Advanced Algorithms – COMS31900

van Emde Boas trees

Raphaël Clifford

Slides by Benjamin Sach

Dictionaries

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such that for any *key* there is at most one pair (*key*, *value*) in the dictionary.

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- Every lookup and every delete takes $O(1)$ *worst-case* time,
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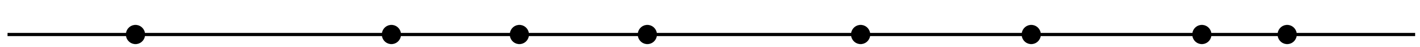

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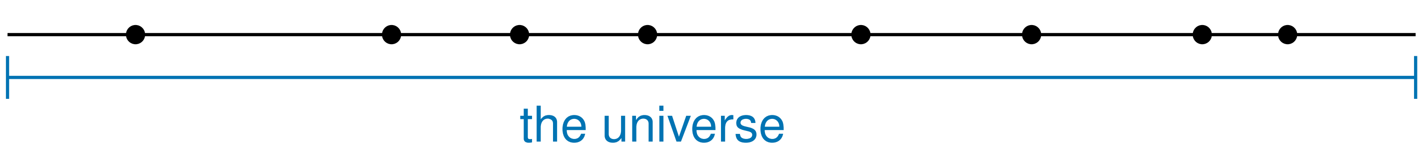

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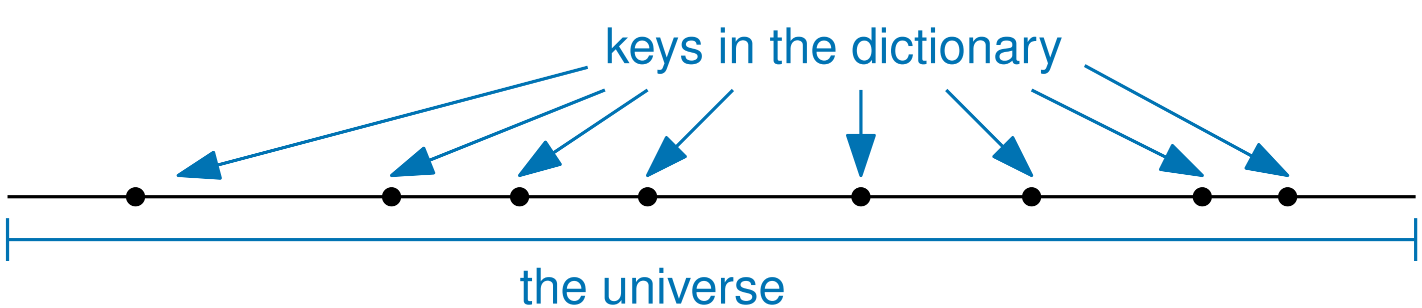

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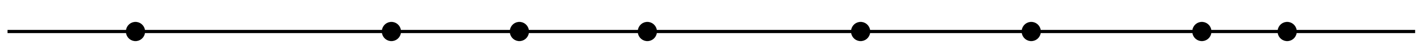

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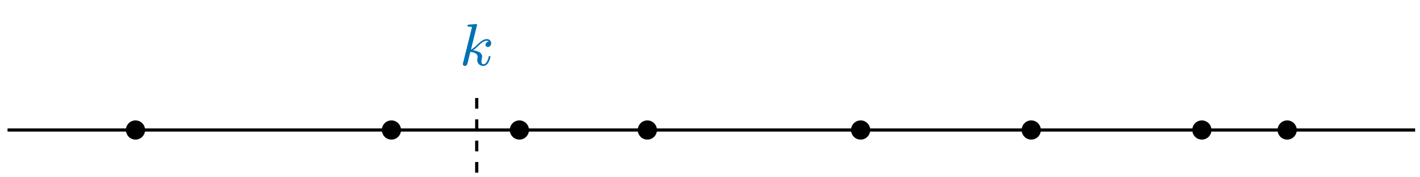

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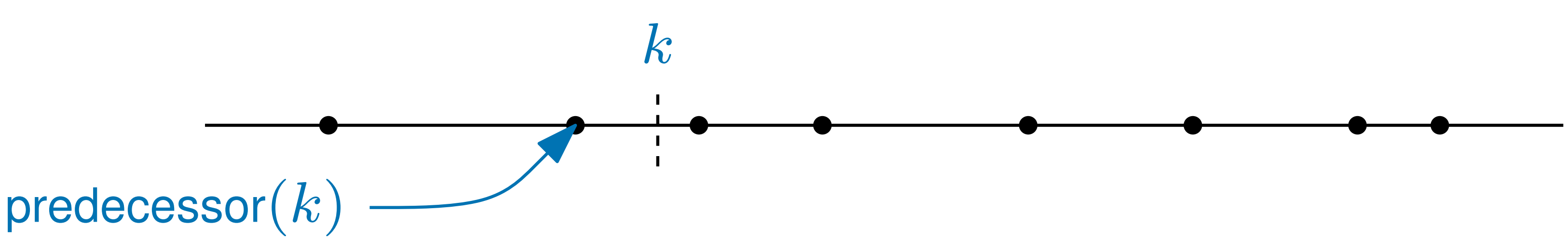

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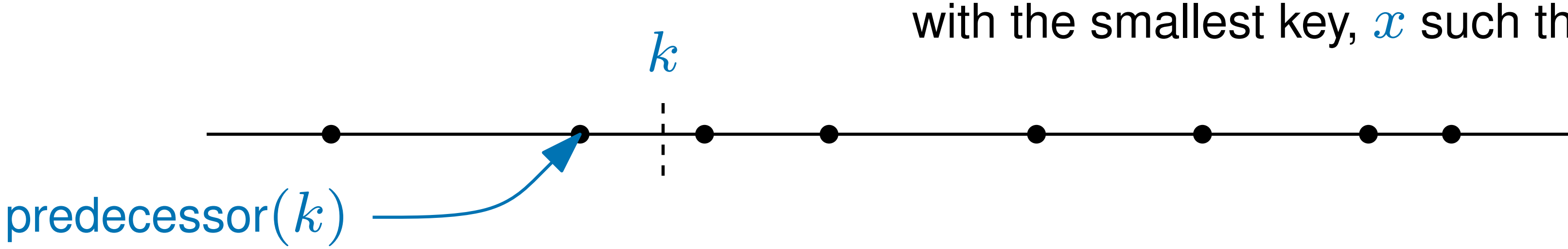

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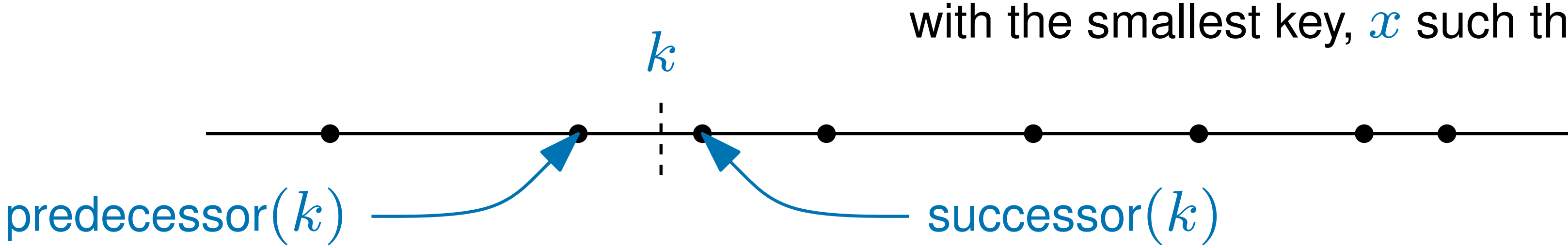

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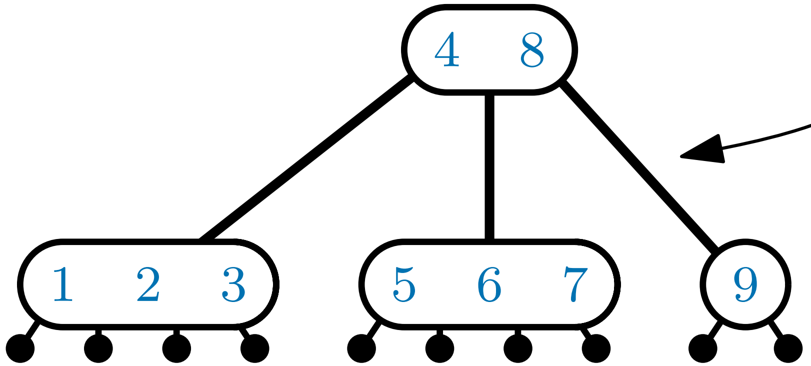
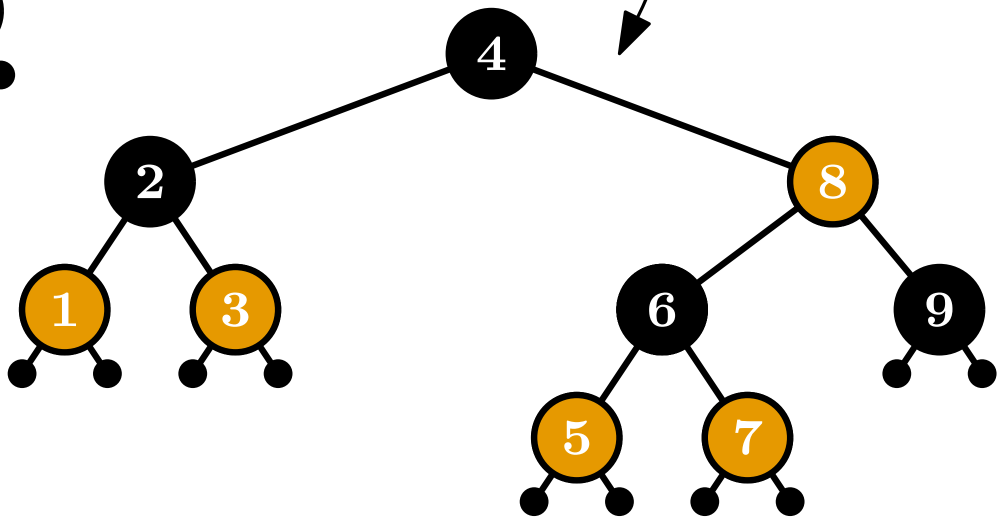
These are very natural operations that the **Hashing**-based solutions
that we have seen are very unsuited to

What could we use instead?

We could use a self-balancing binary search tree. . .
like a 2-3-4 tree, a red-**black** tree or an AVL tree





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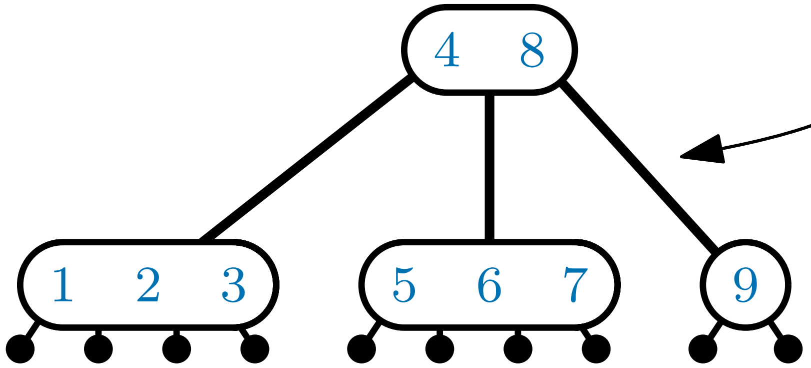

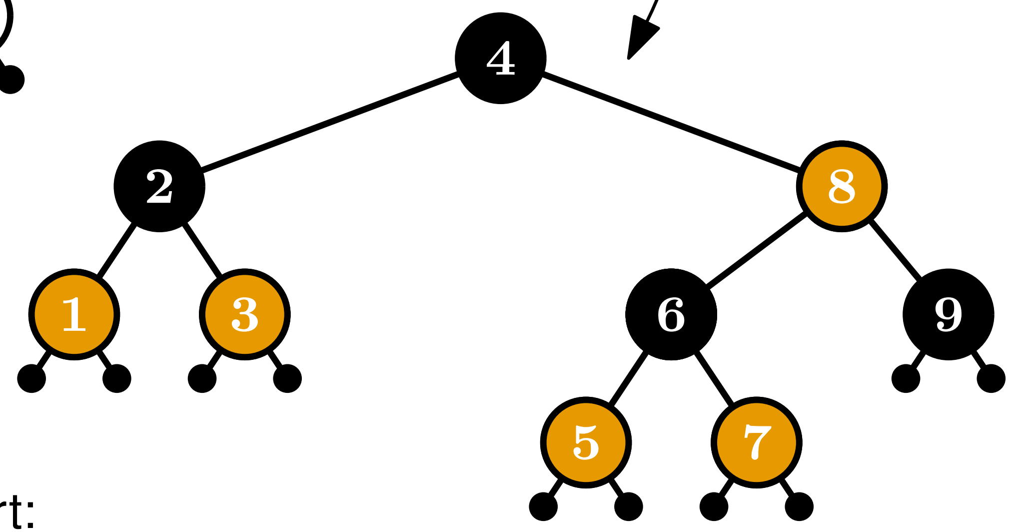

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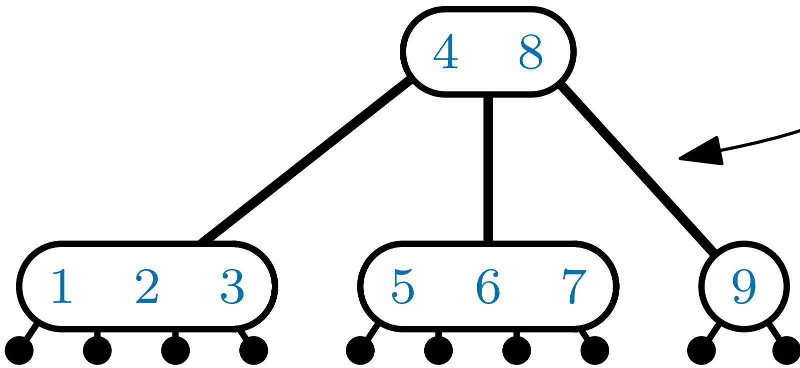

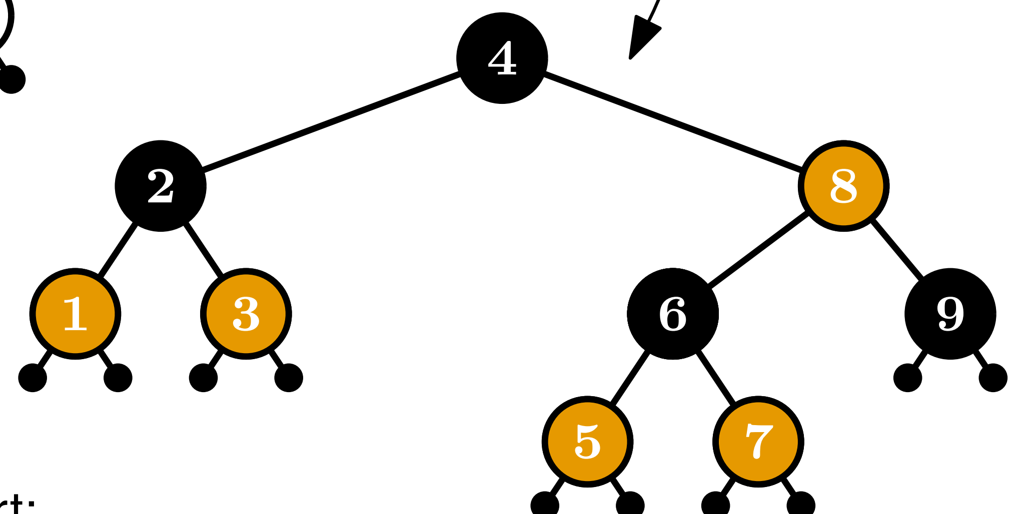

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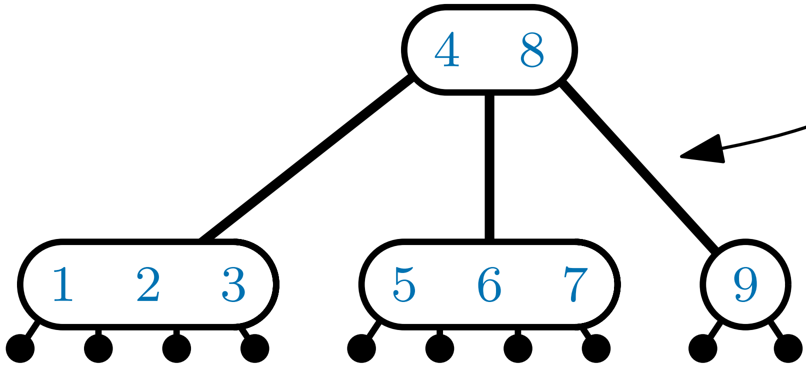

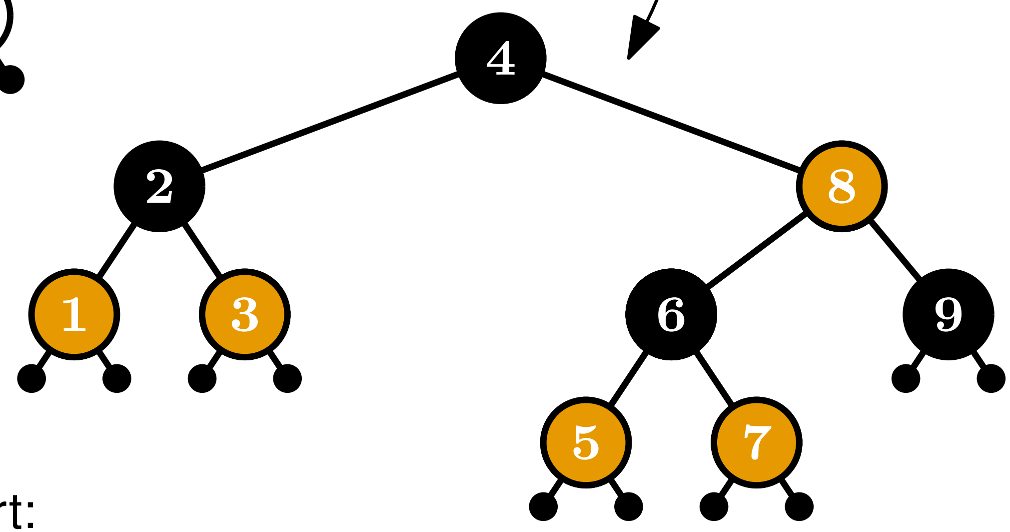

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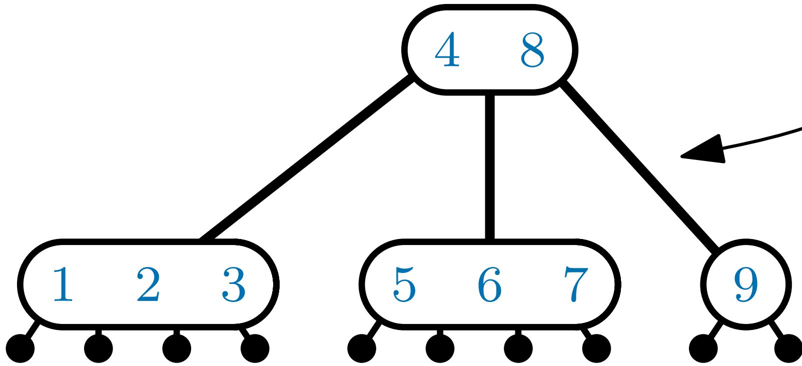

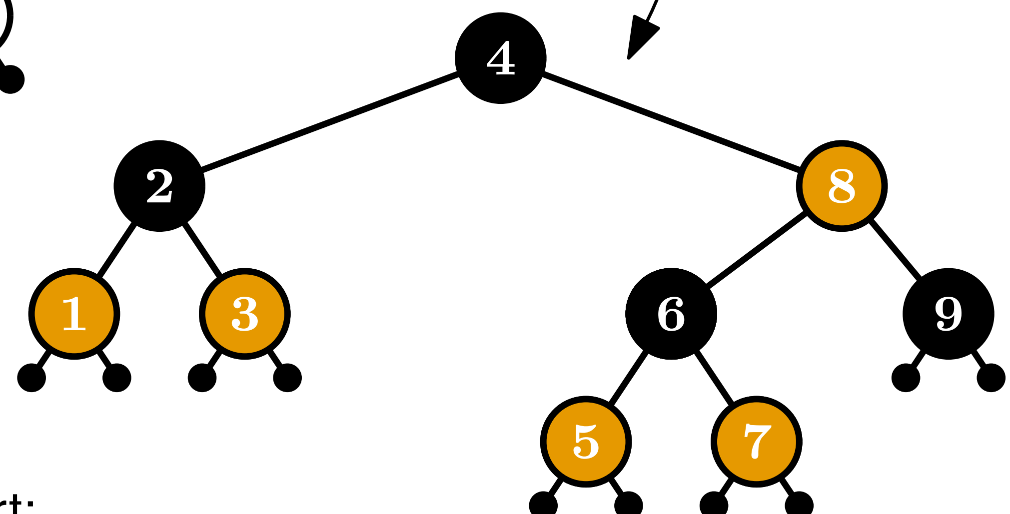

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they are also *deterministic*

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In this lecture, we will see the **van Emde Boas (vEB) tree**
which stores a set S of integer keys from a universe $U = \{1, 2, 3, 4 \dots u\}$ (i.e. $u = |U|$).

