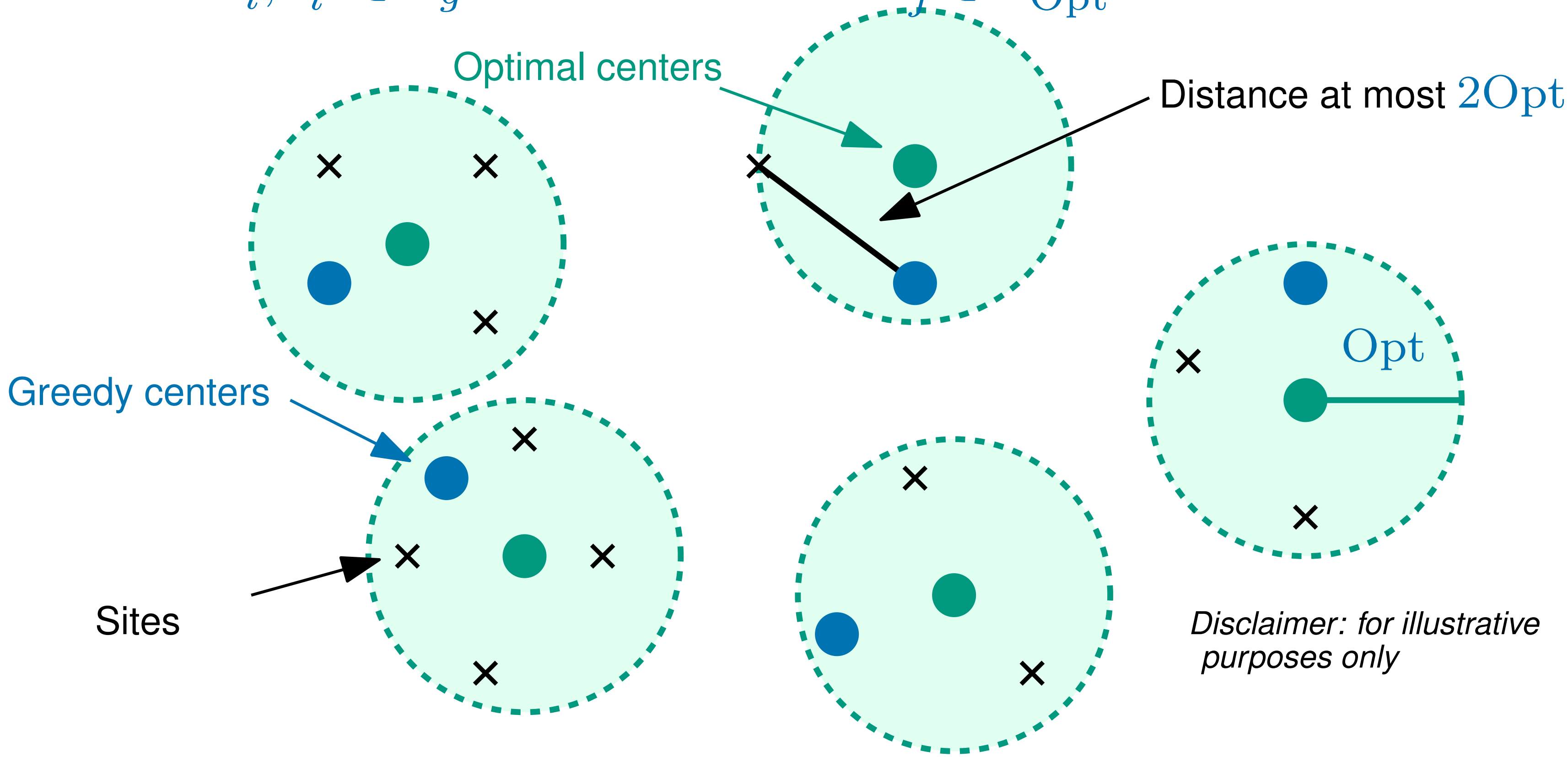

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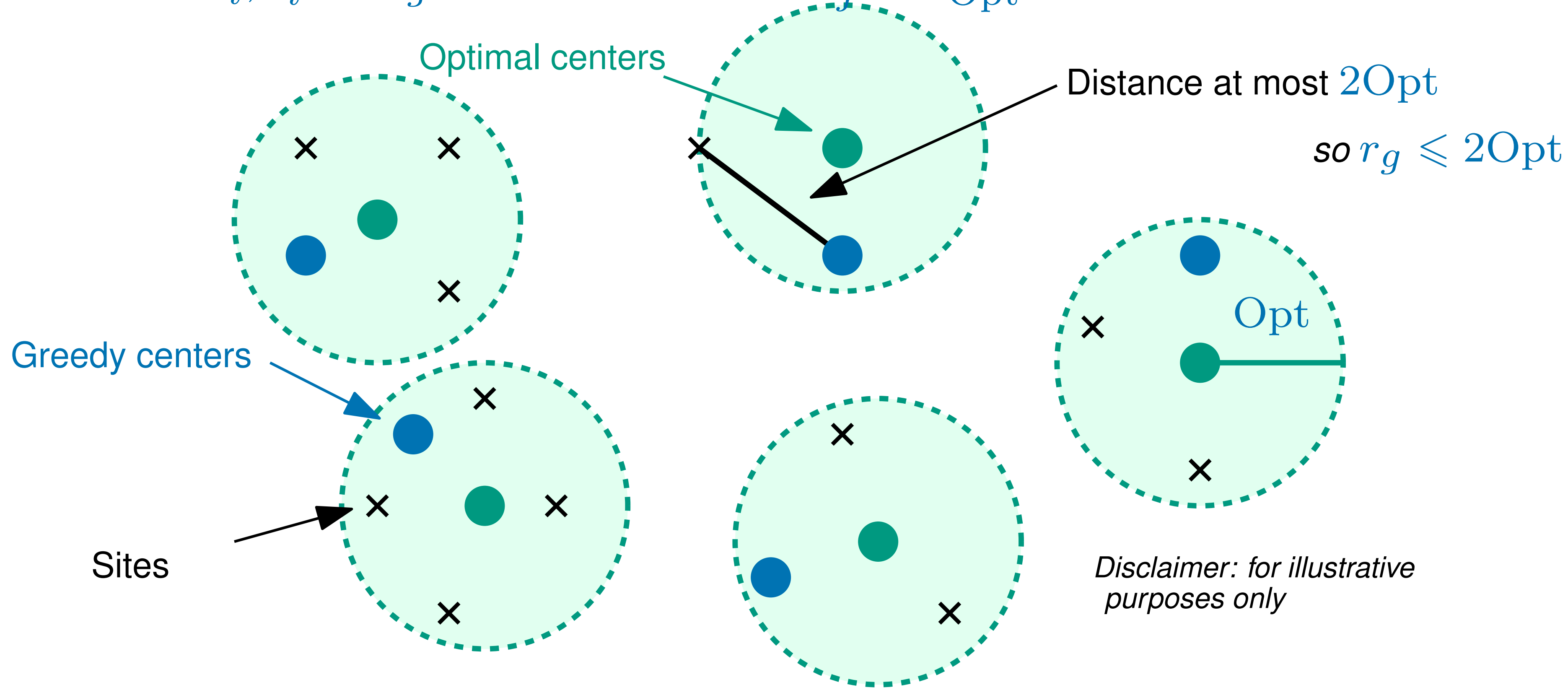

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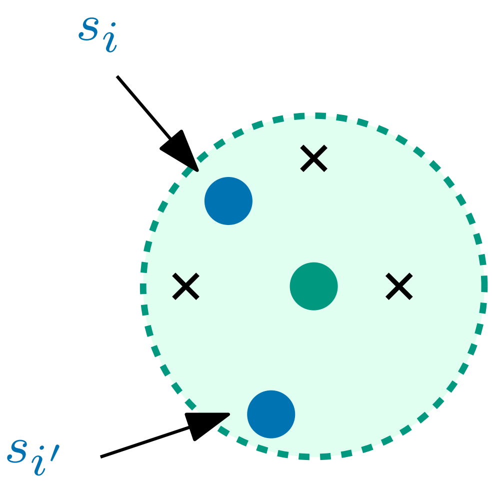

