van Emde Boas trees

Raphaël Clifford

Slides by Benjamin Sach
Dictionaries

In a dynamic dictionary data structure we store \((k, v)\)-pairs such that for any \(k\) there is at most one pair \((k, v)\) in the dictionary.

Three operations are supported:

- \(\text{add}(x, v)\) Add the the pair \((x, v)\) where \(x \in U\), the universe
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**THEOREM**

In the **Cuckoo hashing** scheme:

- Every **lookup** and every **delete** takes \(O(1)\) worst-case time,
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These are very natural operations that the **Hashing**-based solutions that we have seen are very unsuited to
What could we use instead?

We could use a self-balancing binary search tree... like a 2-3-4 tree, a red-black tree or an AVL tree.
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All three of these data structures support:

\[ \text{add}(x, v), \text{lookup}(x), \text{delete}(x), \text{predecessor}(k) \text{ and } \text{successor}(k) \]
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Each in $O(\log n)$ worst case time and $O(n)$ space, where $n$ is the number of elements stored.

They are also deterministic.
van Emde Boas Trees

In this lecture, we will see the van Emde Boas (vEB) tree which stores a set $S$ of integer keys from a universe $U = \{1, 2, 3, 4 \ldots u\}$ (i.e. $u = |U|$).

Five operations will be supported:

- **add($x$)**: Insert the integer $x$ into $S$ (where $x \in U$)
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![Diagram](image)
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**Example:** If $U = \{1, 2, 3, 4 \ldots 100 \cdot n\}$, you get $O(\log \log n)$ time and $O(n)$ space.
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and it is a deterministic data structure.

Example: If $U = \{1, 2, 3, 4 \ldots n^3\}$, you get $O(\log \log n)$ time and $O(n^3)$ space.
Attempt 1: a big array

Build an array of length $u$...
Attempt 1: a big array

Build an array of length $u$ . . . 

$A[i] = 1$ iff $i$ is in $S$
Attempt 1: a big array

Build an array of length $u$…

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$ . . .

$A[i] = 1$ iff $i$ is in $S$

```
A = [0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0]
```

The operations **add**, **delete** and **lookup** all take $O(1)$ time.
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

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Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length \( u \)...

\[
A[i] = 1 \quad \text{iff} \quad i \text{ is in } S
\]

The operations **add**, **delete** and **lookup** all take \( O(1) \) time.
**Attempt 1**: a big array

Build an array of length $u$...

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

The operations add, delete and lookup all take $O(1)$ time.
** Attempt 1: a big array 

Build an array of length \( u \) . . .

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations add, delete and lookup all take \( O(1) \) time.
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Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

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The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!
**Attempt 1: a big array**

Build an array of length \( u \)...

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations **add**, **delete** and **lookup** all take \( O(1) \) time.

...looks good so far!

What about the predecessor operation?
Attempt 1: a big array

Build an array of length $u$...

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!

What about the predecessor operation?
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations *add*, *delete* and *lookup* all take $O(1)$ time.

...looks good so far!

What about the predecessor operation?
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations \textit{add}, \textit{delete} and \textit{lookup} all take $O(1)$ time.

...looks good so far!

What about the \textit{predecessor} operation?

$\text{predecessor}(11)$
**Attempt 1: a big array**

Build an array of length $u$...

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

The operations **add**, **delete** and **lookup** all take $O(1)$ time.

...looks good so far!
**Attempt 1: a big array**

Build an array of length $u$...

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

predecessor(11)

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!
Attempt 1: a big array

Build an array of length \( u \)...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

\[ 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \]

The operations add, delete and lookup all take \( O(1) \) time.

...looks good so far!
**Attempt 1: a big array**

Build an array of length \( u \)...

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations add, delete and lookup all take \( O(1) \) time.  
...looks good so far!

The predecessor and successor operations take \( O(u) \) time.
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!