Five operations will be supported:

add(x)	Insert the integer x into S (where $x \in U$)
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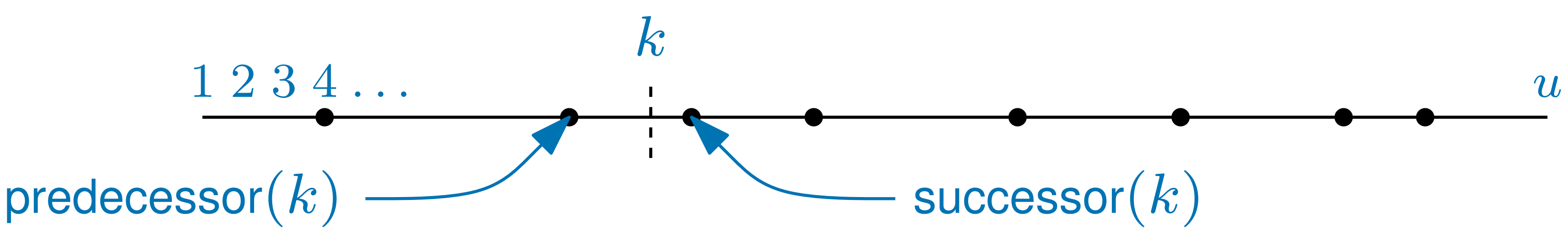

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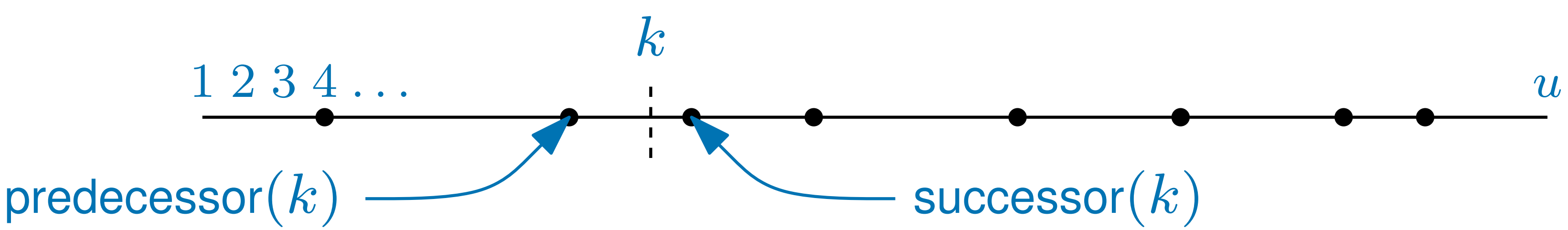

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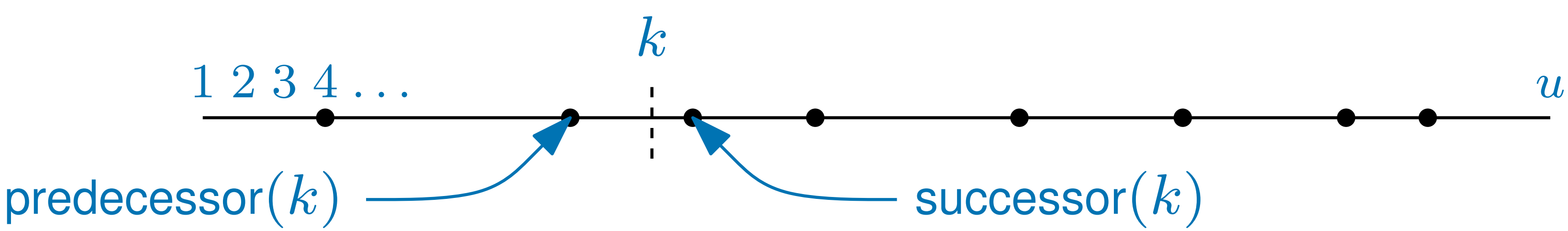

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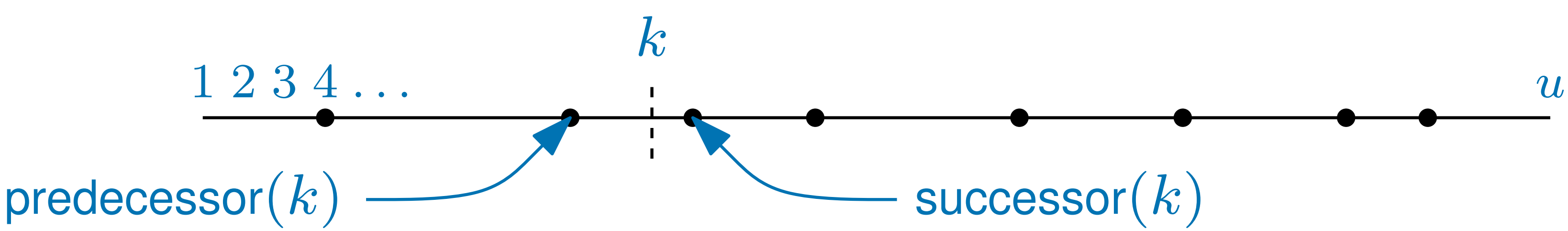

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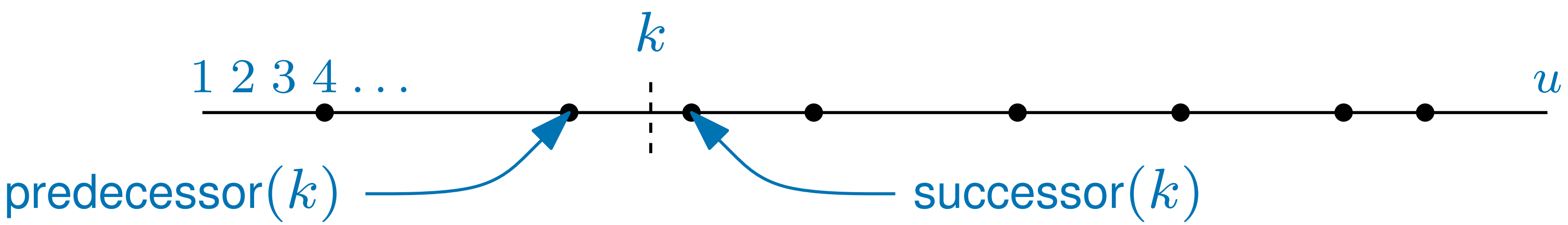
add(x)	Insert the integer x into S (where $x \in U$)
lookup(x)	Return yes if x is in S, or no otherwise.
delete(x)	Remove x from S
predecessor(k)	Return the largest integer x in S such that $x \leqslant k$
successor(k)	Return the smallest integer x in S such that $x \geqslant k$



van Emde Boas Trees

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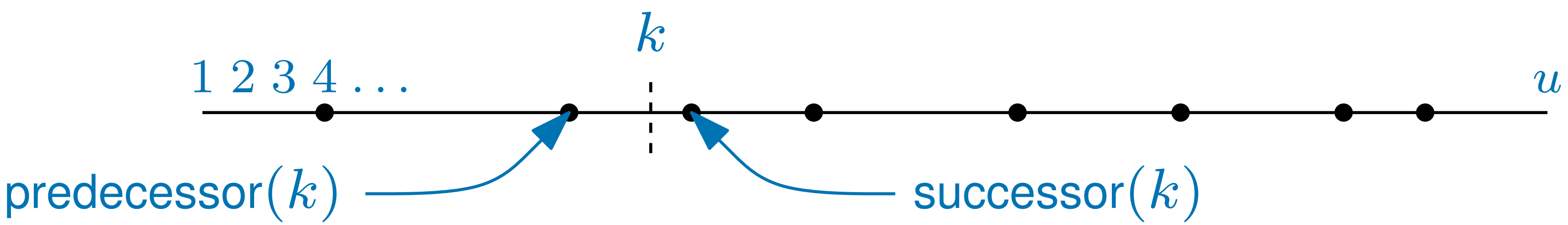

All operations will take $O(\log \log u)$ worst case time

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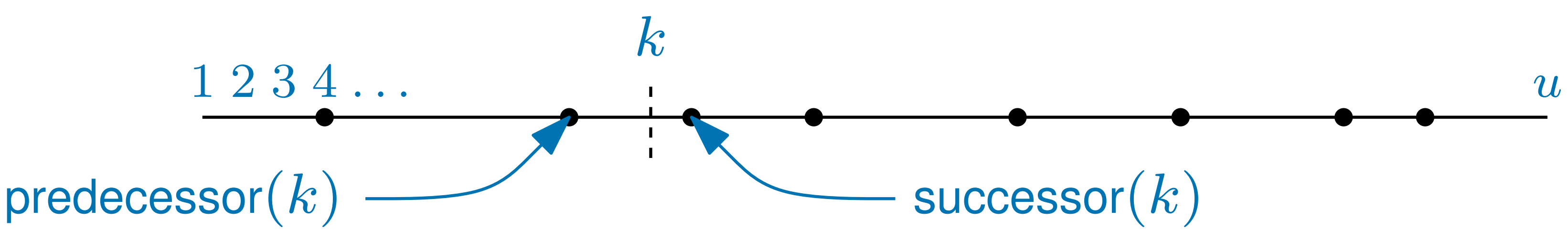

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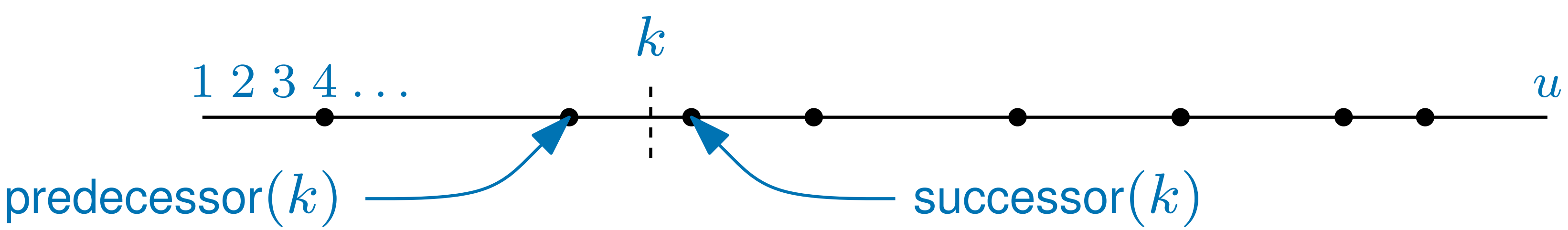

All operations will take $O(\log \log u)$ worst case time and the space used is $O(u)$

and it is a deterministic data structure

Example: If $U = \{1, 2, 3, 4 \dots 100 \cdot n\}$, you get $O(\log \log n)$ time and $O(n)$ space

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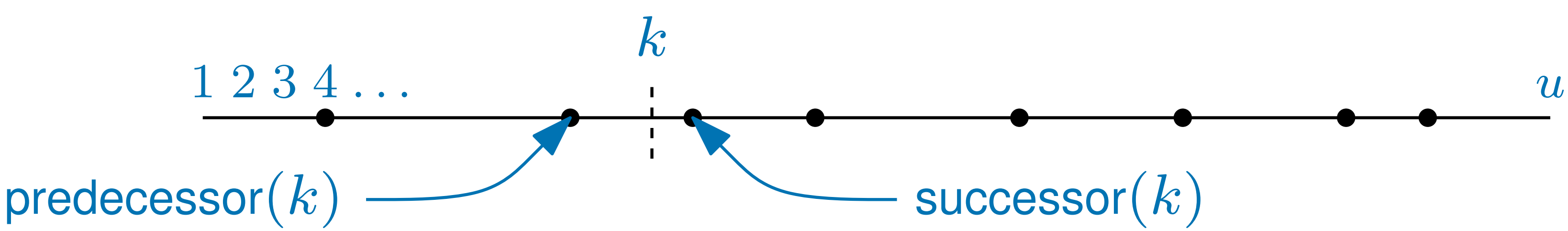

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Example: If $U = \{1, 2, 3, 4 \dots n^2\}$, you get $O(\log \log n)$ time and $O(n^2)$ space

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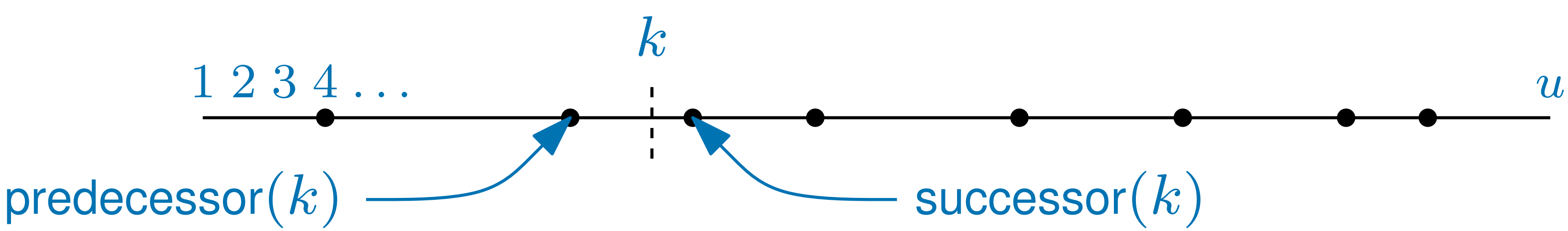

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Example: If $U = \{1, 2, 3, 4 \ldots n^3\}$, you get $O(\log \log n)$ time and $O(n^3)$ space

Attempt 1: a big array

Build an array of length u...

A	0	1	1	1	1	0	1	0	0	0	0	0	0	1	1	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

$\underbrace{\hspace{10em}}_{u}$

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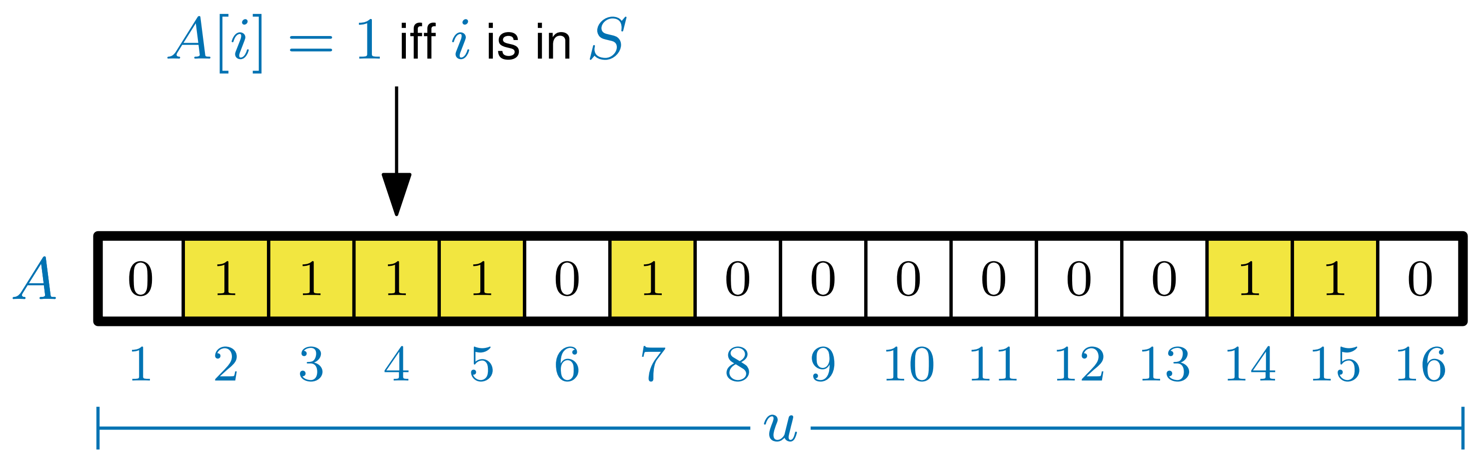

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Build an array of length u...

$A[i] = 1$ iff i is in S

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The operations add, delete and lookup all take $O(1)$ time.

Attempt 1: a big array

Build an array of length u...

add(12)

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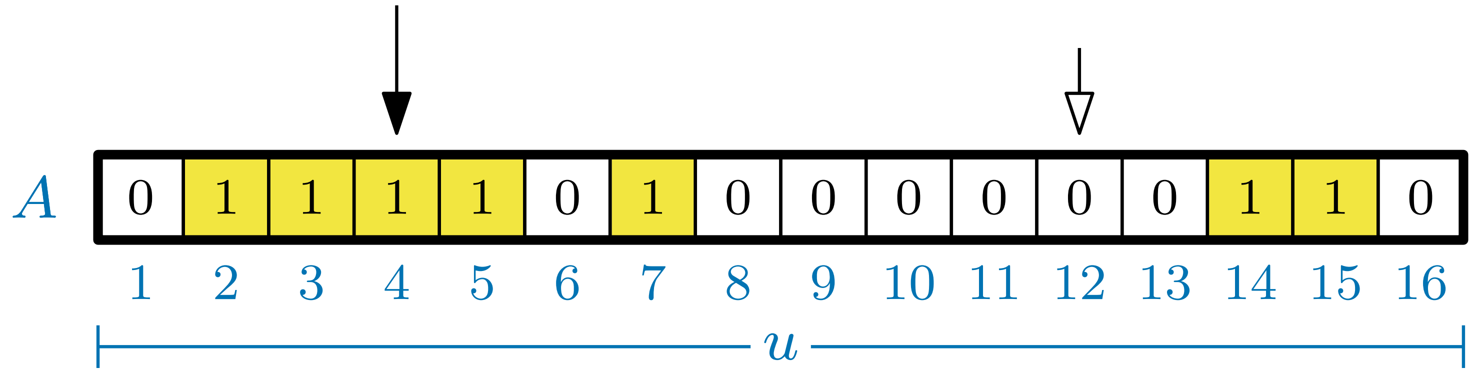

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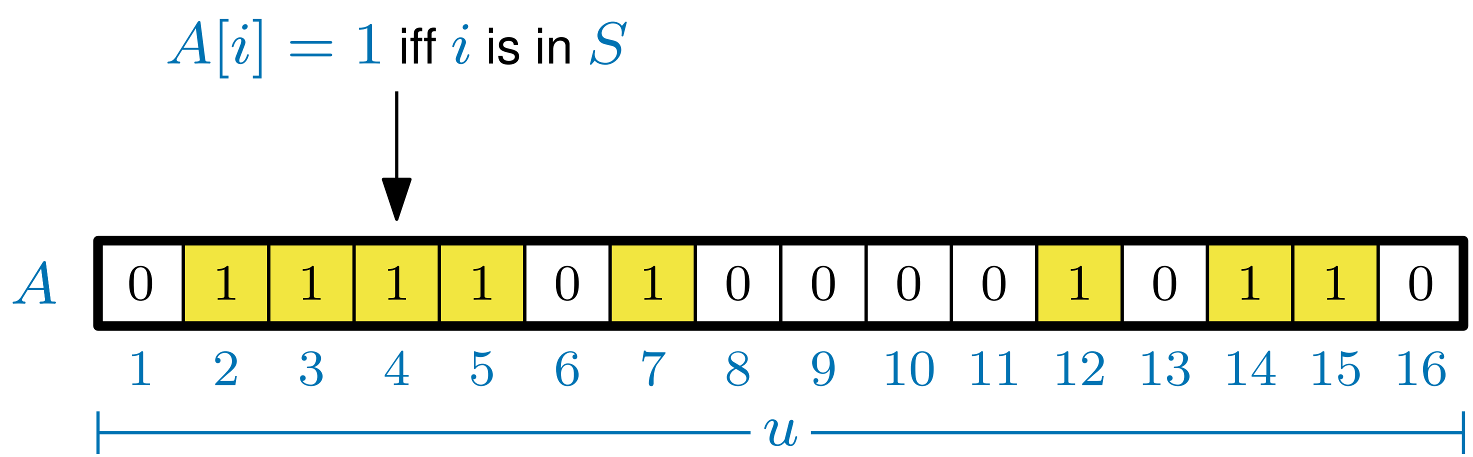

The operations add, delete and lookup all take $O(1)$ time.

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Build an array of length u...

delete(14)

$A[i] = 1$ iff i is in S

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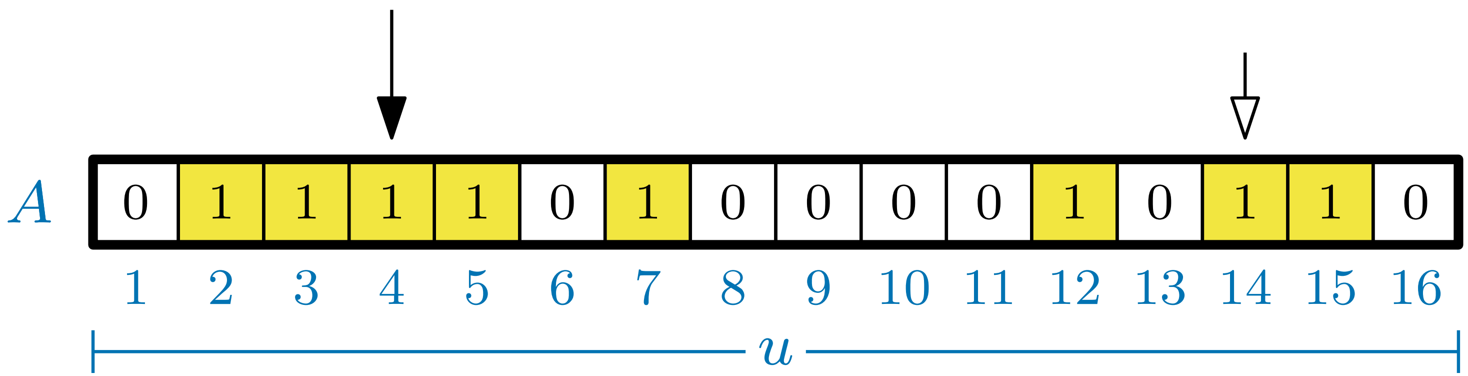

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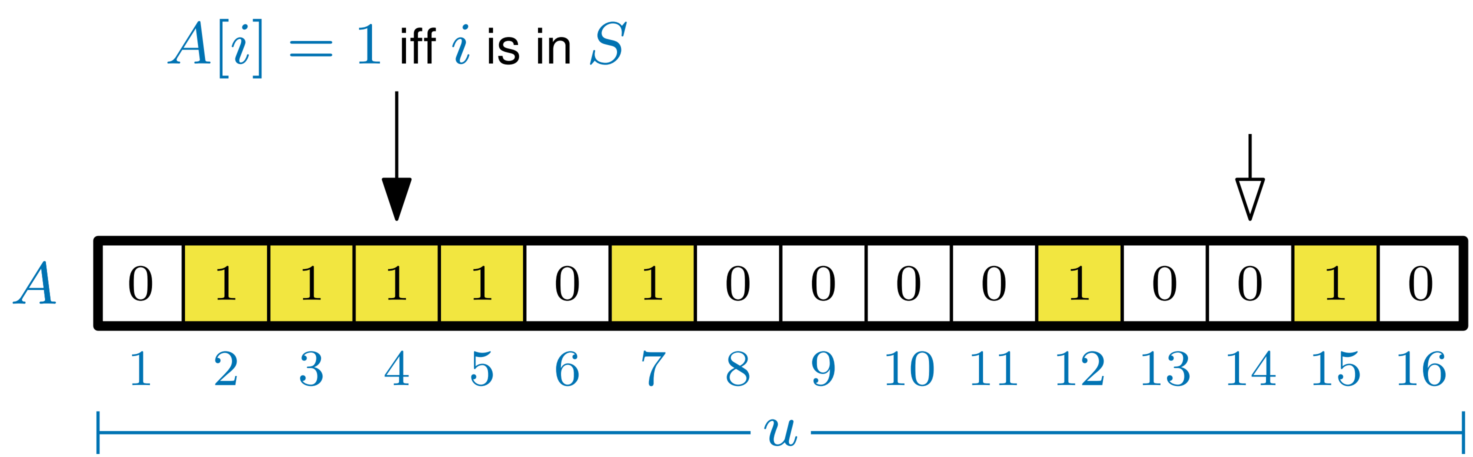

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The operations add, delete and lookup all take $O(1)$ time.