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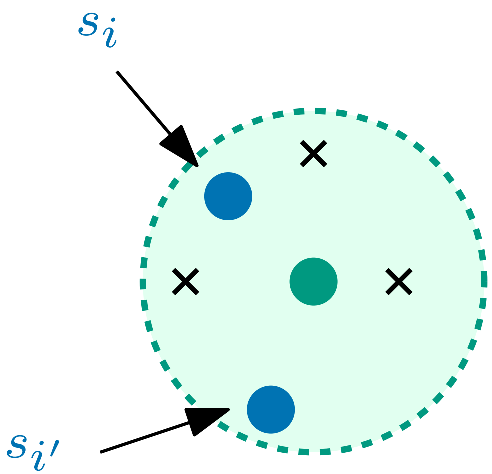
Assume wlog. that Greedy made s_i a center after $s_{i'}$



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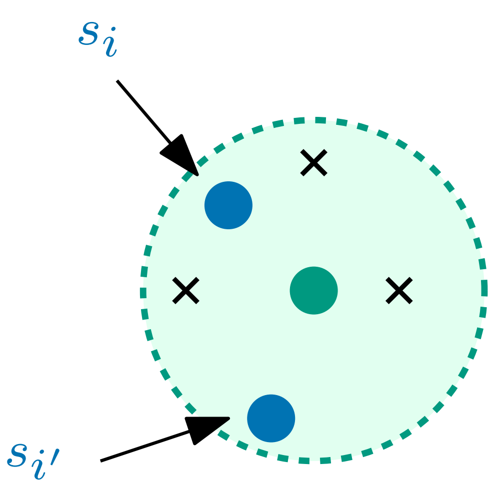
s_i was added as a center because it was