The predecessor and successor operations take $O(u)$ time
**Attempt 1: a big array**

Build an array of length $u$...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

The operations **add**, **delete** and **lookup** all take $O(1)$ time. 
...looks good so far!

The **predecessor** and **successor** operations take $O(u)$ time
...not so good!
Attempt 2: a constant height tree

(on top of a big array)
Attempt 2: a constant height tree

(on top of a big array)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
**Attempt 2:** a constant height tree

*(on top of a big array)*

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u}
\]

\[
u \quad \sqrt{u}
\]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits
**Attempt 2:** a constant height tree

*(on top of a big array)*

### Matrix $A$

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

### Matrix $C$

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
Attempt 2: a constant height tree

(on top of a big array)

$C$ is called the summary of $A$

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
**Attempt 2: a constant height tree**

*(on top of a big array)*

\( C \) is called the *summary* of \( A \)

This is 1 if any bit in the child block is 1

\[
\begin{array}{cccc}
\hline
1 & 1 & 0 & 1 \\
\hline
\end{array}
\]

Split \( A \) into \( \sqrt{u} \) *blocks* each containing \( \sqrt{u} \) bits

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
\hline
\hline
\end{array}
\]
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$.

This is 1 if any bit in the child block is 1.

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2:** a constant height tree
*(on top of a big array)*

$C$ is called the **summary** of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

$C$ is called the *summary* of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits

The *lookup* and *add* operations take $O(1)$ time.
Attempt 2: a constant height tree

(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

add(9)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\( C \) is called the **summary** of \( A \)

- This is 1 if any bit in the child block is 1

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

**Split** \( A \) into \( \sqrt{u} \) **blocks** each containing \( \sqrt{u} \) bits

**The lookup and add operations take** \( O(1) \) **time.**
**Attempt 2**: a constant height tree

*(on top of a big array)*

$C$ is called the *summary* of $A$

This is 1 if any bit in the child block is 1

Add(9)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2: a constant height tree**

*(on top of a big array)*

$C$ is called the *summary* of $A$

This is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

$C$ is called the *summary* of $A$.

This is 1 if any bit in the child block is 1.

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

C is called the summary of A

this is 1 if any bit in the child block is 1

delete(7)

Split A into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\(C\) is called the **summary** of \(A\)

- This is 1 if any bit in the child block is 1
- The lookup and add operations take \(O(1)\) time.
- The operations delete, predecessor and successor take \(O(\sqrt{u})\) time.

Split \(A\) into \(\sqrt{u}\) blocks each containing \(\sqrt{u}\) bits.
**Attempt 2:** a constant height tree

(on top of a big array)

\( C \) is called the *summary* of \( A \)

this is 1 if any bit in the child block is 1

to determine this bit you have to look through this block

delete(7)

Split \( A \) into \( \sqrt{u} \) *blocks* each containing \( \sqrt{u} \) bits

The *lookup* and *add* operations take \( O(1) \) time.

The operations *delete*, *predecessor* and *successor* take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

C

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

\[\sqrt{u}\]

\[u\]

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the **summary** of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the \textit{summary} of \( A \).

This is 1 if any bit in the child block is 1.

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits.

The \textit{lookup} and \textit{add} operations take \( O(1) \) time.

The operations \textit{delete}, \textit{predecessor} and \textit{successor} take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

\(C\) is called the summary of \(A\)

This is 1 if any bit in the child block is 1

to determine this bit
you have to look through this block

Split \(A\) into \(\sqrt{u}\) blocks each containing \(\sqrt{u}\) bits

The lookup and add operations take \(O(1)\) time.