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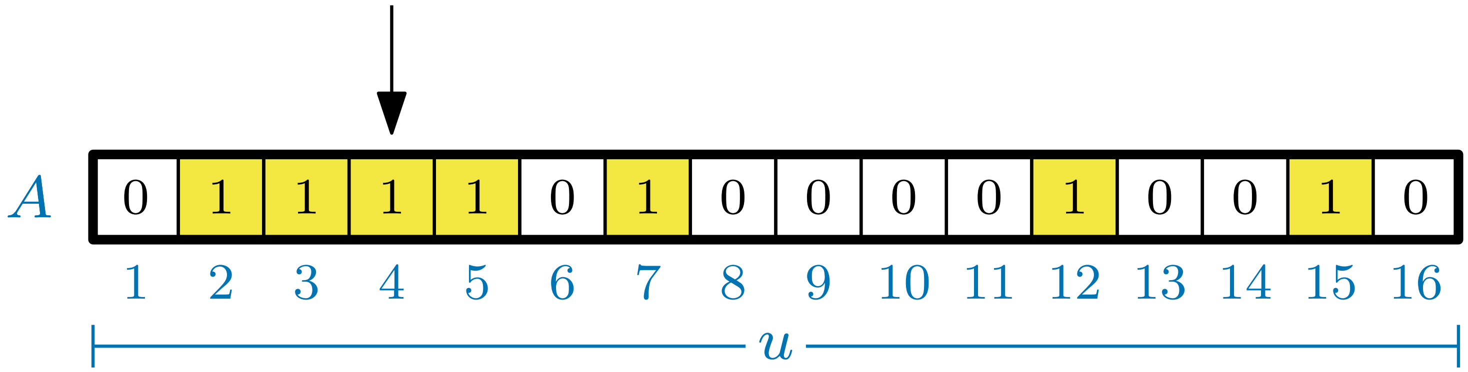
Build an array of length u...

lookup(11)

$A[i] = 1$ iff i is in S



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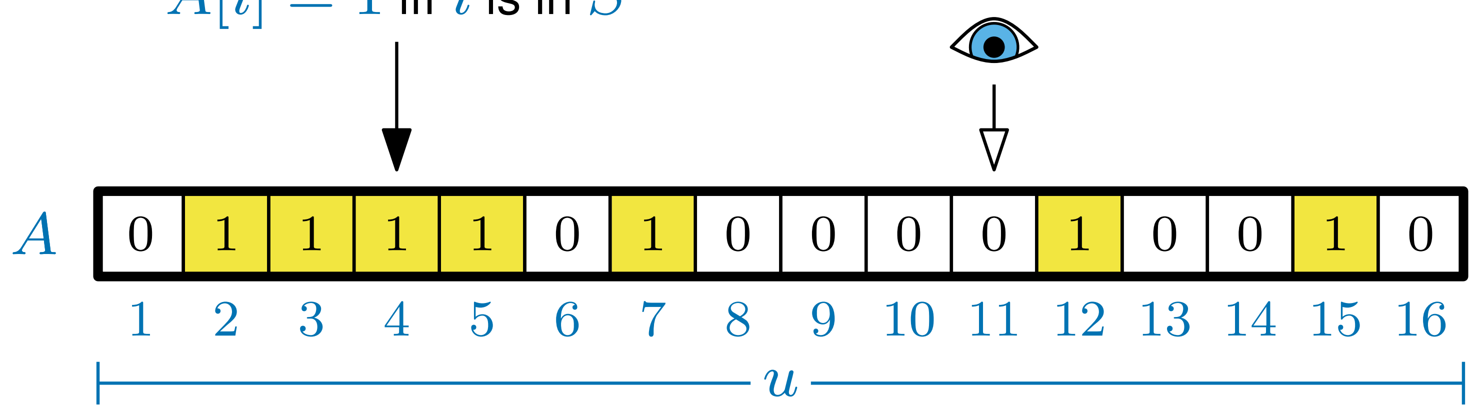

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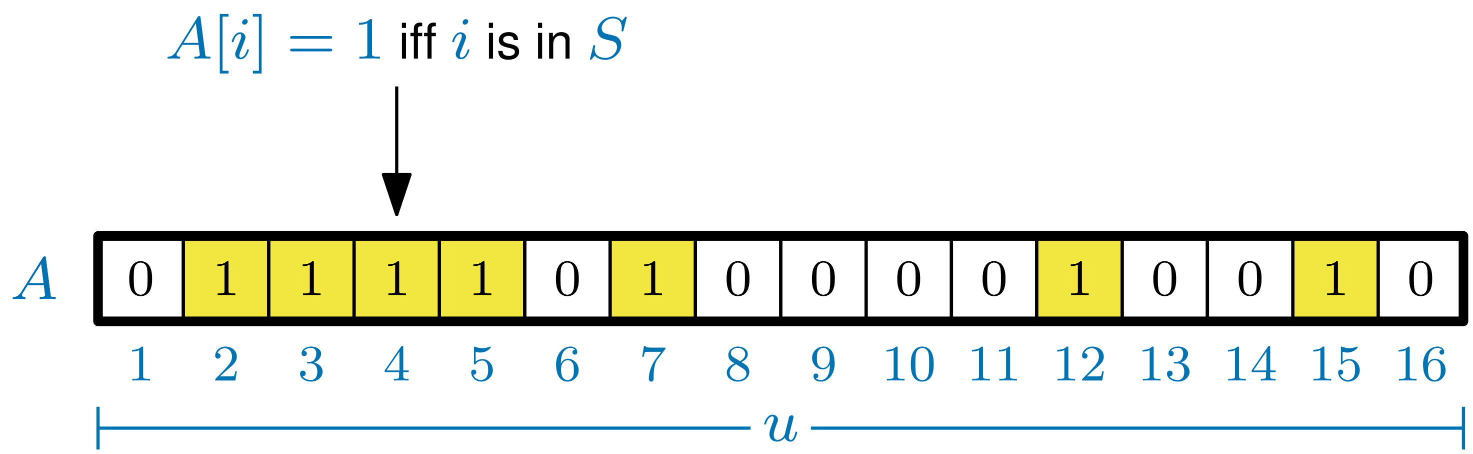

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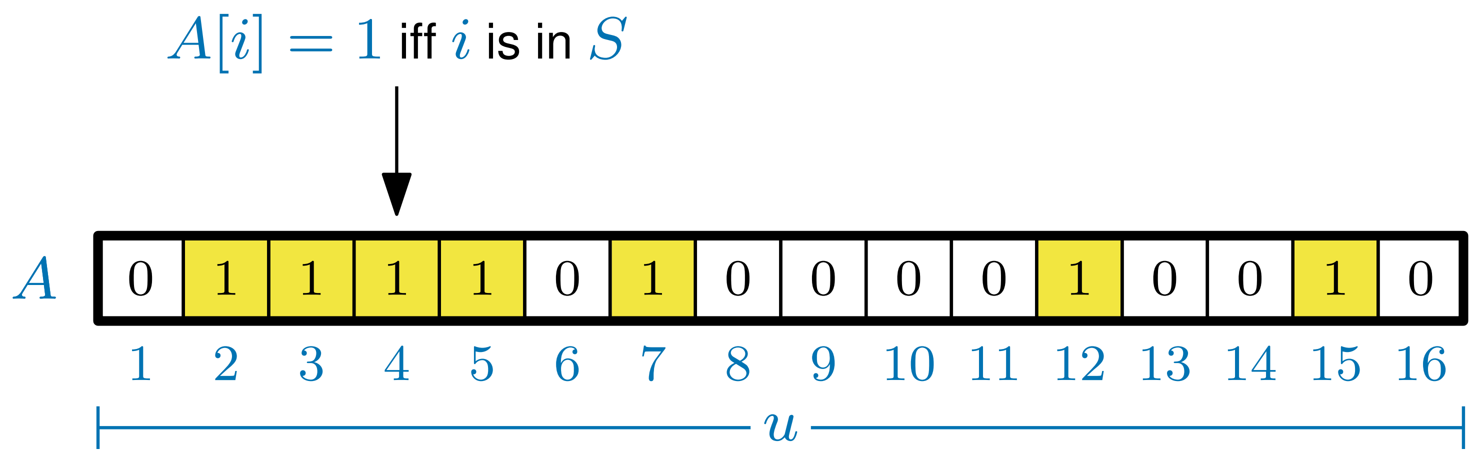

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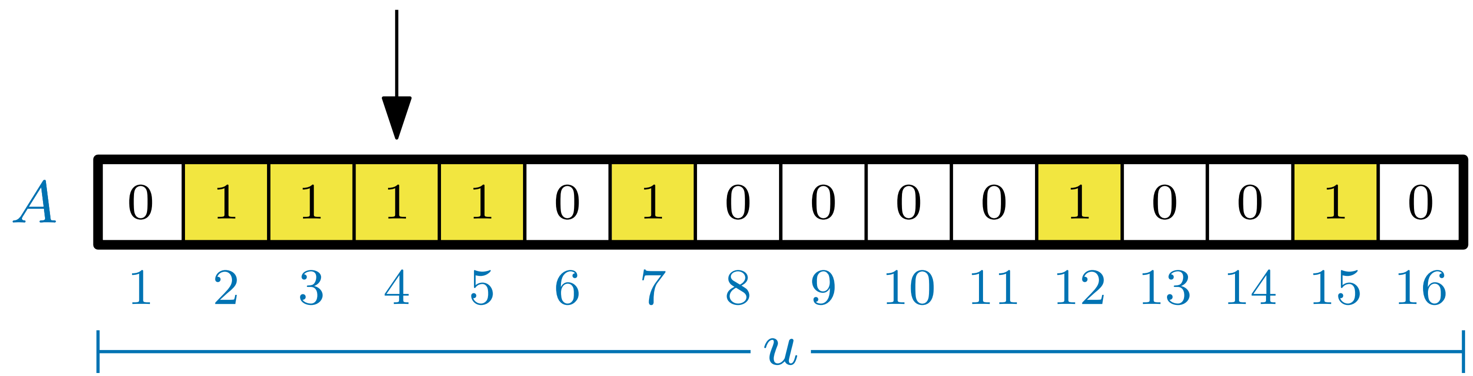

The operations add, delete and lookup all take $O(1)$ time.

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What about the predecessor operation?

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Build an array of length u. . .

predecessor(11)

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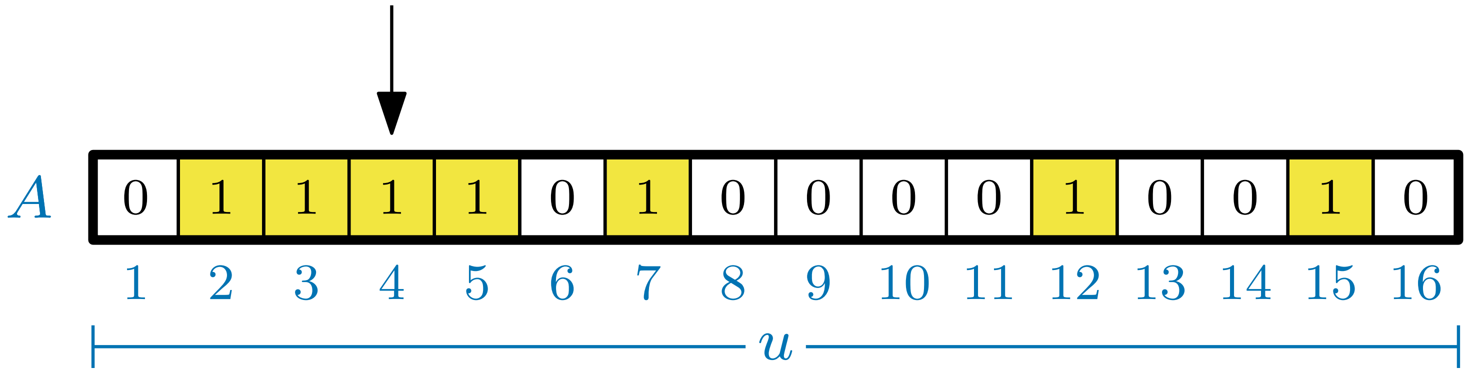

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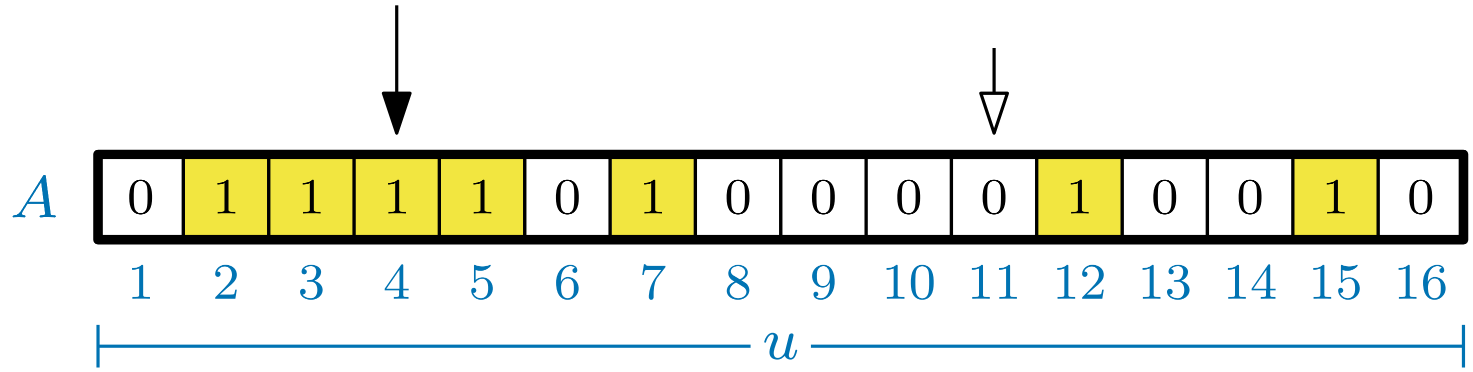

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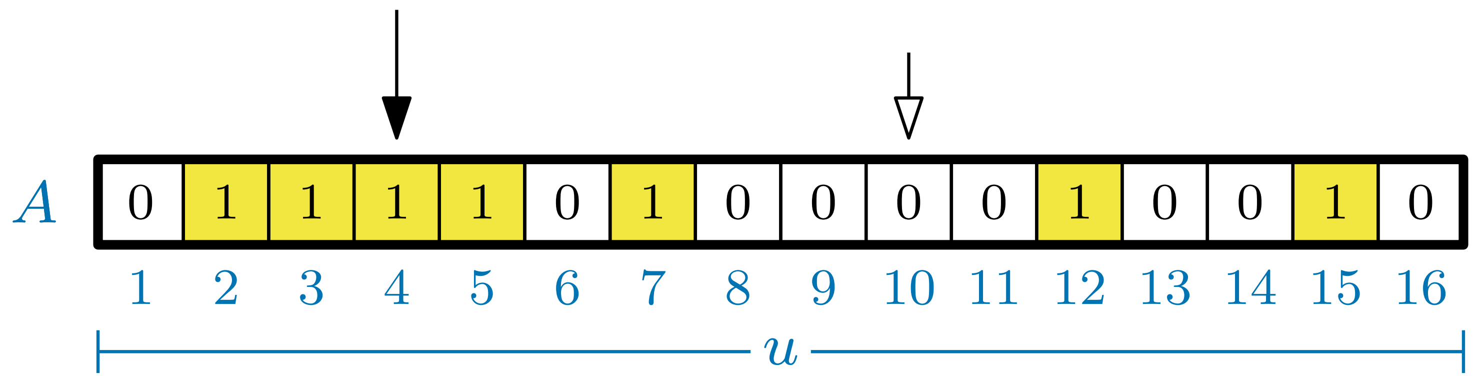

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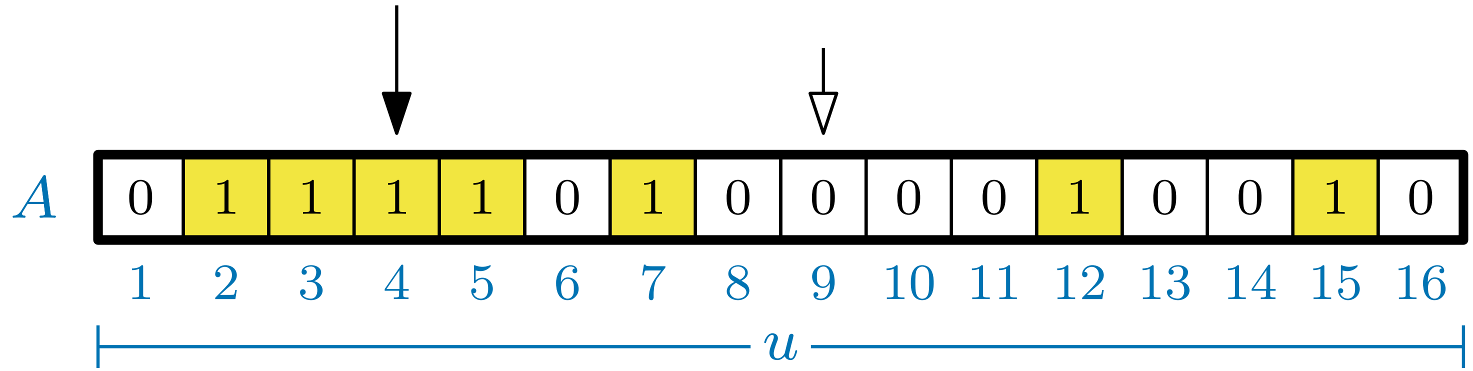

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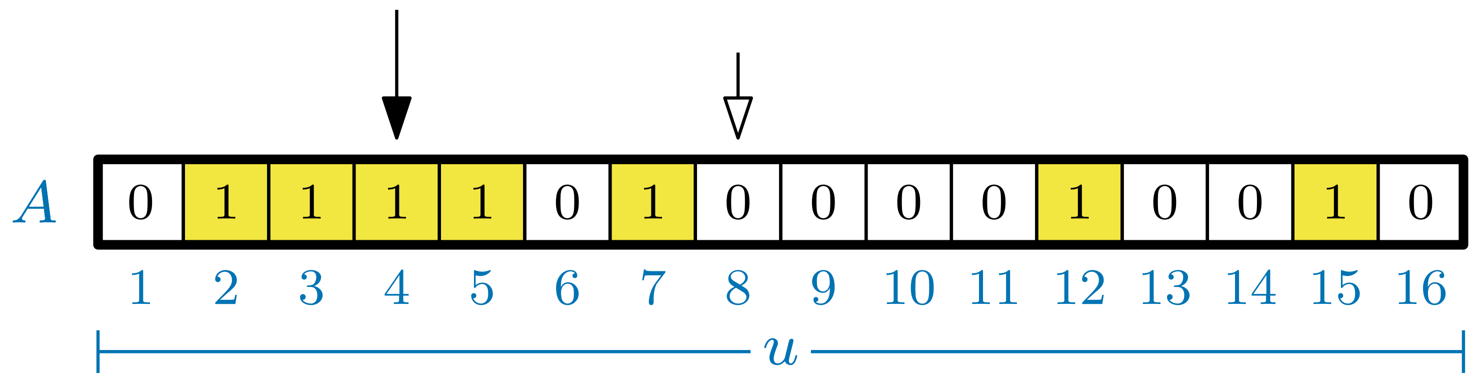