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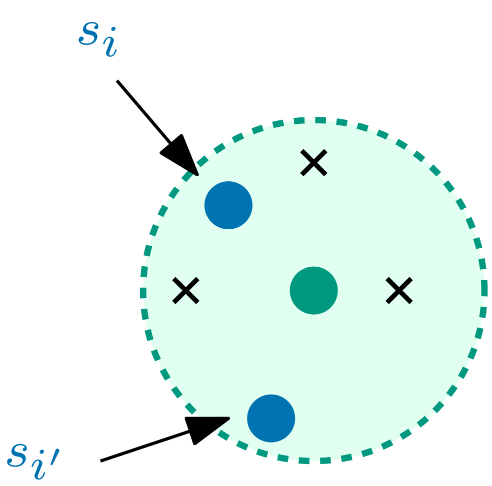

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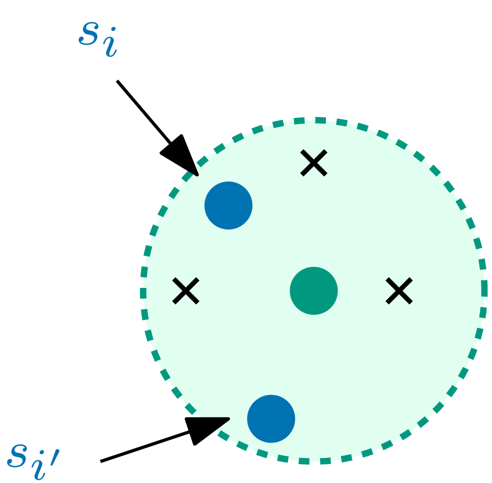
However, s_i is at most $2\mathrm{Opt}$ away from $s_{i'}$



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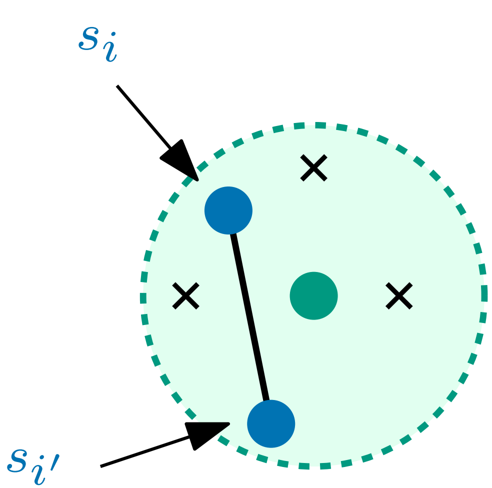

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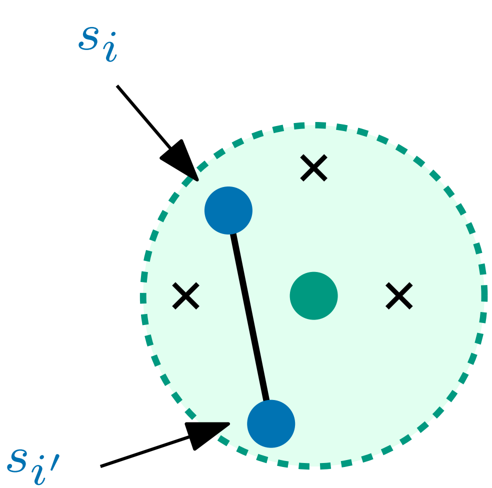
So even before adding s_i as a center, all sites were $\leqslant 2\mathrm{Opt}$ away from a Greedy center



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However, s_i is at most 2Opt away from $s_{i'}$

So even before adding s_i as a center, all sites
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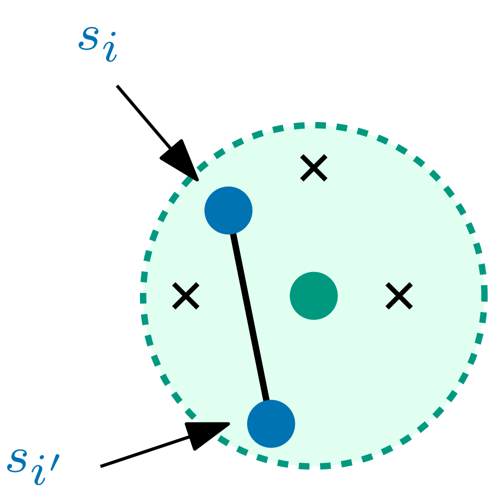
Therefore, $r_g \leqslant 2\text{Opt}$



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- Distance function d is a metric iff

  $d(x, y) = d(y, x)$, $d(x, y) \geqslant 0$

  ($d(x, y) = 0$ iff $x = y$) and $d(x, z) \leqslant d(x, y) + d(y, z)$

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- For $d = L_1$ or $d = L_\infty$, the problem is not α-approximable with $\alpha < 2$