The operations delete, predecessor and successor take \(O(\sqrt{u})\) time.
**Attempt 2: a constant height tree**

(On top of a big array)

\( C \) is called the summary of \( A \)

this is 1 if any bit in the child block is 1

\[ C \]

\[ \sqrt{u} \]

\[ \begin{array}{cccc}
1 & 1 & 0 & 1 \\
\end{array} \]

\[ A \]

\[ \begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array} \]

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\( C \) is called the *summary* of \( A \)

this is 1 if any bit in the child block is 1

\[ \sqrt{u} \]

**predecessor(14)**

\[ C \]

\[ \begin{array}{cccc}
1 & 1 & 0 & 1 \\
\end{array} \]

\[ \begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array} \]

\[ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \]

\[ \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \]

Split \( A \) into \( \sqrt{u} \) *blocks* each containing \( \sqrt{u} \) bits

The *lookup* and *add* operations take \( O(1) \) time.

The operations *delete*, *predecessor* and *successor* take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree

(on top of a big array)

\( C \) is called the summary of \( A \)

this is 1 if any bit in the child block is 1

\[ \text{predecessor}(14) \]

\( C \)

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\( C \) is called the *summary* of \( A \)

[this is 1 if any bit in the child block is 1]

\[ \text{predecessor}(14) \]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the \textit{summary} of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\( C \) is called the \textit{summary} of \( A \)

- this is 1 if any bit in the child block is 1
- \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
** Attempt 2: a constant height tree  
(on top of a big array) 

$C$ is called the summary of $A$  

this is 1 if any bit in the child block is 1  

** predecessor(14) **

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\(C\) is called the **summary** of \(A\)

- this is 1 if any bit in the child block is 1

Split \(A\) into \(\sqrt{u}\) **blocks** each containing \(\sqrt{u}\) bits

- The lookup and add operations take \(O(1)\) time.
- The operations delete, predecessor and successor take \(O(\sqrt{u})\) time.
Attempt 2: a constant height tree

(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor($14$)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

$C$ is called the *summary* of $A$. This is 1 if any bit in the child block is 1.

In the worst case we look at all of $C$ and all of two blocks.

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\(C\) is called the **summary** of \(A\)

In the worst case we look at all of \(C\) and all of two blocks *(successor is the same)*

\[
\text{predecessor}(14)
\]

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}
\]

Split \(A\) into \(\sqrt{u}\) blocks each containing \(\sqrt{u}\) bits

The lookup and add operations take \(O(1)\) time.

The operations delete, predecessor and successor take \(O(\sqrt{u})\) time.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

There is a whole lot more universe in here.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

there is a whole lot more universe in here
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

there is a whole lot more universe in here
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

There is a whole lot more universe in here.

$\text{Diagram:} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \\
\underbrace{\ldots} \quad \underbrace{\ldots} \quad \underbrace{\ldots} \\
u \quad \underbrace{\ldots} \quad \sqrt{u}$
An abstract view

Split the universe $\mathbb{U}$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

we can think of each block as a ‘little’ universe of size $\sqrt{u}$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

we can think of each block as a ‘little’ universe of size $\sqrt{u}$

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$.

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$.

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$.

*(this is just to deal with the offset from the start of the real universe)*
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$

(this is just to deal with the offset from the start of the real universe)
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots, \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$

We also build a summary data structure $C$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$i$ is stored in $C$ iff $B[i]$ is non-empty
Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

For block $i$, we build a data structure $B[i]$ which stores elements from \( \{1, 2, 3, \ldots, \sqrt{u}\} \)

\[ x \text{ is stored in } B[i] \text{ iff } (x + (i - 1))\sqrt{u} \in S \]

We also build a summary data structure $C$ which stores elements from \( \{1, 2, 3, \ldots, \sqrt{u}\} \)

\[ i \text{ is stored in } C \text{ iff } B[i] \text{ is non-empty} \]
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

*Recursion!*
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$
An abstract view

Split the universe \( U \) into \( \sqrt{u} \) blocks each associated with \( \sqrt{u} \) elements

How should we build \( B[1], B[2], \ldots B[\sqrt{u}] \) and \( C \)?

Recursion!