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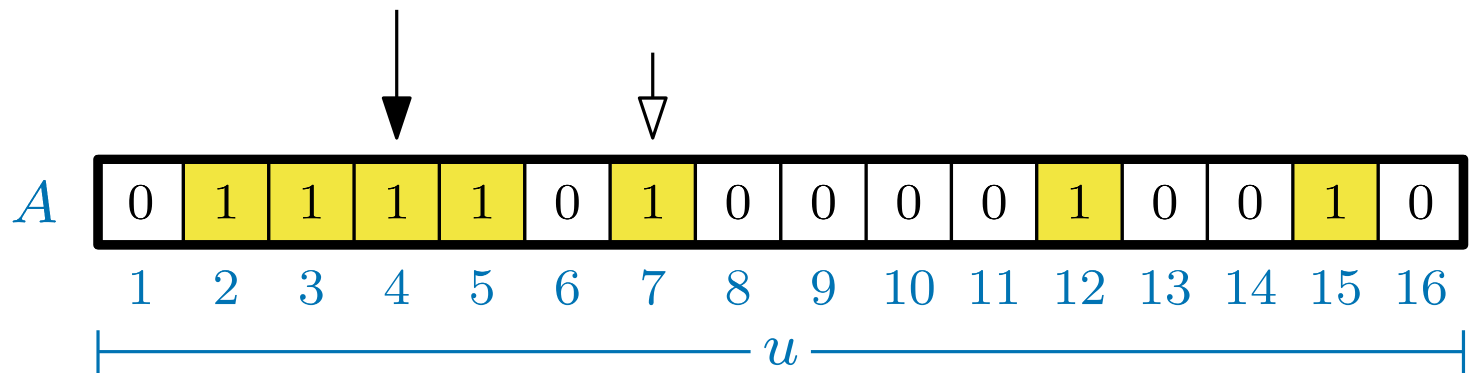

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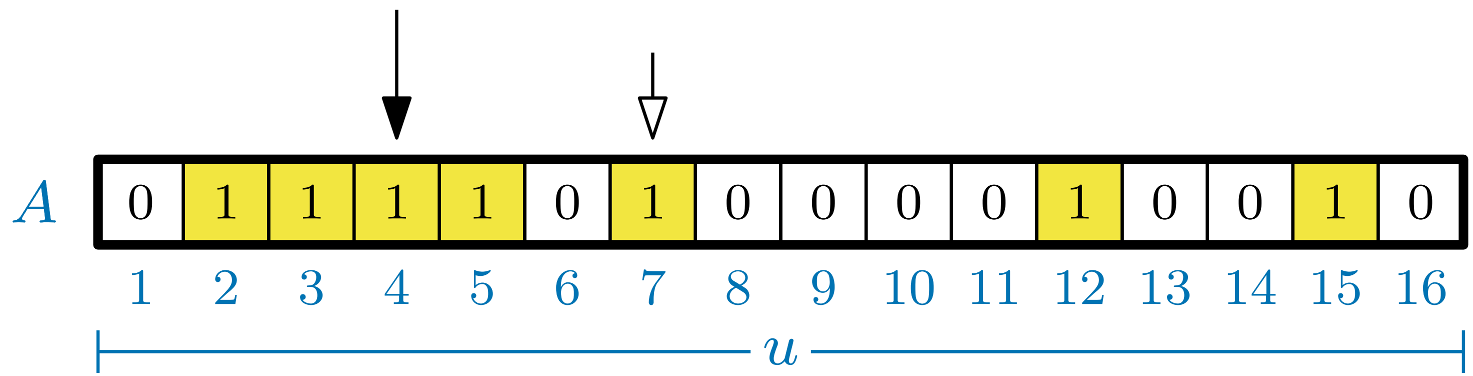

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The predecessor and successor operations take $O(u)$ time

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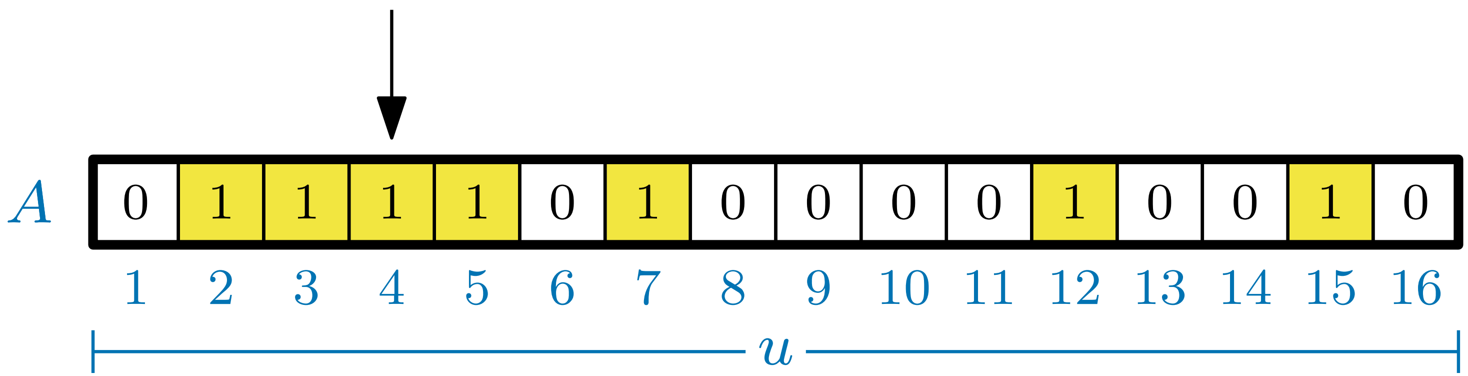

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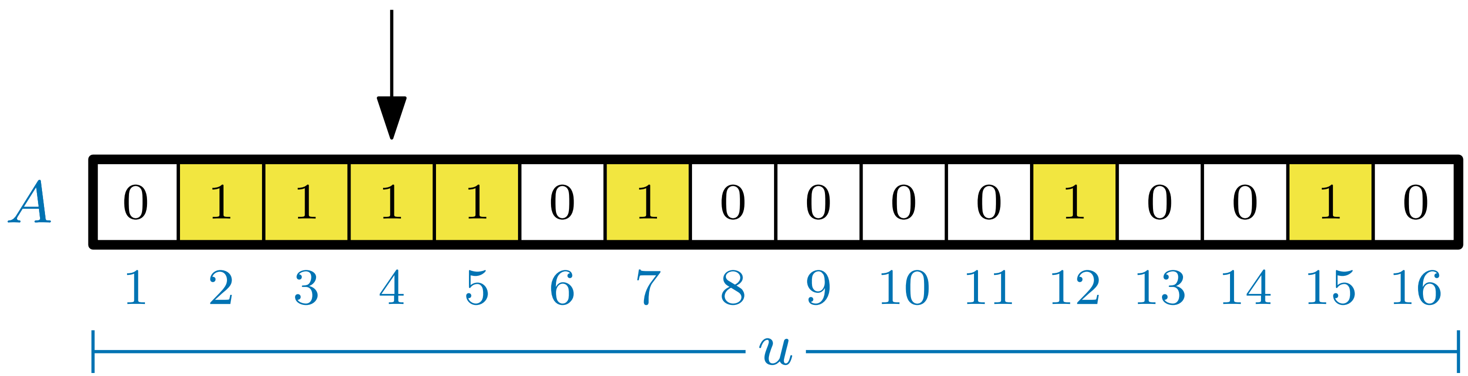

The operations add, delete and lookup all take $O(1)$ time.

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The predecessor and successor operations take $O(u)$ time

...not so good!

Attempt 2: a constant height tree

(on top of a big array)

A	0	1	1	1	1	0	1	0	0	0	0	1	0	0	1	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

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Split A into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits

Attempt 2: a constant height tree

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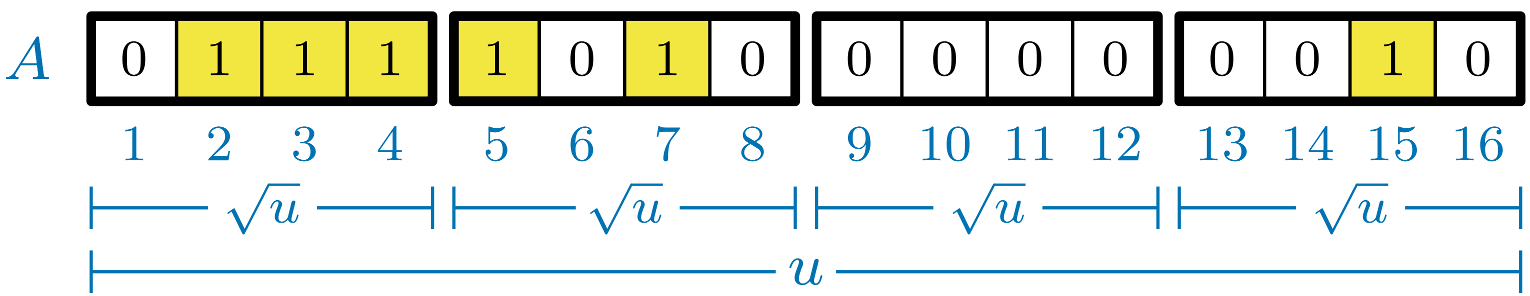

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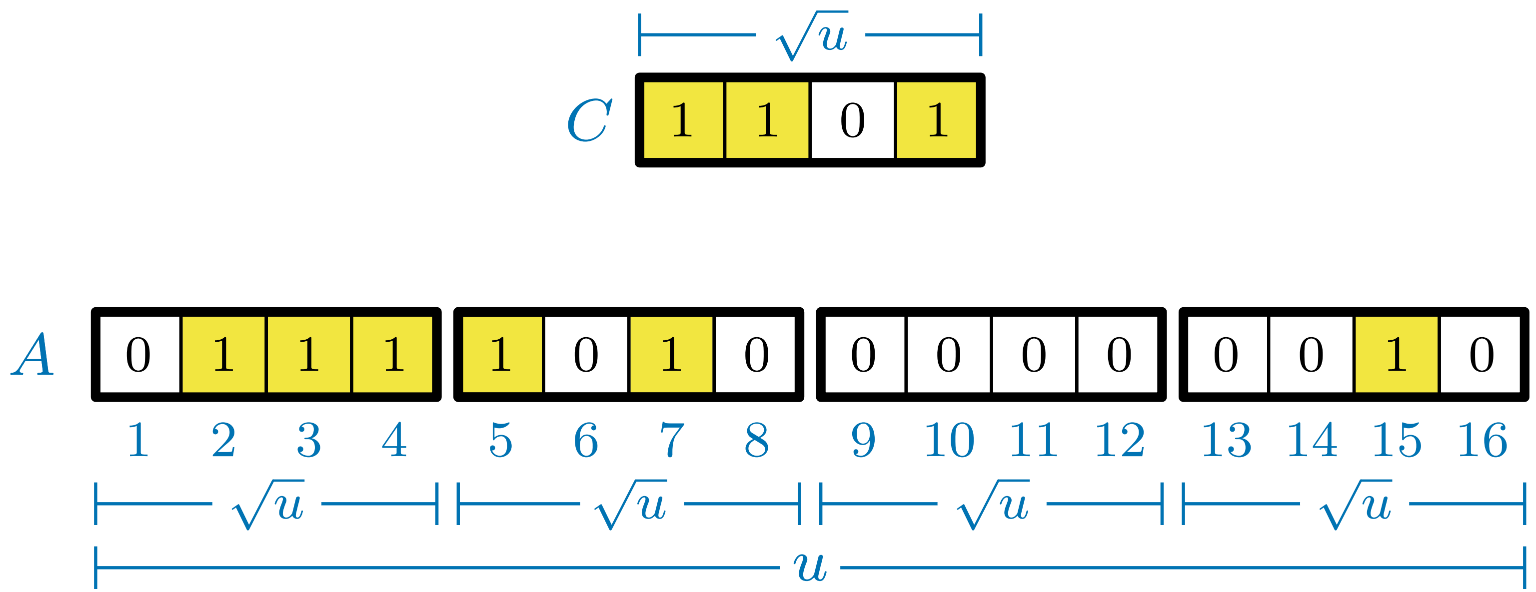

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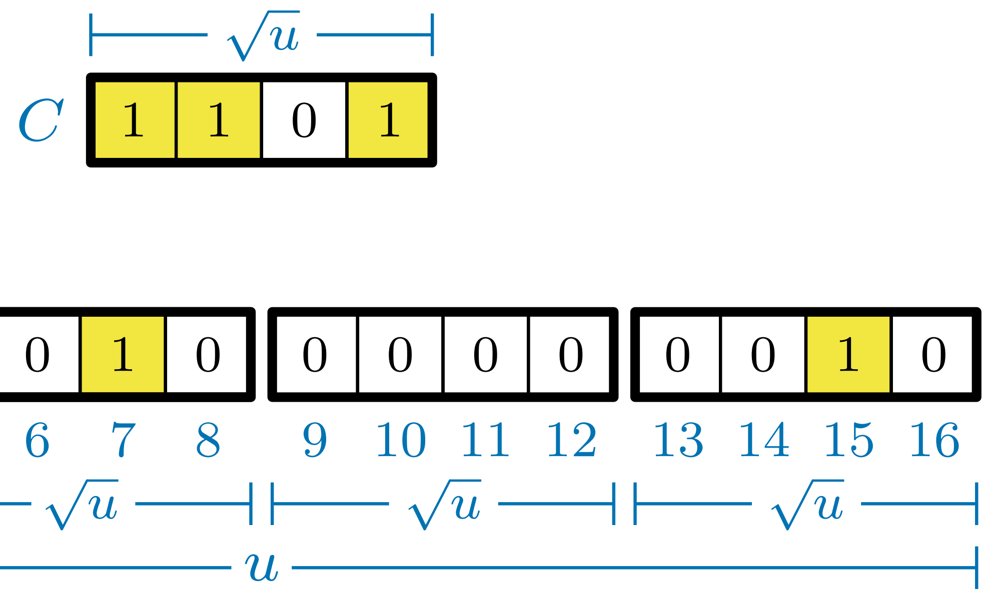
(on top of a big array)

C is called the *summary* of A



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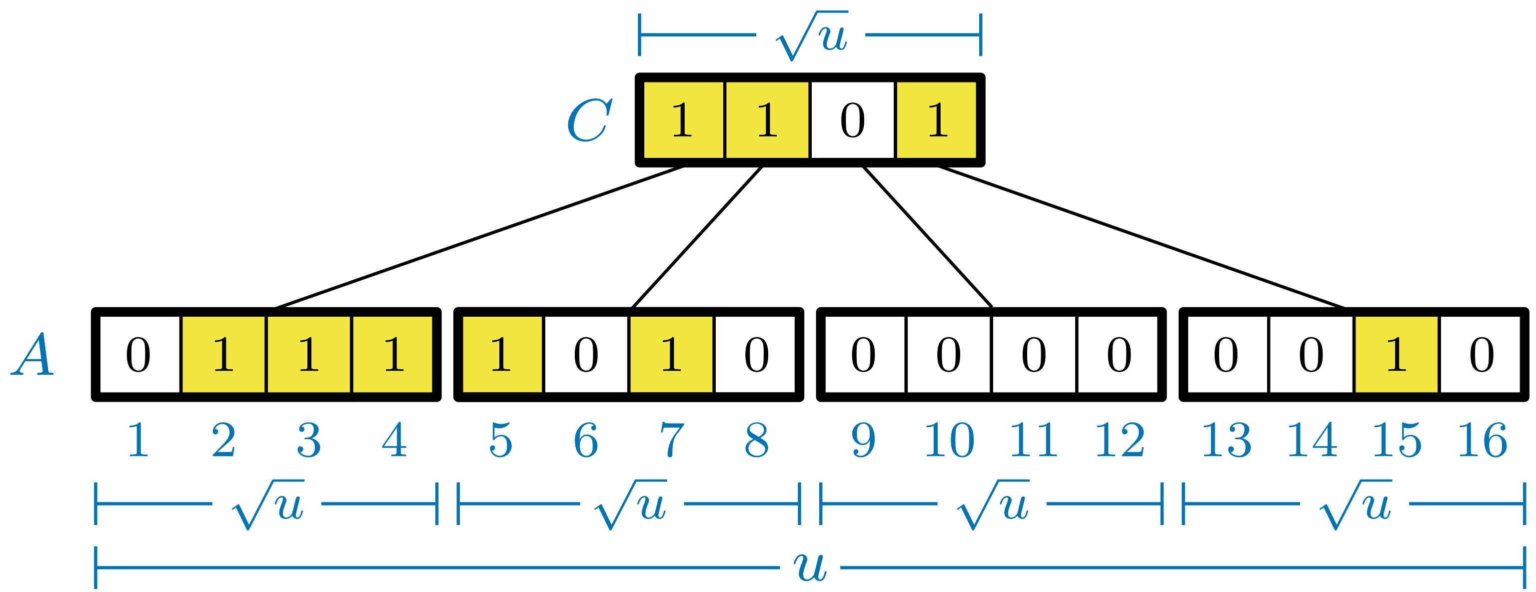

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this is 1 if
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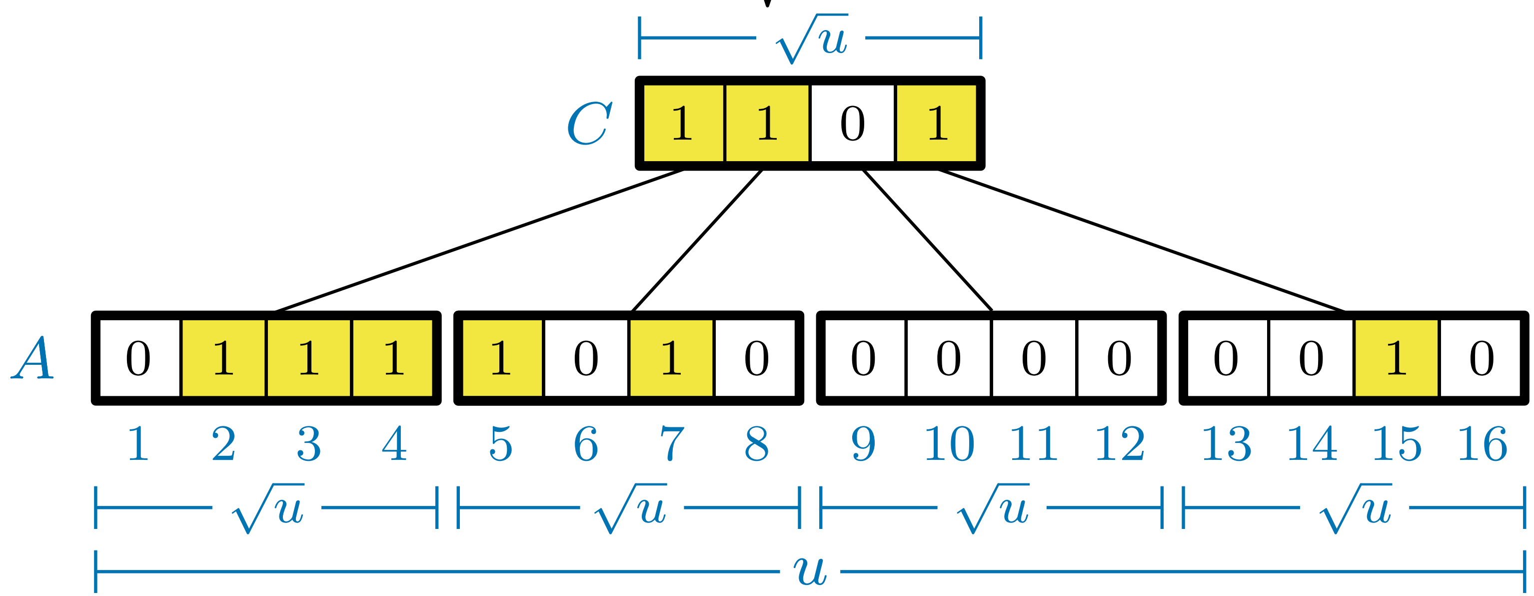

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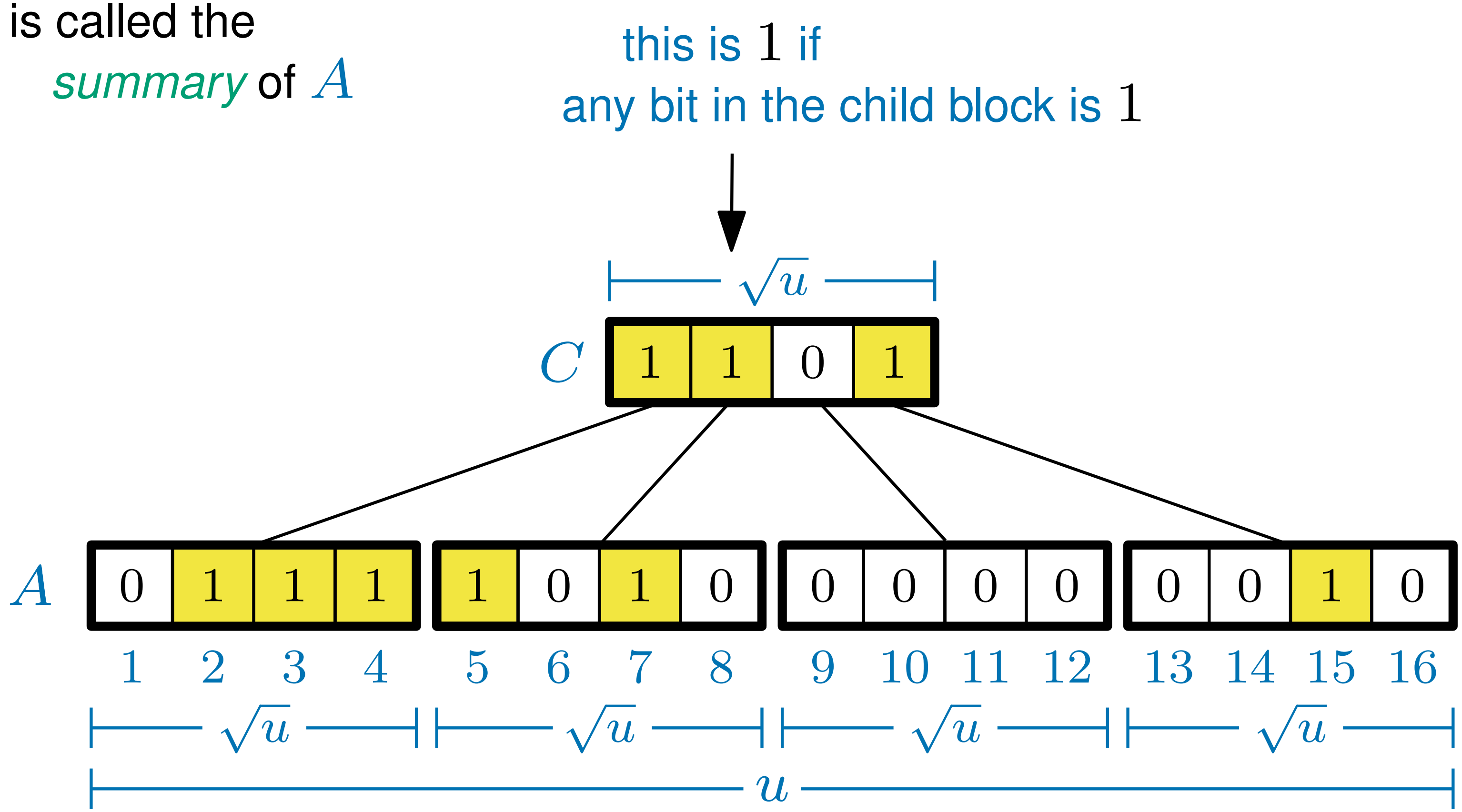