Each \( B[i] \) has universe \( \{1, 2, 3, \ldots \sqrt{u}\} \)

We recursively split this into \( 4^{\sqrt{u}} \) blocks each associated with \( 4^{\sqrt{u}} \) elements...
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$

We recursively split this into $\sqrt[4]{u}$ blocks each associated with $\sqrt[4]{u}$ elements... eventually (after some more work), this will lead to an $O(\log \log u)$ time solution.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

$C \quad \sqrt{u} \quad B[1] \quad B[2] \quad B[3] \quad B[\sqrt{u}]$

$\sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u}$

$u$
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

How do we perform the operations?
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in
   (this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in
(this takes $O(1)$ time with a little bit twiddling)

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Attempt 3: Recursion

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Attempt 3: Recursion

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Attempt 3: Recursion

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To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

(this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$

(suitably adjusting the offset from the start of $B[i]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

![Diagram of blocks and universe]

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.
   (this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$.

**Step 3** Add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$).
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

We actually insert $x'$ where

$$x = (x' + (i - 1)\sqrt{u})$$

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

(this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in
   (this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in
   (this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{predecessor}(x)$:

Step 1 Determine which $B[i]$ the element $x$ belongs in

Step 2 Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

Step 3 If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$
Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$).

**Step 3** If $x$ has no predecessor in $B[i]$:
- Compute $j = \text{predecessor}(i)$ in $C'$
- Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

$$\begin{array}{c}
\text{C} \\
\sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \\
\sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \\
u \\
\end{array}$$

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$  
Compute the predecessor of $x$ in $B[j]$  
(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

![Diagram of splits and blocks]

To perform $\text{predecessor}(x)$:

1. **Step 1** Determine which $B[i]$ the element $x$ belongs in.
2. **Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$).
3. **Step 3** If $x$ has no predecessor in $B[i]$:
   - Compute $j = \text{predecessor}(i)$ in $C$.
   - Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

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   - Compute $j = \text{predecessor}(i)$ in $C$.
   - Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform `predecessor(x)`:  

**Step 1** Determine which $B[i]$ the element $x$ belongs in  

**Step 2** Compute the predecessor of $x$ in $B[i]$  

(suitably adjusting the offset from the start of $B[i]$)  

**Step 3** If $x$ has no predecessor in $B[i]$:  

Compute $j = \text{predecessor}(i)$ in $C$  

Compute the predecessor of $x$ in $B[j]$  

(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

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Compute $j = \text{predecessor}(i)$ in $C$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

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   - Compute $j = \text{predecessor}(i)$ in $C$.
   - Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
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**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:

Compute $j = \text{predecessor}(i)$ in $C$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

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**Step 3** If $x$ has no predecessor in $B[i]$: Compute $j = \text{predecessor}(i)$ in $C$.

Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$  
Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

$x$ has no predecessor in here

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:
Compute $j = \text{predecessor}(i)$ in $C$
Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$). 

**Step 3** If $x$ has no predecessor in $B[i]$:

- Compute $j = \text{predecessor}(i)$ in $C$.
- Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

$C$
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations lookup, delete and successor can all also be defined in a similar, recursive manner.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations lookup, delete and successor can all also be defined in a similar, recursive manner.

How efficient are the operations?
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{add}(x)$:

1. **Step 1** Determine which $B[i]$ the element $x$ belongs in (this takes $O(1)$ time with a little bit twiddling).

2. **Step 2** If $B[i]$ is empty, add $i$ to $C$.

3. **Step 3** Add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$).
Attempt 3: Recursion

To perform \texttt{add}(x):

\textbf{Step 1} Determine which \(B[i]\) the element \(x\) belongs in

\(\text{this takes } O(1) \text{ time with a little bit twiddling}\)

\textbf{Step 2} If \(B[i]\) is empty, \textit{add} \(i\) to \(C\)

\textbf{Step 3} \textit{add} \(x\) to \(B[i]\)

(suitably adjusting the offset from the start of \(B[i]\))

Split the universe \(U\) into \(\sqrt{u}\) \textit{blocks} each associated with \(\sqrt{u}\) elements

\[\begin{array}{cccc}
C & & & \\
\sqrt{u} & \sqrt{u} & \sqrt{u} & \sqrt{u} \\
\hline
\sqrt{u} & \sqrt{u} & \sqrt{u} & \sqrt{u} \\
\hline
\end{array}\]
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

(this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$

(suitably adjusting the offset from the start of $B[i]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform $\text{predecessor}(x)$:

1. **Step 1** Determine which $B[i]$ the element $x$ belongs in.
2. **Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$).
3. **Step 3** If $x$ has no predecessor in $B[i]$: Compute $j = \text{predecessor}(i)$ in $C$.
   - Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations `lookup`, `delete` and `successor` can all also be defined in a similar, recursive manner.

How efficient are the operations?
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, **recursive** manner.

How efficient are the operations?

The **add** operation makes up to two recursive calls and the **predecessor** operation makes up to three
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

![Diagram of blocks](image)

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, recursive manner.

**How efficient are the operations?**

The **add** operation makes up to two recursive calls and the **predecessor** operation makes up to three.

Each recursive call could in turn make multiple recursive calls...
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

![Diagram showing the universe $U$ split into $\sqrt{u}$ blocks]

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, recursive manner

**How efficient are the operations?**

The **add** operation makes up to two recursive calls and the **predecessor** operation makes up to three.

Each recursive call could in turn make multiple recursive calls... *this could get out of hand!*
A closer look at predecessor

To perform \textit{predecessor}(x):

\textbf{Step 1} Determine which $B[i]$ the element $x$ belongs in

\textbf{Step 2} Compute the \textit{predecessor} of $x$ in $B[i]$

\textbf{Step 3} If $x$ has no \textit{predecessor} in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$
Return the \textit{predecessor} of $x$ in $B[j]$
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** Compute the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x \) has no predecessor in \( B[i] \): Compute \( j = \text{predecessor}(i) \) in \( C \)

Return the predecessor of \( x \) in \( B[j] \)
A closer look at predecessor

Observation 1: if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

\[ x \text{ has a predecessor in } B[i] \text{ iff } x \geq \text{the minimum in } B[i] \]

To perform \text{predecessor}(x):

\textbf{Step 1} Determine which \( B[i] \) the element \( x \) belongs in

\textbf{Step 2} Compute the predecessor of \( x \) in \( B[i] \)

\textbf{Step 3} If \( x \) has no predecessor in \( B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \)

Return the predecessor of \( x \) in \( B[j] \)
A closer look at predecessor

**Observation 1:** if $x$ has a predecessor in $B[i]$ we only make one recursive call

$x$ has a predecessor in $B[i]$ iff $x \geq$ the minimum in $B[i]$

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$  

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$ 
Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

Observation 1: if $x$ has a predecessor in $B[i]$ we only make one recursive call

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

**Step 3** If $x < \text{the minimum in } B[i]$:

    Compute $j = \text{predecessor}(i)$ in $C$

    Return the predecessor of $x$ in $B[j]$

$x$ has a predecessor in $B[i]$ iff $x \geq \text{the minimum in } B[i]$. 

Observation 1: if $x$ has a predecessor in $B[i]$ we only make one recursive call.
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

\[ x \text{ has a predecessor in } B[i] \iff x \geq \text{the minimum in } B[i] \]

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \text{the minimum in } B[i] \):

Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \text{the minimum in } B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \)

Return the predecessor of \( x \) in \( B[j] \)
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):
   - Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \) the minimum in \( B[i] \):
   - Compute \( j = \) predecessor \((i)\) in \( C\)
   - Return the predecessor of \( x \) in \( B[j] \)

\( x \) has a predecessor in \( B[i] \) iff \( x \geq \) the minimum in \( B[i] \)
A closer look at predecessor

**Observation 1:** if $x$ has a predecessor in $B[i]$ we only make one recursive call.

$x$ has a predecessor in $B[i]$ iff $x \geq$ the minimum in $B[i]$.