Split A into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

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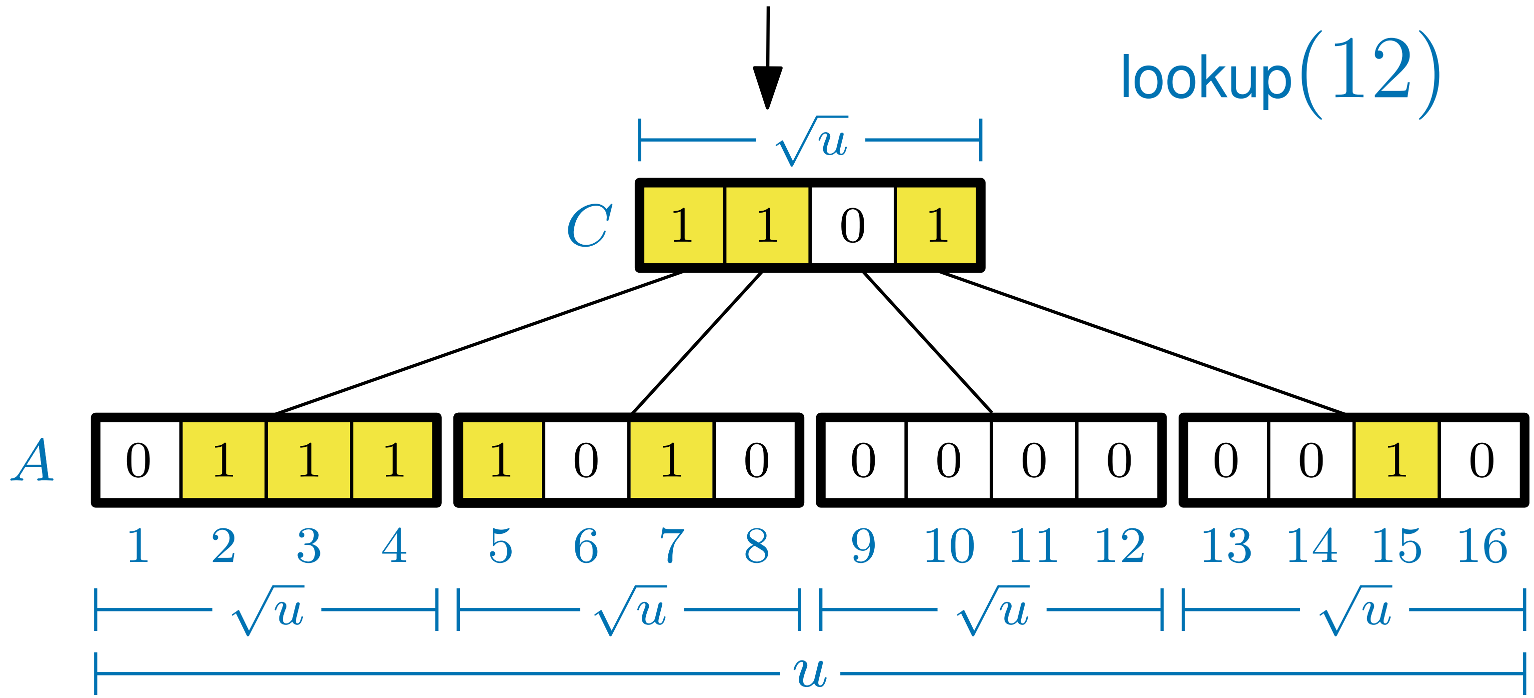
lookup(12)



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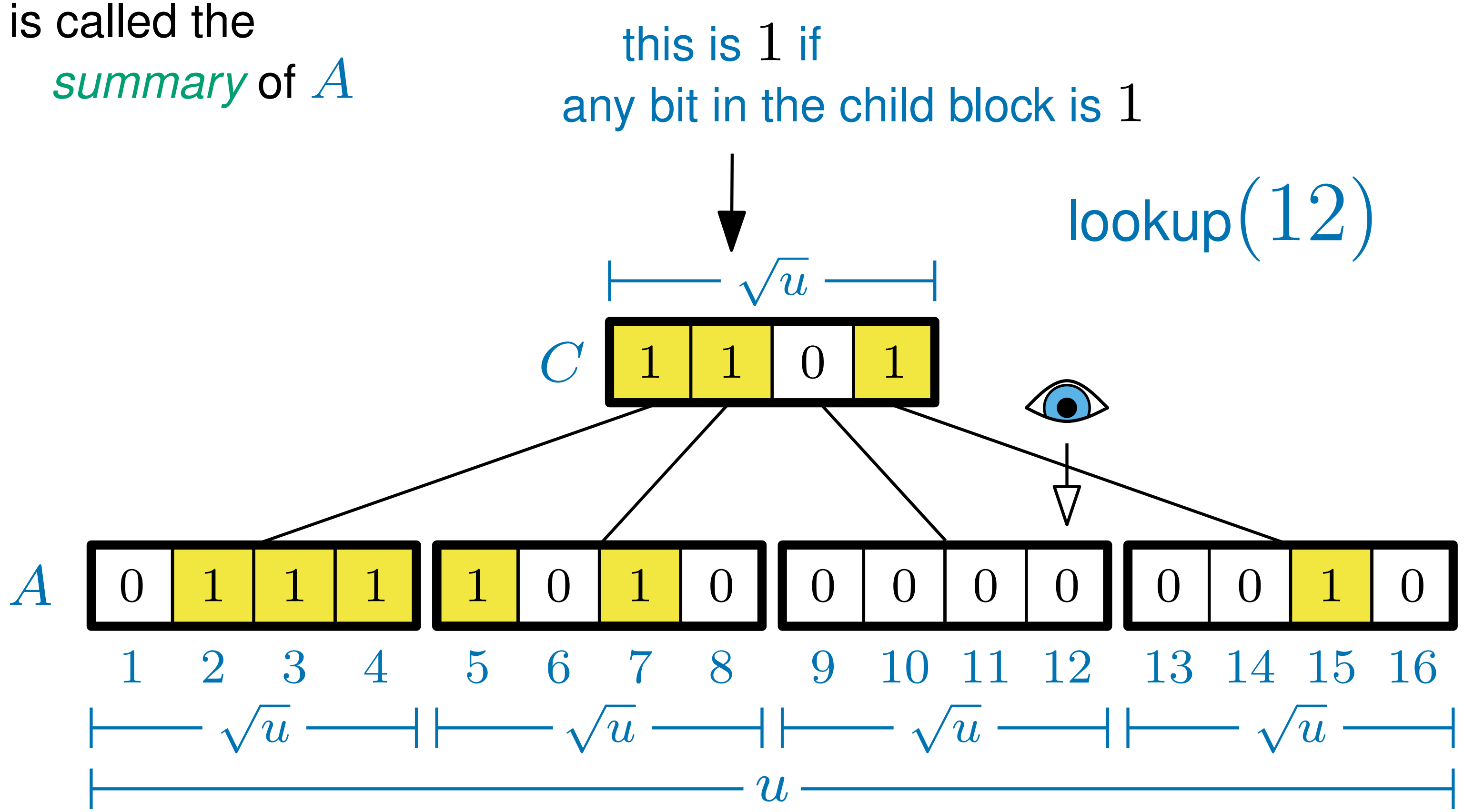

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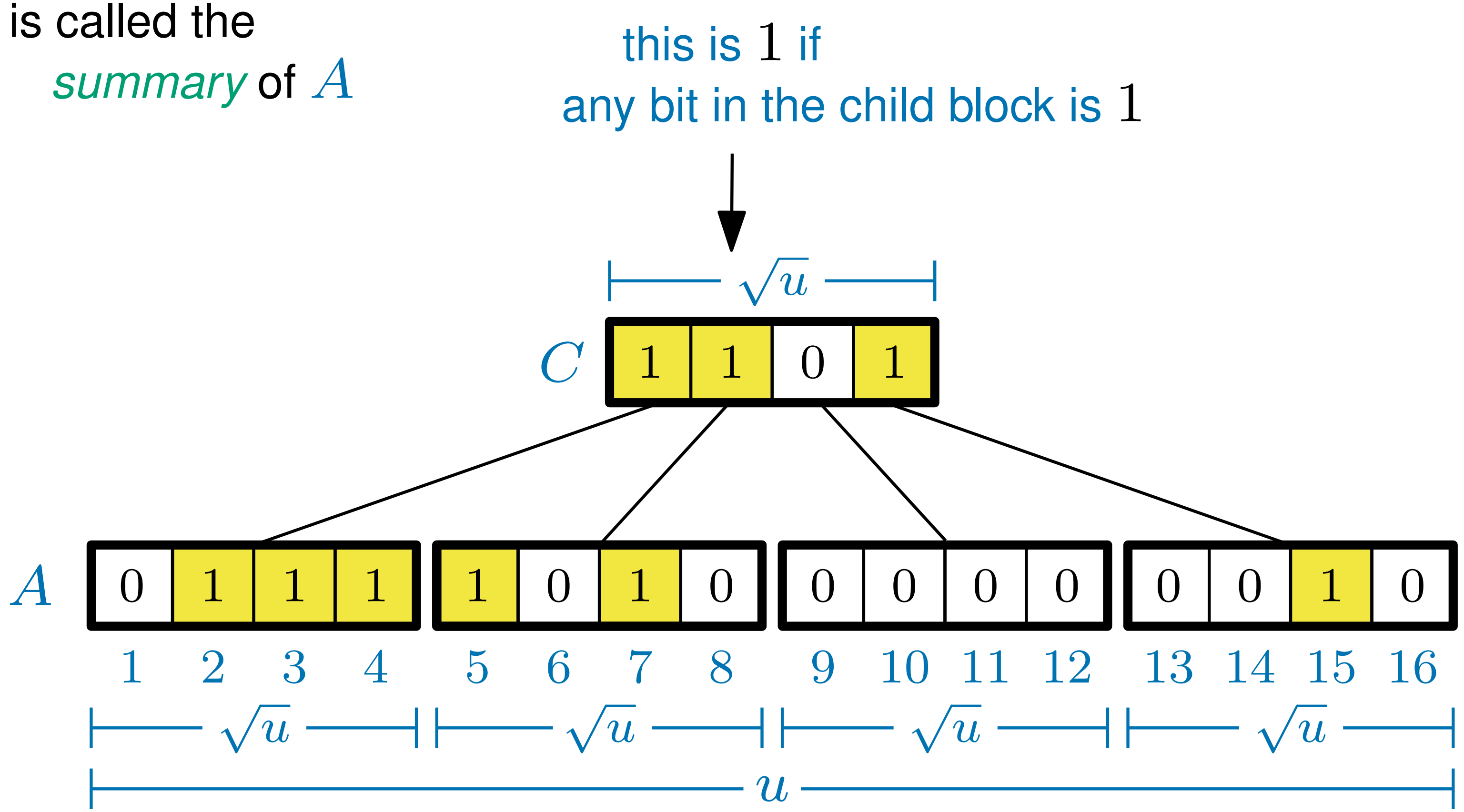

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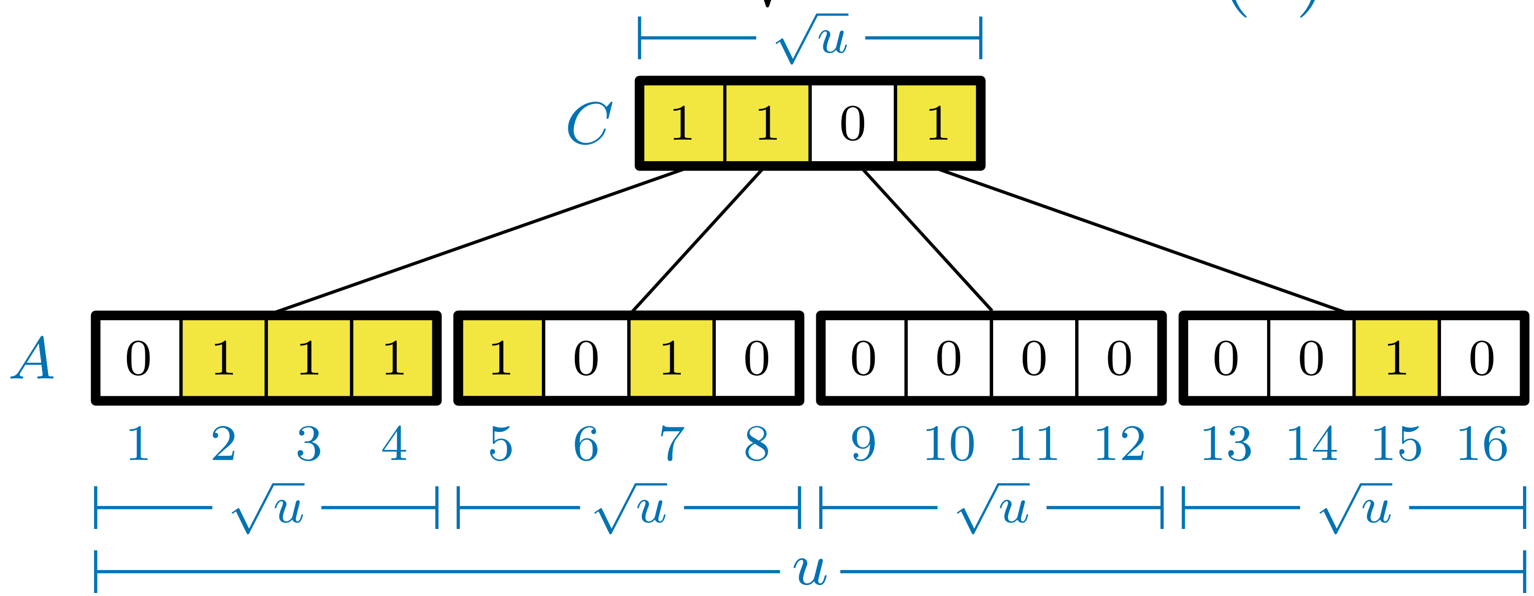
add(9)



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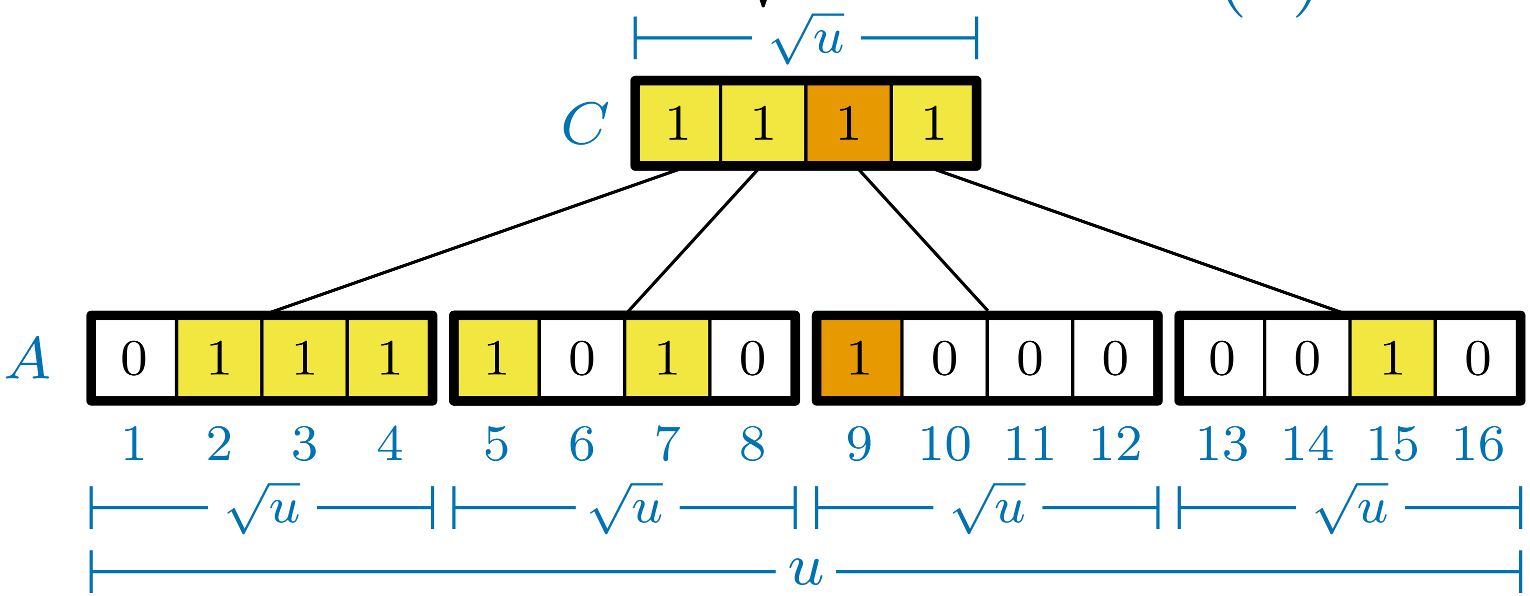
add(9)



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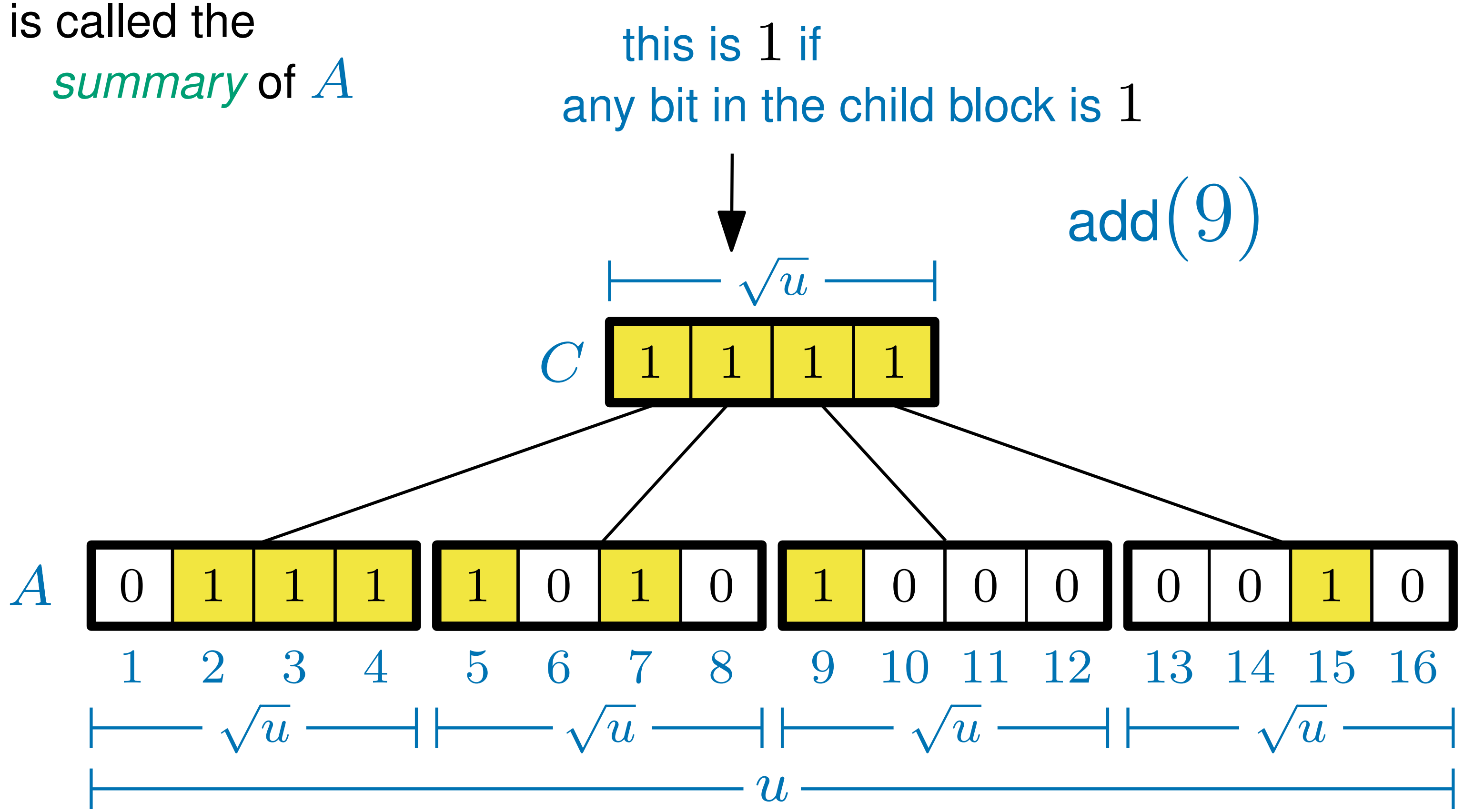

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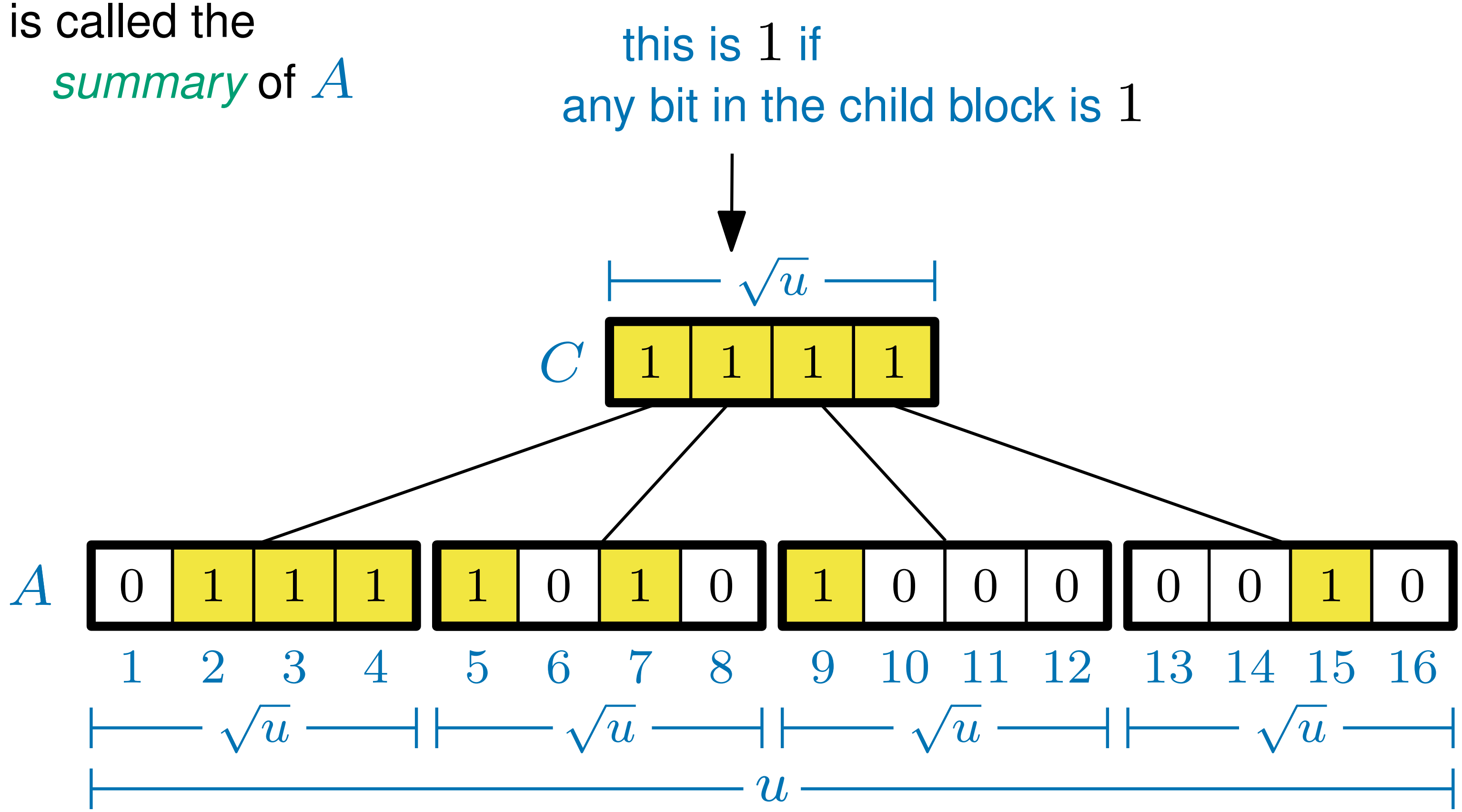

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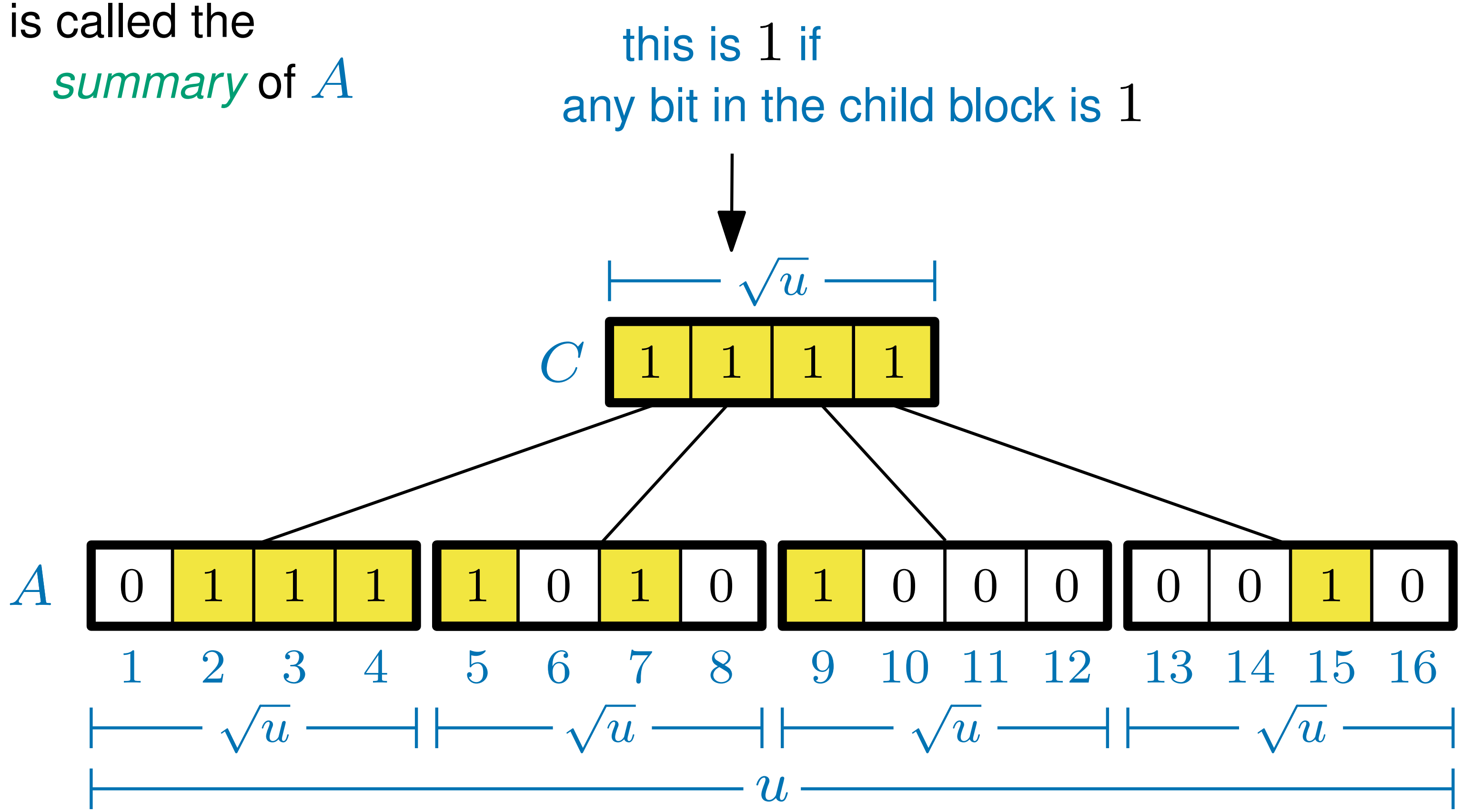
Attempt 2: a constant height tree

(on top of a big array)

C is called the *summary* of A



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this is 1 if
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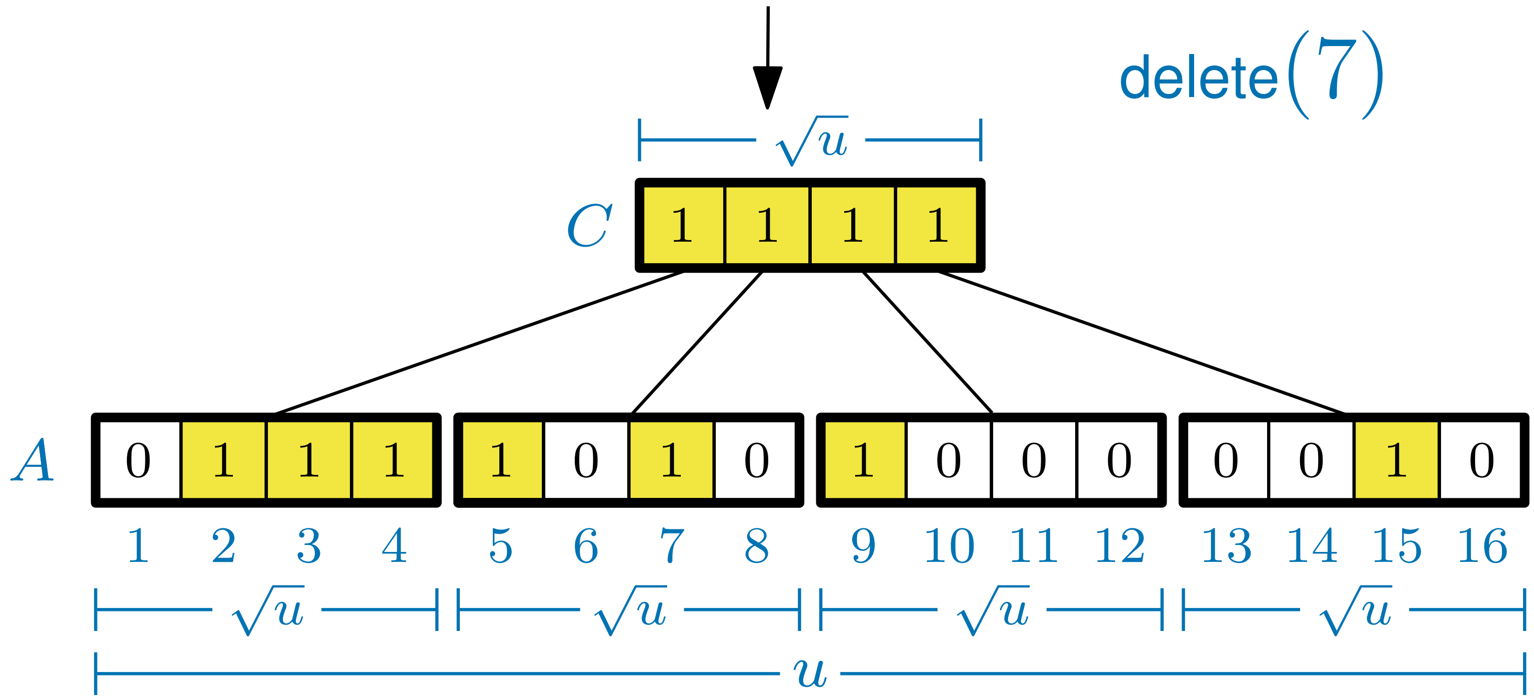
delete(7)



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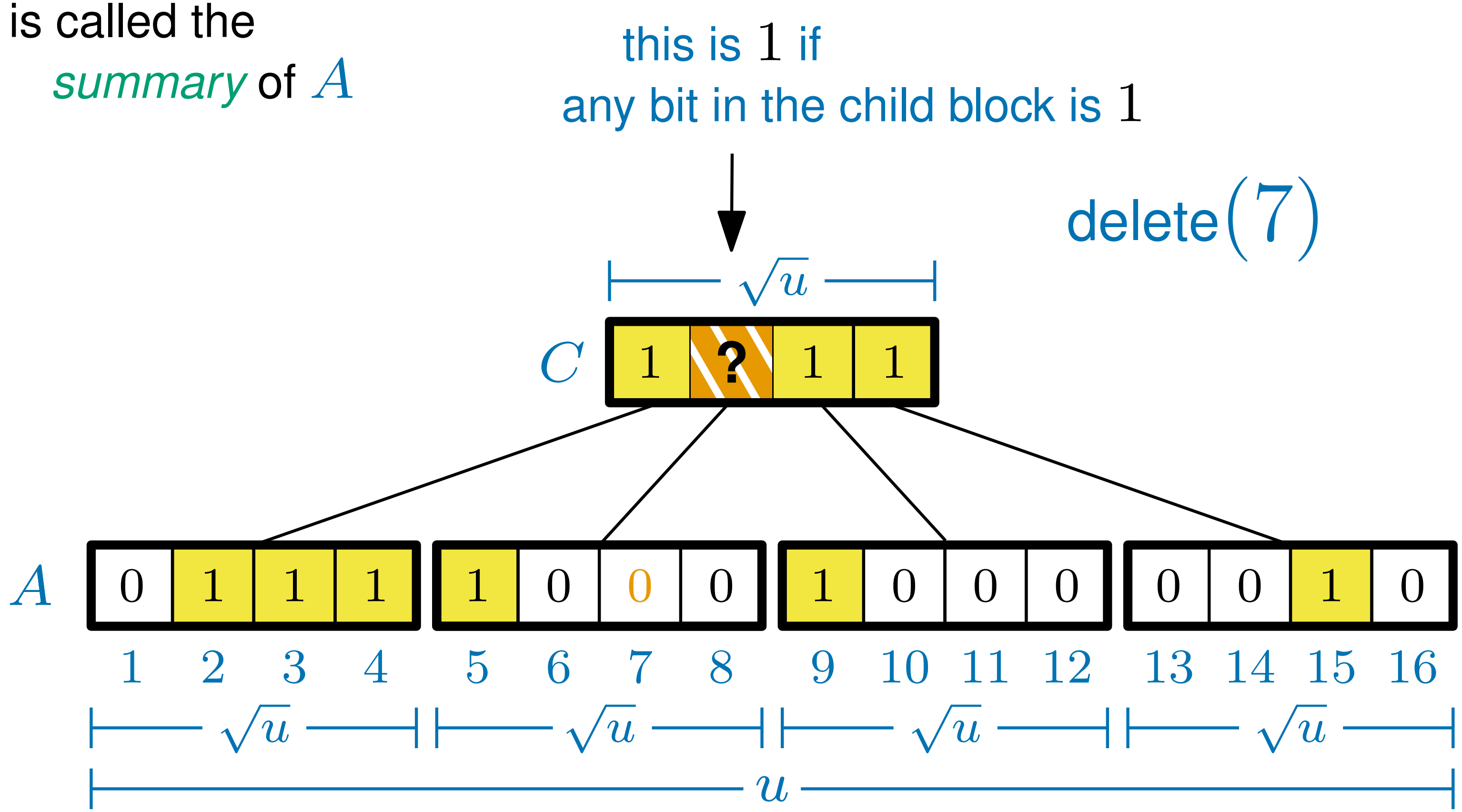

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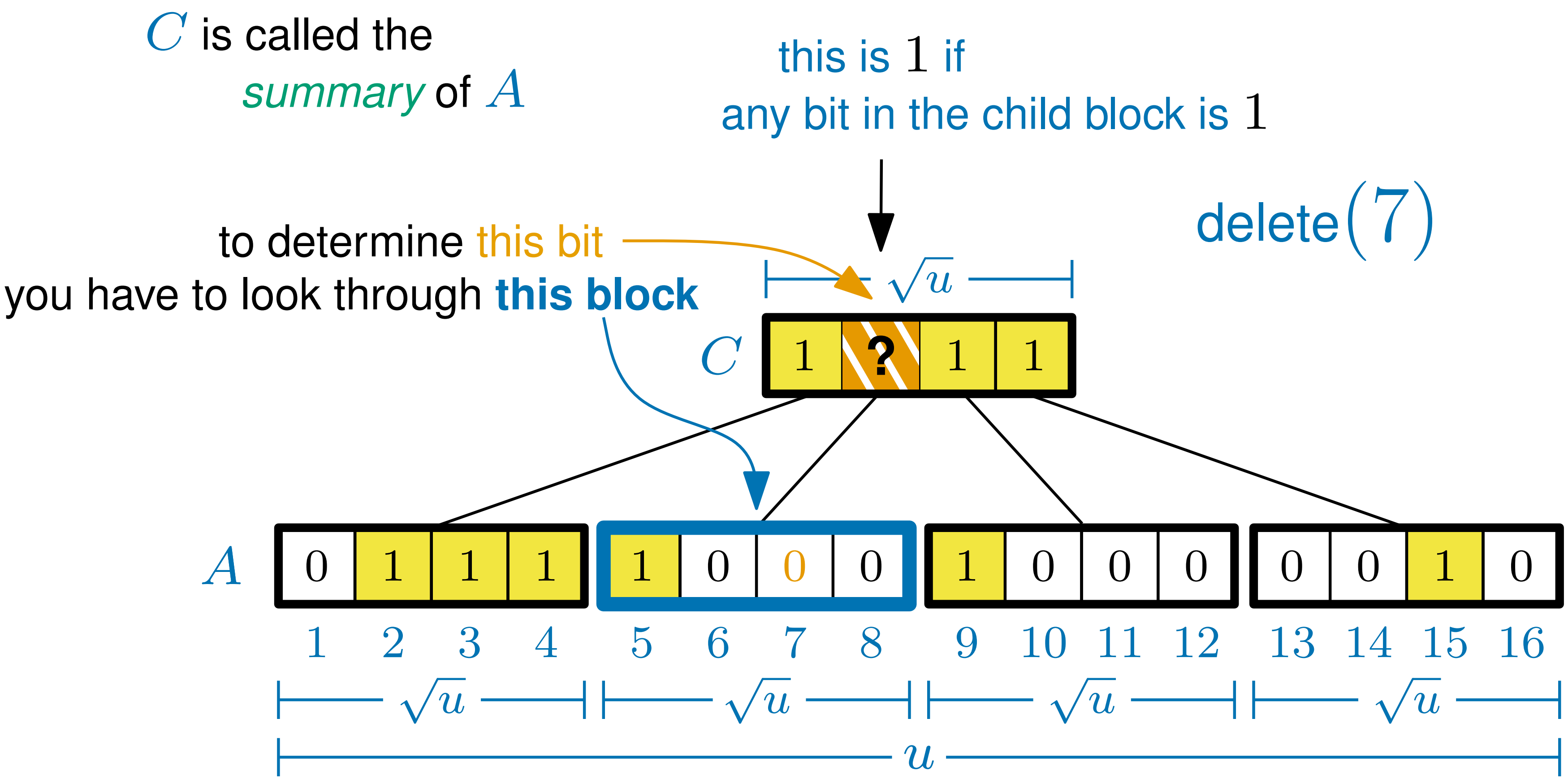

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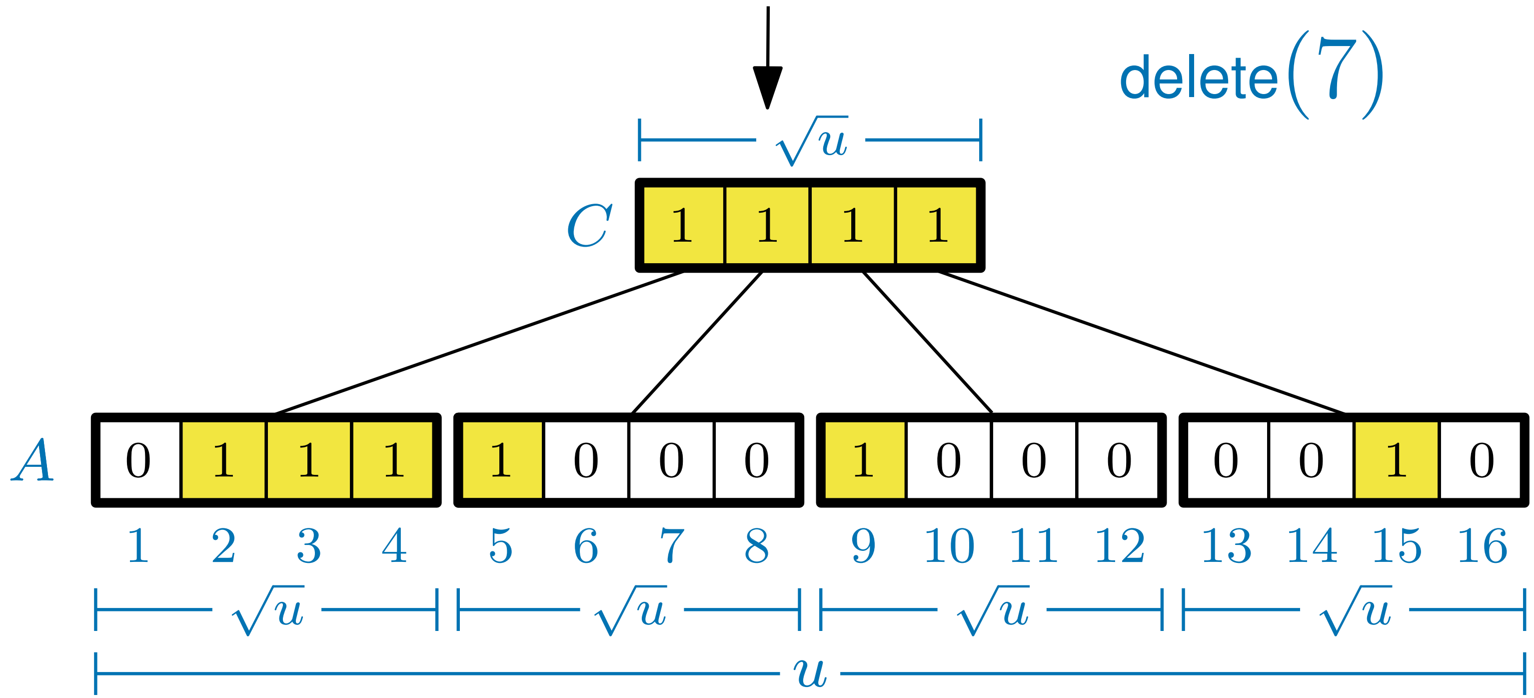