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** If $x \geq$ the minimum in $B[i]$:
   - Return the predecessor of $x$ in $B[i]$.

**Step 3** If $x <$ the minimum in $B[i]$:
   - Compute $j = \text{predecessor}(i)$ in $C$.
   - Return the predecessor of $x$ in $B[j]$.

Now we make at most two recursive calls (ignoring finding the minimum).
A closer look at predecessor

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in.

**Step 2** If \( x \geq \text{the minimum in } B[i] \):

Return the \text{predecessor} of \( x \) in \( B[i] \)

**Step 3** If \( x < \text{the minimum in } B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \)

Return the \text{predecessor} of \( x \) in \( B[j] \)
A closer look at predecessor

To perform \texttt{predecessor}(x):

\textbf{Step 1} Determine which \( B[i] \) the element \( x \) belongs in

\textbf{Step 2} If \( x \geq \) the minimum in \( B[i] \):

\begin{itemize}
  \item Return the \texttt{predecessor} of \( x \) in \( B[i] \)
\end{itemize}

\textbf{Step 3} If \( x < \) the minimum in \( B[i] \):

\begin{itemize}
  \item Compute \( j = \texttt{predecessor}(i) \) in \( C \)
  \item Return the \texttt{predecessor} of \( x \) in \( B[j] \)
\end{itemize}

we need to get rid of one of these recursive calls
To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):

Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \) the minimum in \( B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \)

Return the predecessor of \( x \) in \( B[j] \)
A closer look at predecessor

Observation 2: In **Step 3**, the predecessor of \( x \) in \( B[j] \) is the maximum in \( B[j] \)

To perform `predecessor(x)`:

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):

- Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x \) < the minimum in \( B[i] \):

- Compute \( j = \text{predecessor}(i) \) in \( C \)
- Return the predecessor of \( x \) in \( B[j] \)

---

We need to get rid of one of these recursive calls
A closer look at predecessor

**Observation 2:** In **Step 3**, the predecessor of \( x \) in \( B[j] \) is the maximum in \( B[j] \)

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):

Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \) the minimum in \( B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \)

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A closer look at predecessor

Observation 2: In Step 3, the predecessor of $x$ in $B[j]$ is the maximum in $B[j]$.

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** If $x \geq$ the minimum in $B[i]$: Return the predecessor of $x$ in $B[i]$.

**Step 3** If $x <$ the minimum in $B[i]$: Compute $j = \text{predecessor}(i)$ in $C$.

Return the maximum in $B[j]$.
A closer look at predecessor

**Observation 2:** In **Step 3**, the predecessor of \( x \) in \( B[j] \) is the maximum in \( B[j] \)

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \text{the minimum in } B[i] \):
    - Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x \ < \text{the minimum in } B[i] \):
    - Compute \( j = \text{predecessor}(i) \) in \( C \)
    - Return the maximum in \( B[j] \)

**Observation 2:** In **Step 3**, the predecessor of \( x \) in \( B[j] \) is the maximum in \( B[j] \)
Finally: van Emde Boas Trees

So that we can find the min/max quickly we store them separately...
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Remember that each $B[i]$ and $C$ are also vEB (van Emde Boas) trees each over the universe $\{1, 2, 3, \ldots, \sqrt{u}\}$
Finally: van Emde Boas Trees

So that we can find the min/max quickly we store them separately...

Remember that each \( B[i] \) and \( C \) are also vEB (van Emde Boas) trees each over the universe \( \{1, 2, 3, \ldots, \sqrt{u}\} \)

In particular \( B[i] \) also stores it’s min/max elements separately

so recovering the minimum or maximum in \( B[i] \) (or \( C \)) takes \( O(1) \) time
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so recovering the minimum or maximum in $B[i]$ (or $C$) takes $O(1)$ time.