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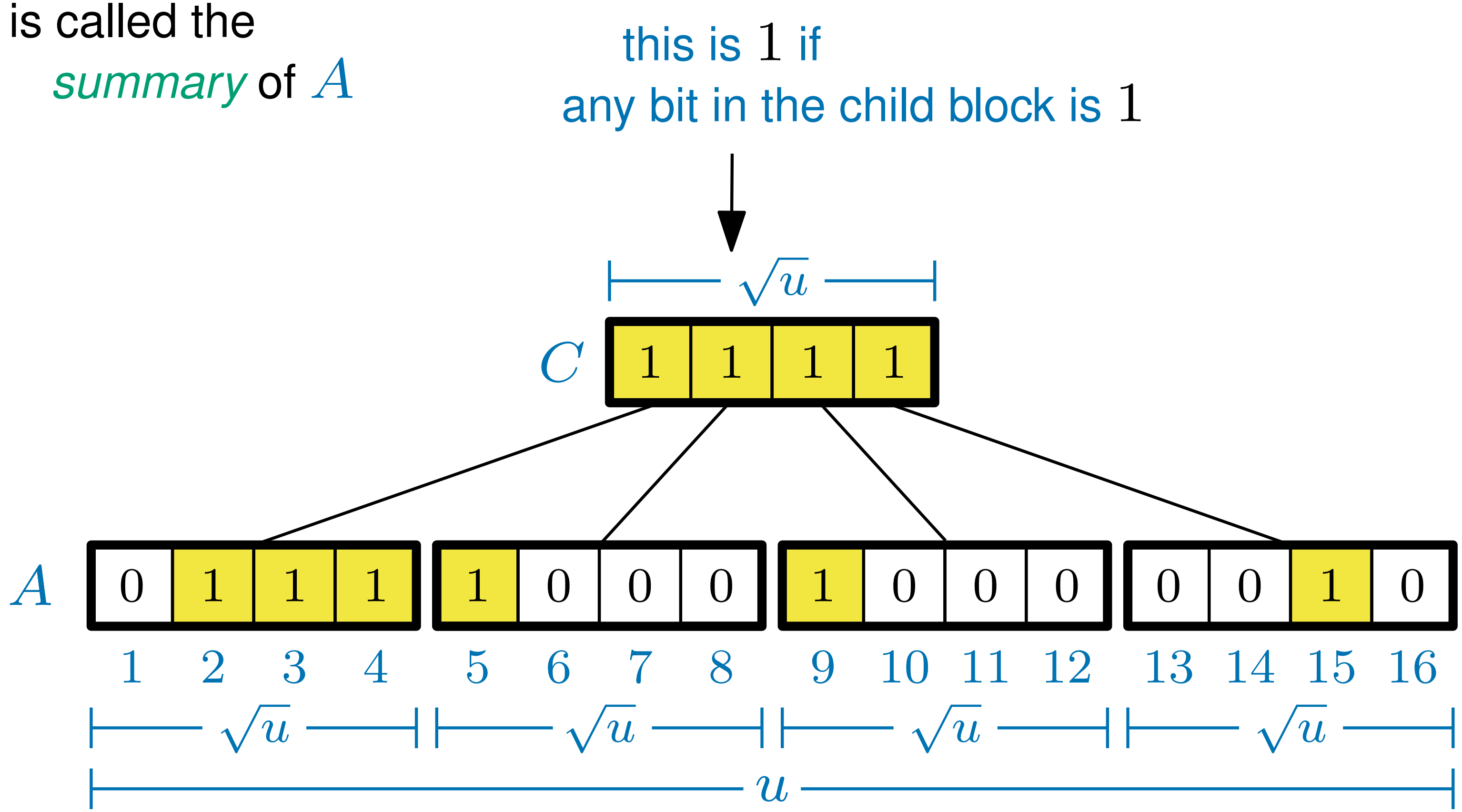

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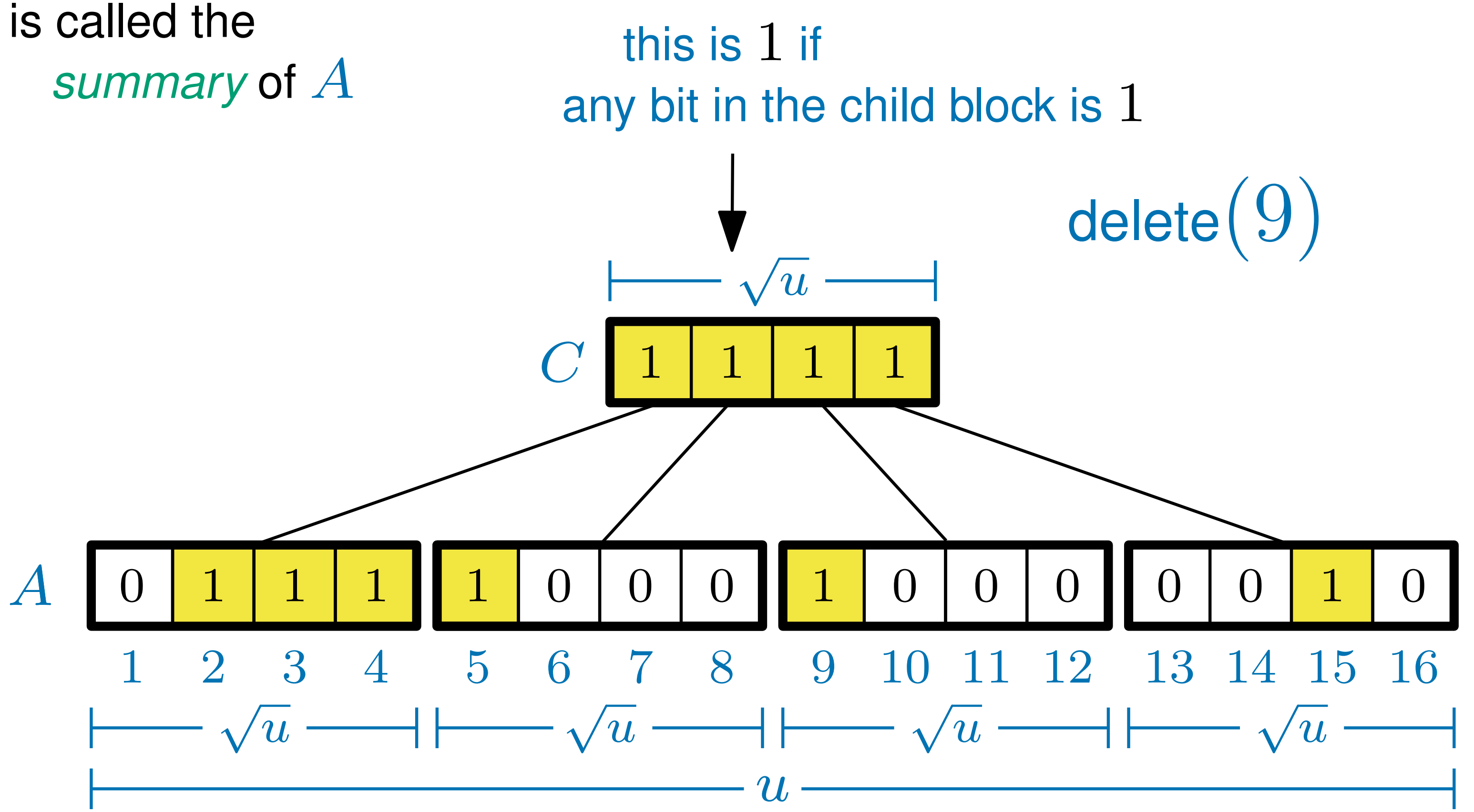

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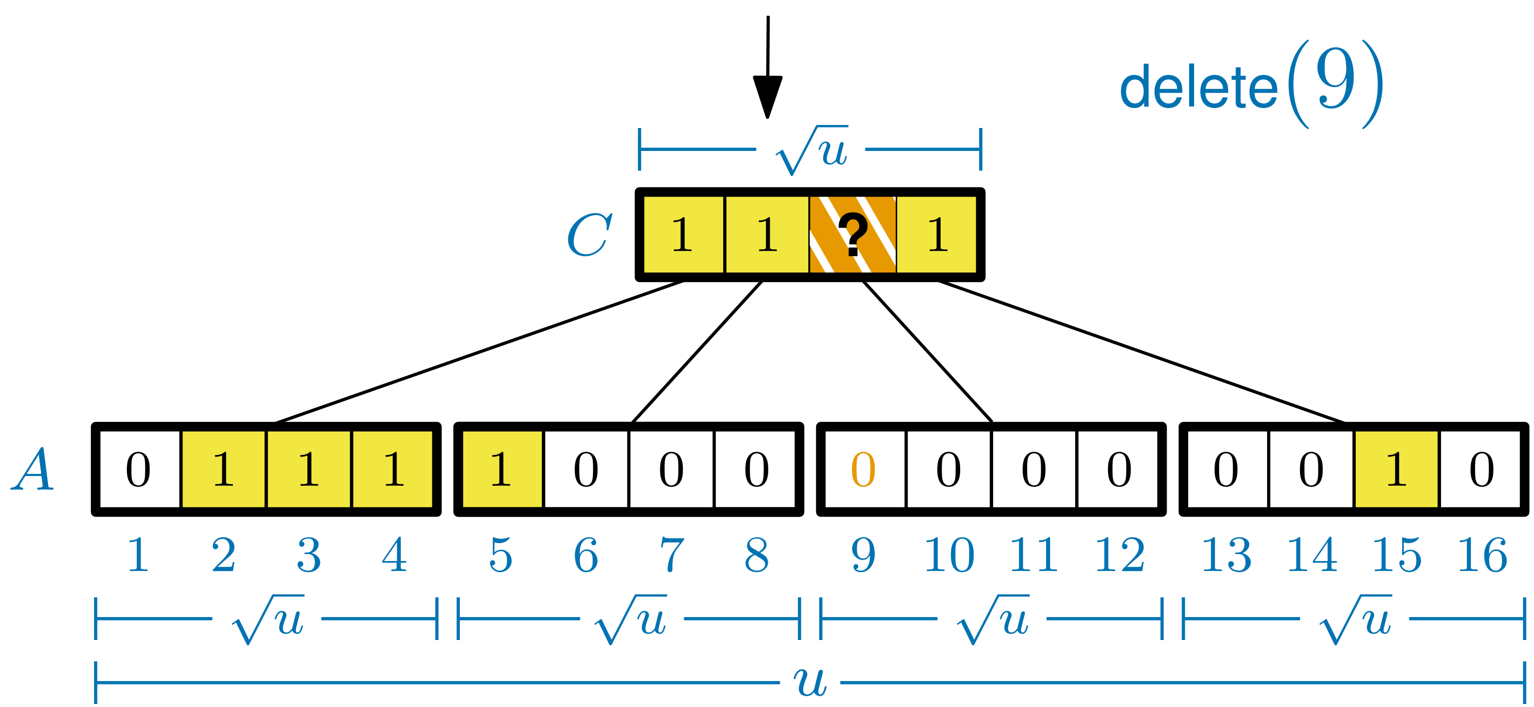

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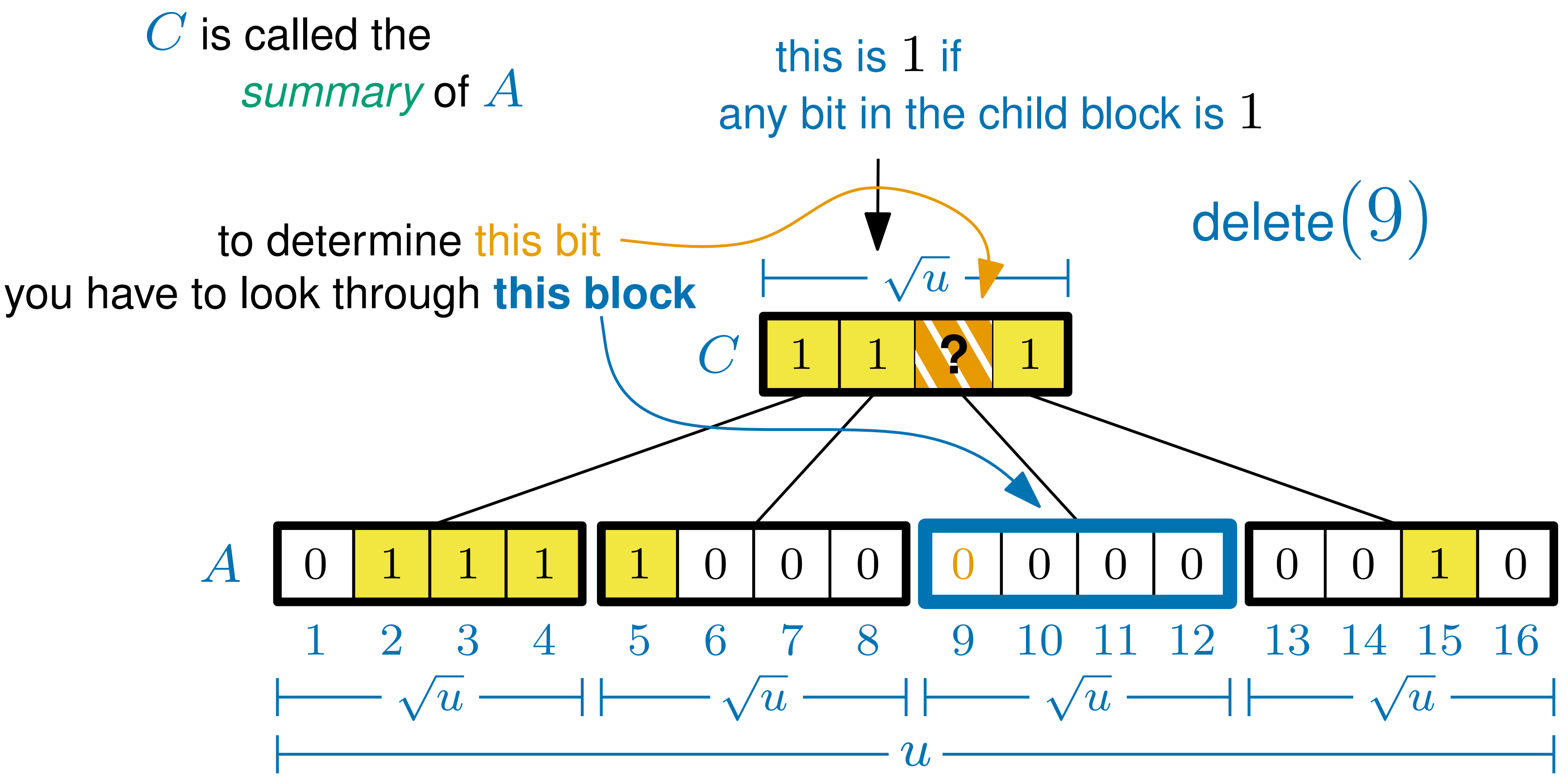

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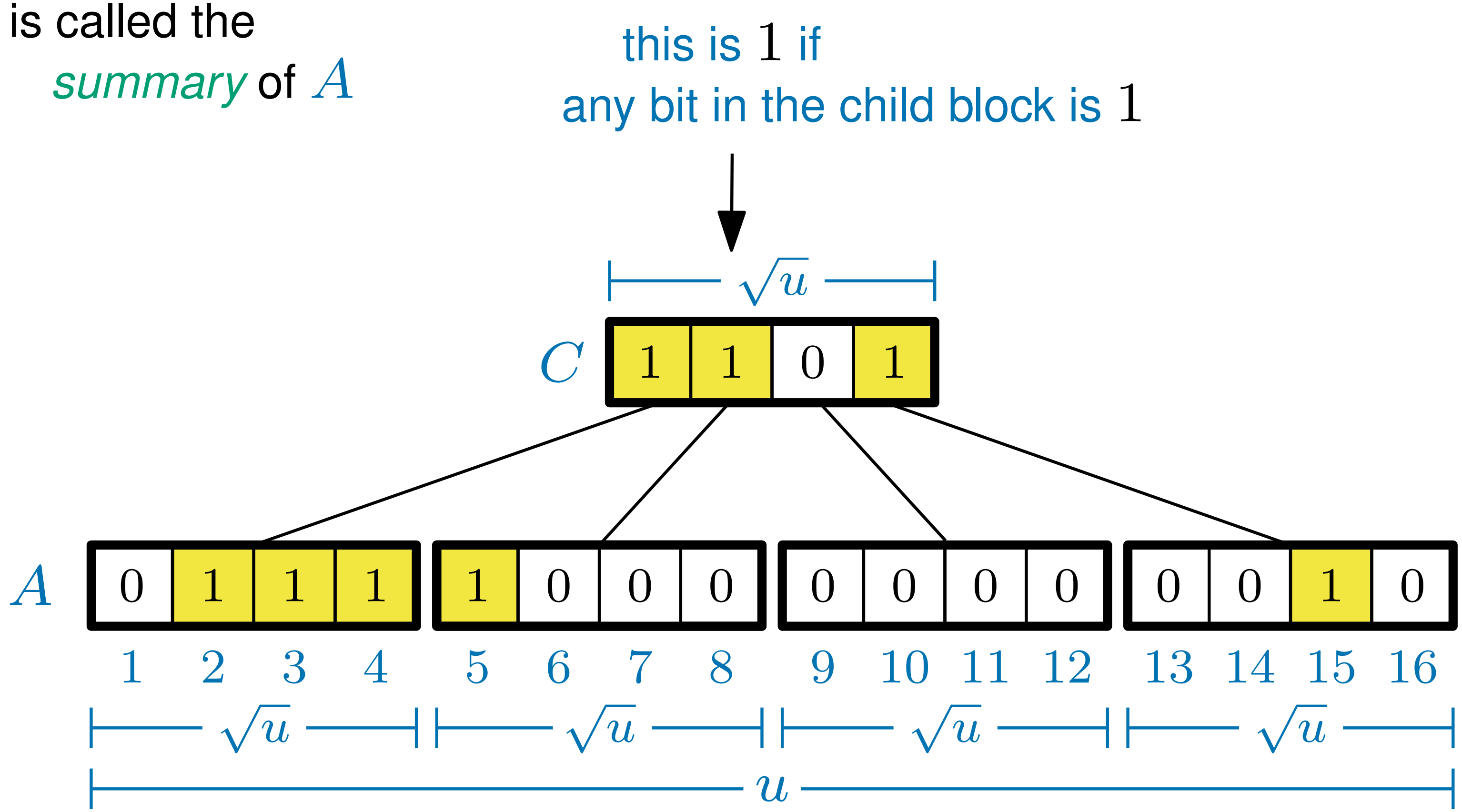

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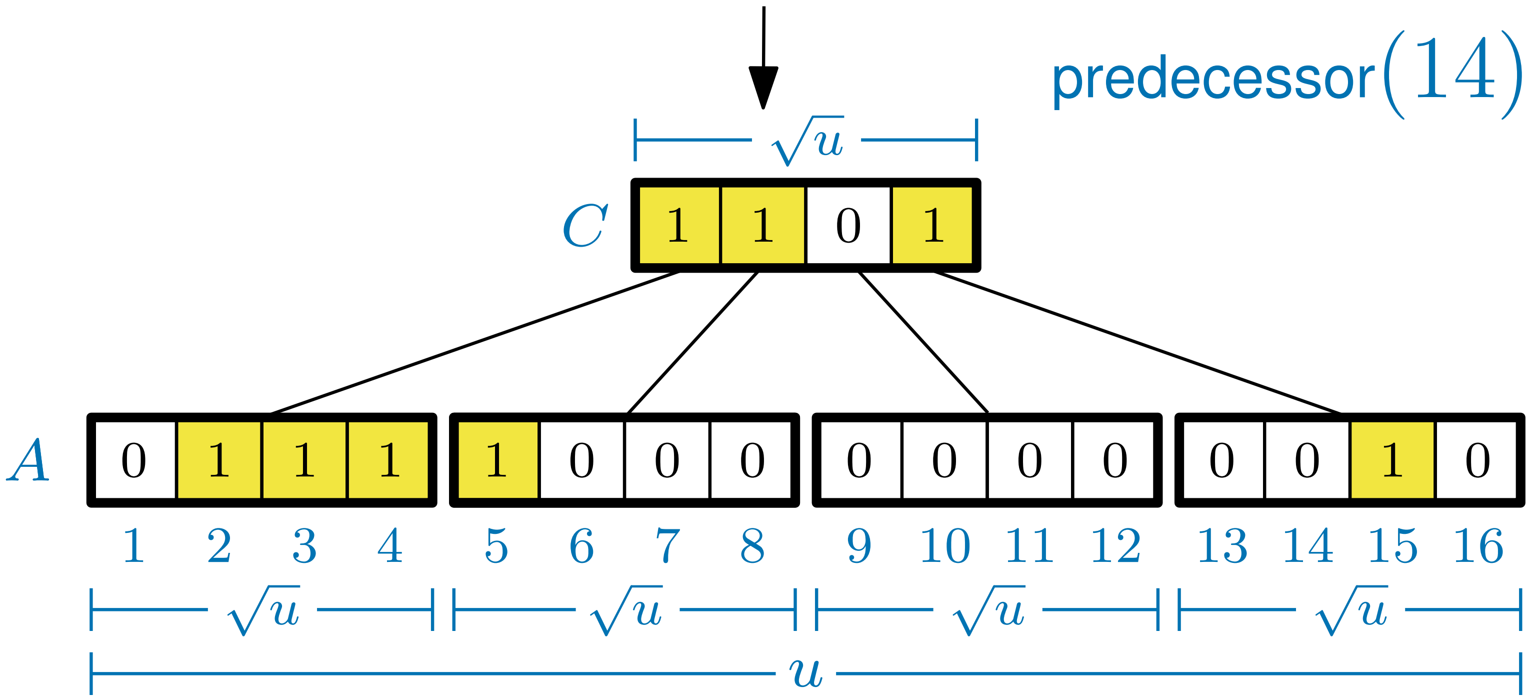