There is one more important thing, the minimum is not also stored in $B[i]$.

this allows us to avoid making multiple recursive calls when adding an element.
Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( B[i] \) is empty, add \( i \) to \( C \)

and set the min and max in \( B[i] \) to \( x \) (adjusting the offset)

**Step 3** If \( B[i] \) is not empty, add \( x \) to \( B[i] \)
Another look at add

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $B[i]$ is empty, add $i$ to $C$ and set the min and max in $B[i]$ to $x$ (adjusting the offset)

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Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( B[i] \) is empty, add \( i \) to \( C \) and set the min and max in \( B[i] \) to \( x \) *adjusting the offset*

**Step 3** If \( B[i] \) is not empty, add \( x \) to \( B[i] \)

we make one recursive call
Another look at add

To perform add($x$):

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $B[i]$ is empty, add $i$ to $C$ and set the min and max in $B[i]$ to $x$ (*adjusting the offset*)

**Step 3** If $B[i]$ is not empty, add $x$ to $B[i]$

we make one recursive call
Another look at add

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $B[i]$ is empty, add $i$ to $C$

and set the min and max in $B[i]$ to $x$ (*adjusting the offset*)

**Step 3** If $B[i]$ is not empty, add $x$ to $B[i]$
Another look at add

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Another look at add

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Another look at add

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Another look at add

To perform \text{add}(x):

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\textbf{Step 3} If \(B[i]\) is not empty, add \(x\) to \(B[i]\)
Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

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we make one recursive call

this is not recursive
Another look at add

To perform add(x):

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $B[i]$ is empty, add $i$ to $C$ and set the min and max in $B[i]$ to $x$ *(adjusting the offset)*

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Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( B[i] \) is empty, add \( i \) to \( C \)

and set the min and max in \( B[i] \) to \( x \) *adjusting the offset*

**Step 3** If \( B[i] \) is not empty, add \( x \) to \( B[i] \)
Another look at `add`:

To perform `add(x)`:

**Step 1** Determine which `B[i]` the element `x` belongs in.

**Step 2** If `B[i]` is empty, add `i` to `C` and set the min and max in `B[i]` to `x` (*adjusting the offset*).

**Step 3** If `B[i]` is not empty, add `x` to `B[i]`.

But what happens when the min/max change?
Another look at add

To perform add\((x)\):

Step 1 Determine which \(B[i]\) the element \(x\) belongs in

Step 2 If \(B[i]\) is empty, add \(i\) to \(C\) and set the min and max in \(B[i]\) to \(x\) (adjusting the offset)

Step 3 If \(B[i]\) is not empty, add \(x\) to \(B[i]\)

Now we always make exactly one recursive call but what happens when the min/max change?

the min is only stored here

\[\begin{align*}
\text{min} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \text{max} \\
37 & \quad \times & \quad \times & \quad \times & \quad \times & \quad \times & \quad \times & \quad \times & \quad 483 \\
x & \quad \text{min} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \sqrt{u} & \quad \text{max}
\end{align*}\]
Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( B[i] \) is empty, add \( i \) to \( C \)

and set the min and max in \( B[i] \) to \( x \) *(adjusting the offset)*

**Step 3** If \( B[i] \) is not empty, add \( x \) to \( B[i] \)

Now we always make exactly one recursive call but what happens when the min/max change?

The min is only stored here
Another look at add

To perform add\((x)\):

**Step 0** If \(x < \text{min}\) then swap \(x\) and \(\text{min}\)

**Step 1** Determine which \(B[i]\) the element \(x\) belongs in

**Step 2** If \(B[i]\) is empty, add \(i\) to \(C\)

and set the \(\text{min}\) and \(\text{max}\) in \(B[i]\) to \(x\) *(adjusting the offset)*

**Step 3** If \(B[i]\) is not empty, add \(x\) to \(B[i]\)

**Step 4** Update the \(\text{max}\)

---

The min is only stored here.

Now we always make exactly one recursive call but what happens when the \(\text{min}/\text{max}\) change?
Time Complexity

the min is only stored here

C

$\sqrt{u}$


37

min

$\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$ u max

483

the min is only stored here
We have seen that the operations \textit{add} and \textit{predecessor} can be defined so that they make only one recursive call.
Time Complexity

The operations `lookup`, `delete` and `successor` can all also be defined in a similar, recursive manner so that they make only one recursive call.

We have seen that the operations `add` and `predecessor` can be defined so that they make only one recursive call.
We have seen that the operations \textit{add} and \textit{predecessor} can be defined so that they make only one recursive call.

The operations \textit{lookup}, \textit{delete} and \textit{successor} can all also be defined in a similar, \textit{recursive} manner so that they make only one recursive call.

\textit{How long do the operations take?}
Time Complexity

the min is only stored here

\begin{itemize}
\end{itemize}

\begin{itemize}
\item min
\item $\sqrt{u}$ \hfill $\sqrt{u}$ \hfill $\sqrt{u}$ \hfill $\sqrt{u}$
\end{itemize}

\begin{itemize}
\item $u$
\end{itemize}

\begin{itemize}
\item max
\end{itemize}

the min is only stored here

\begin{itemize}
\item $C$
\end{itemize}

only stored here

\begin{itemize}
\item $\sqrt{u}$
\end{itemize}

\begin{itemize}
\item $\min$
\item $\max$
\end{itemize}
Let $T(u)$ be the time complexity of the add operation (where $u$ is the universe size).
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Using substitution and the master method you can show that… $T(u) = O(\log \log u)$
Let $T(u)$ be the time complexity of the predecessor operation (where $u$ is the universe size)

We have that, $T(u) = T(\sqrt{u}) + O(1)$

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Let $T(u)$ be the time complexity of the \textit{predecessor} operation
(where \(u\) is the universe size)

We have that, $T(u) = T(\sqrt{u}) + O(1)$

Using substitution and the master method you can show that\ldots $T(u) = O(\log \log u)$

\textit{this holds for all the operations}
Space Complexity

the min is only stored here


\[
\begin{array}{c}
\text{min} \\
\hline
\sqrt{u} && \sqrt{u} && \sqrt{u} \\
\hline
u
\end{array}
\]

\[
\begin{array}{c}
\text{max} \\
\hline
\sqrt{u} \\
\hline
\end{array}
\]
Space Complexity

Let $Z(u)$ be the space used by a vEB tree over a universe of size $u$.
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We have that, $Z(u) = (\sqrt{u} + 1) \cdot Z(\sqrt{u}) + O(1)$
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We have that, $Z(u) = (\sqrt{u} + 1) \cdot Z(\sqrt{u}) + O(1)$

If you solve this you get that... $Z(u) = O(u)$
van Emde Boas Trees

The van Emde Boas (vEB) tree stores a set $S$ of integer keys from a universe $U = \{1, 2, 3, 4 \ldots u\}$ (i.e. $u = |U|$).

Five operations are supported:

- $\text{add}(x)$ Insert the integer $x$ into $S$ (where $x \in U$)
- $\text{lookup}(x)$ Return yes if $x$ is in $S$, or no otherwise.
- $\text{delete}(x)$ Remove $x$ from $S$
- $\text{predecessor}(k)$ Return the largest integer $x$ in $S$ such that $x \leq k$
- $\text{successor}(k)$ Return the smallest integer $x$ in $S$ such that $x \geq k$

All operations take $O(\log \log u)$ worst case time and the space used is $O(u)$
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The space can be improved to $O(n)$ using hashing (see y-fast trees)