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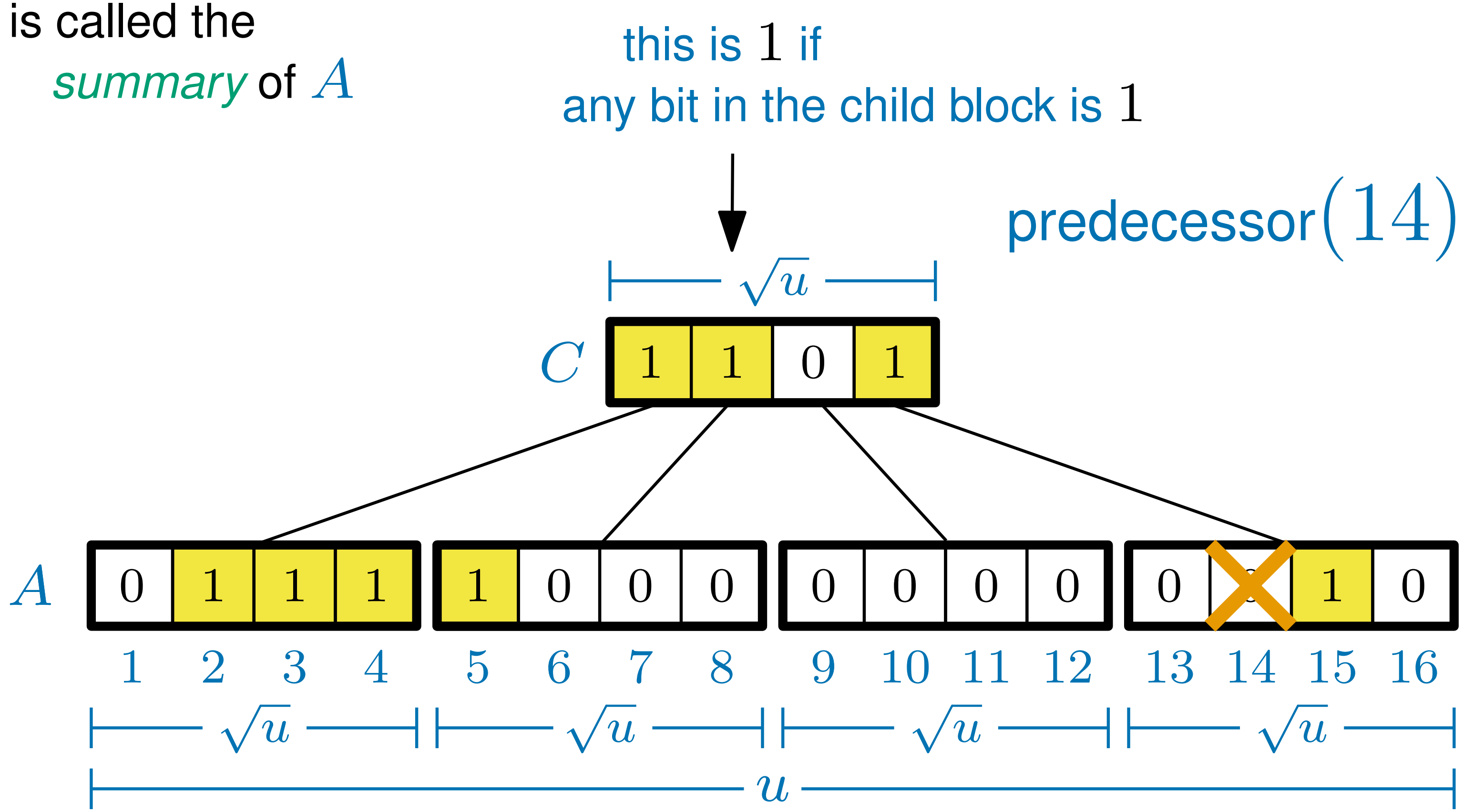

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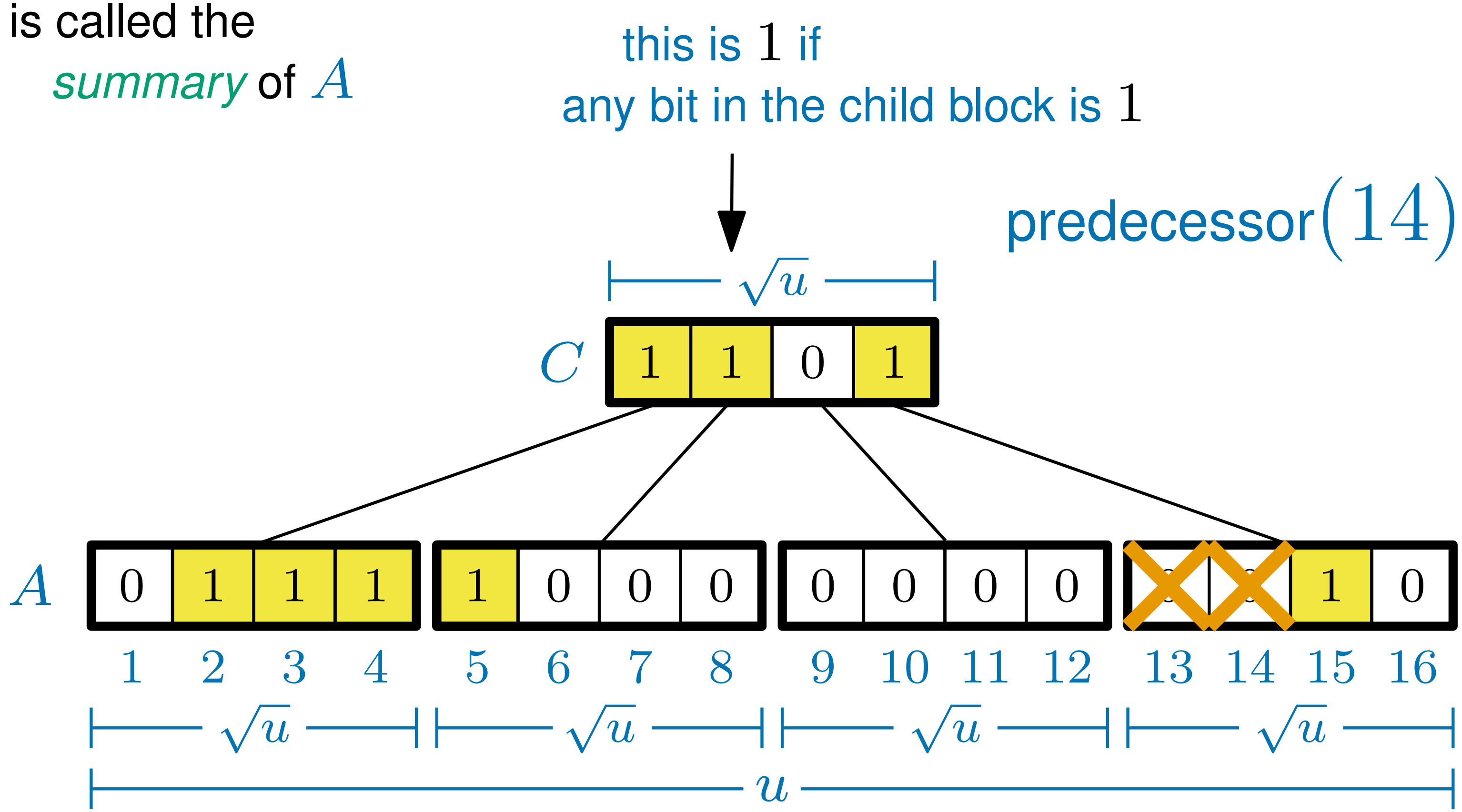

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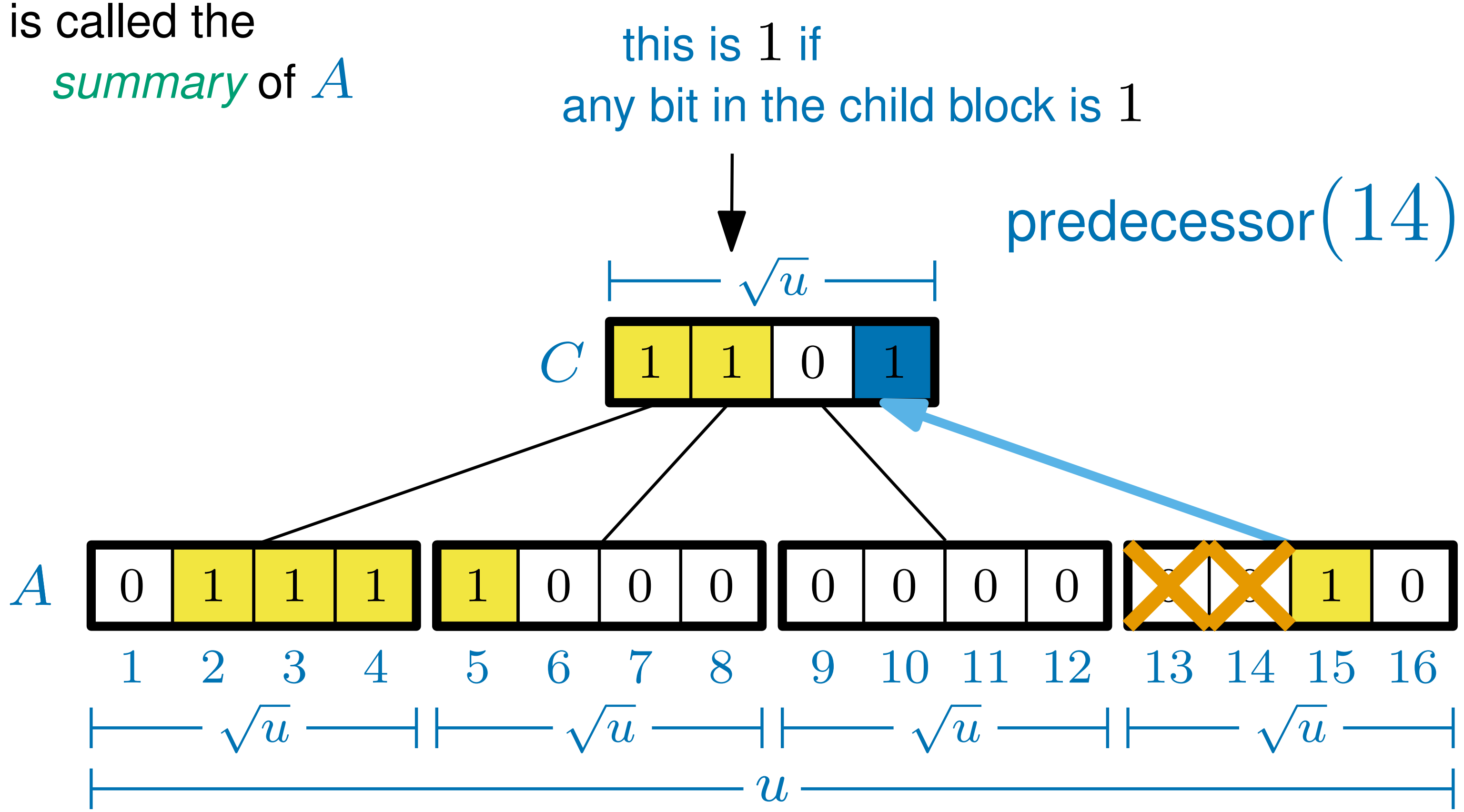

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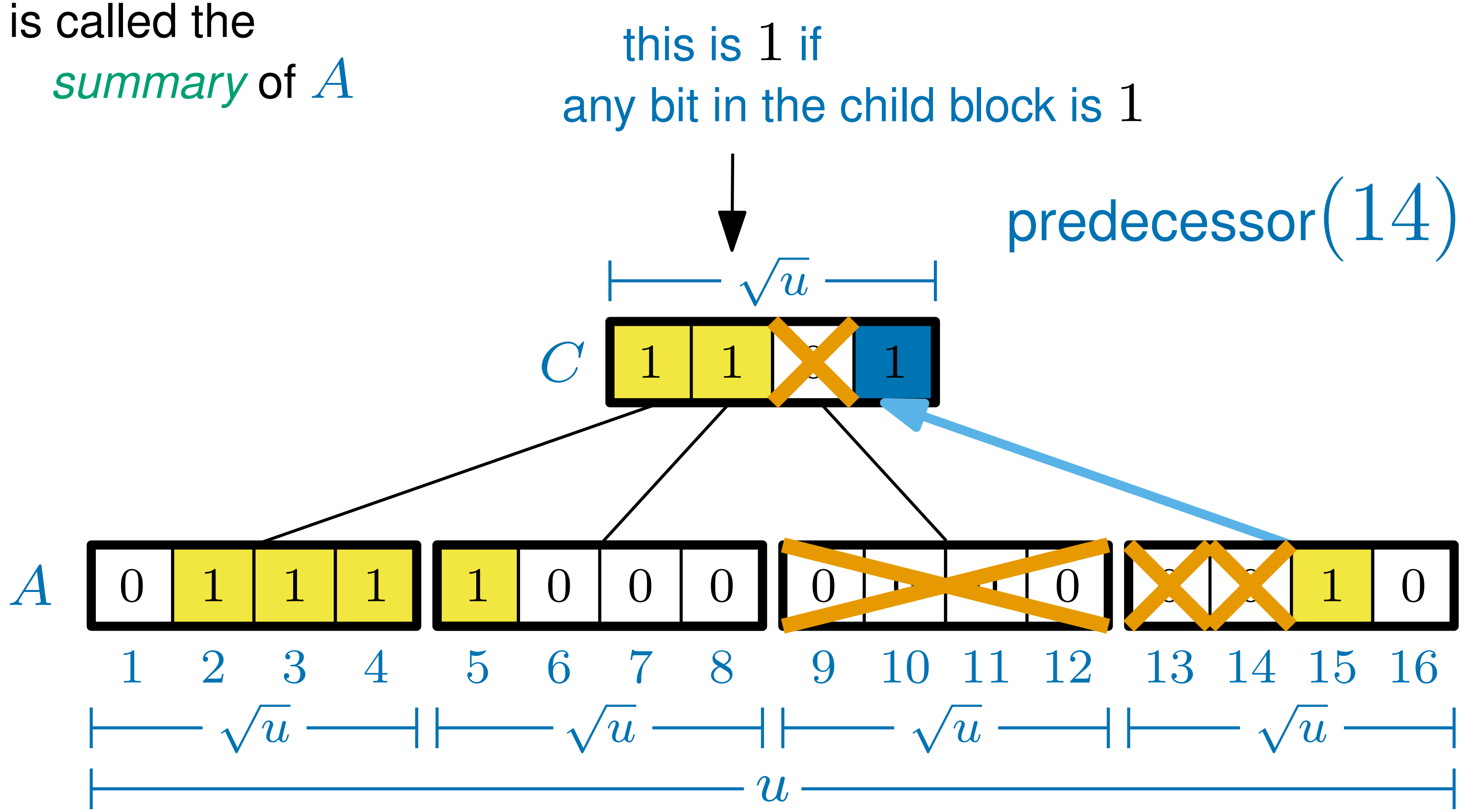

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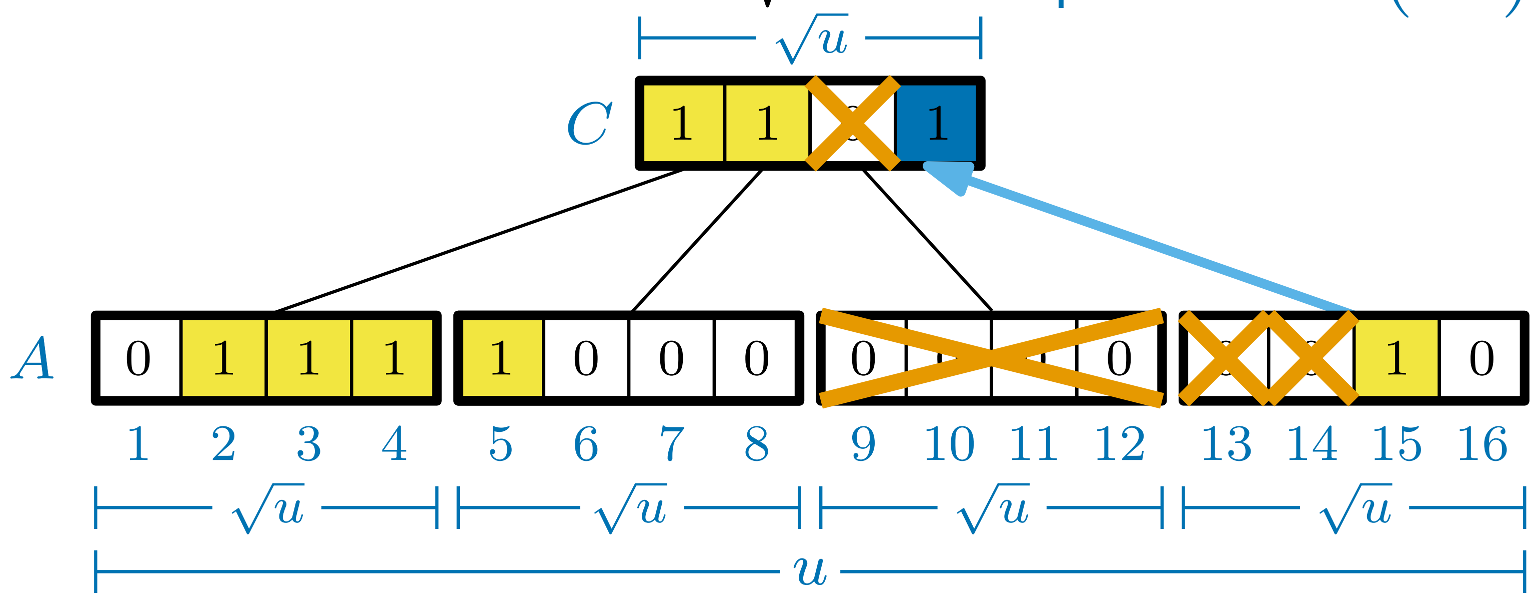

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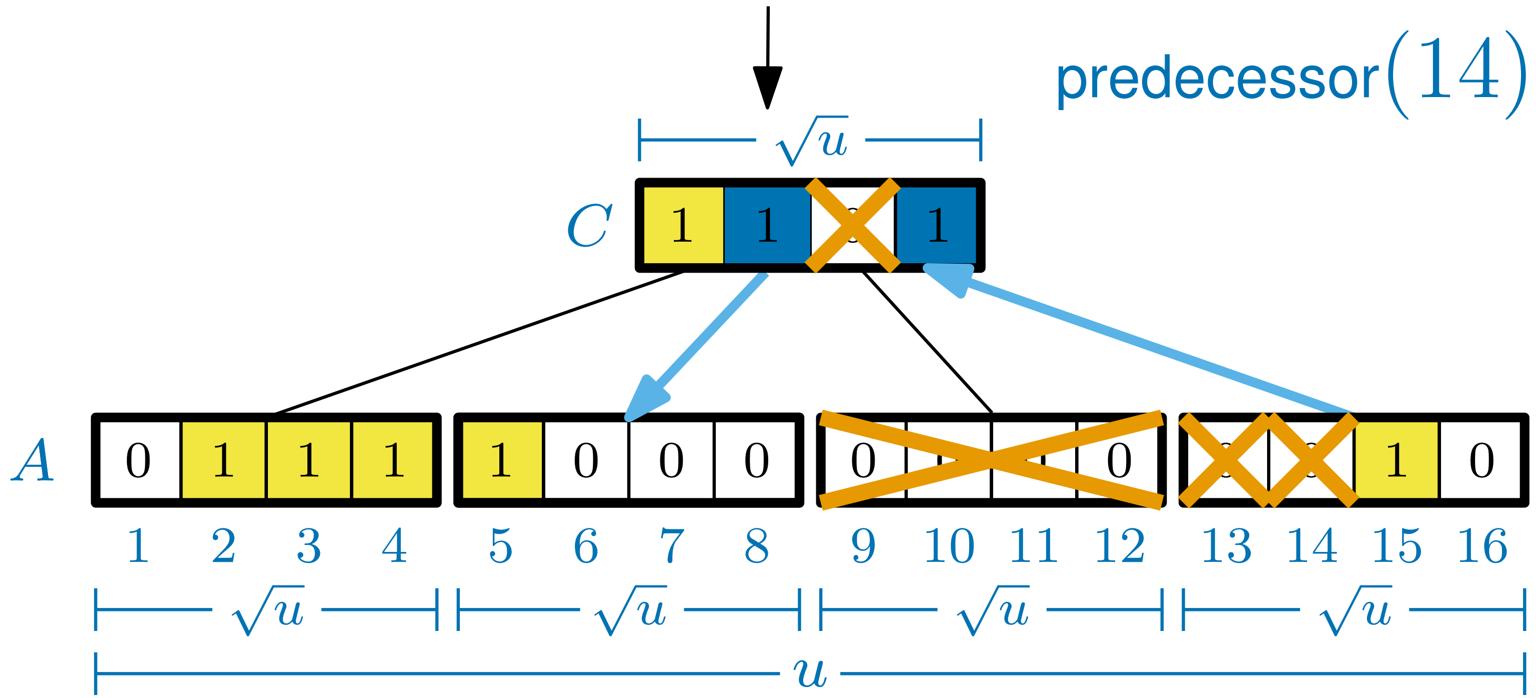

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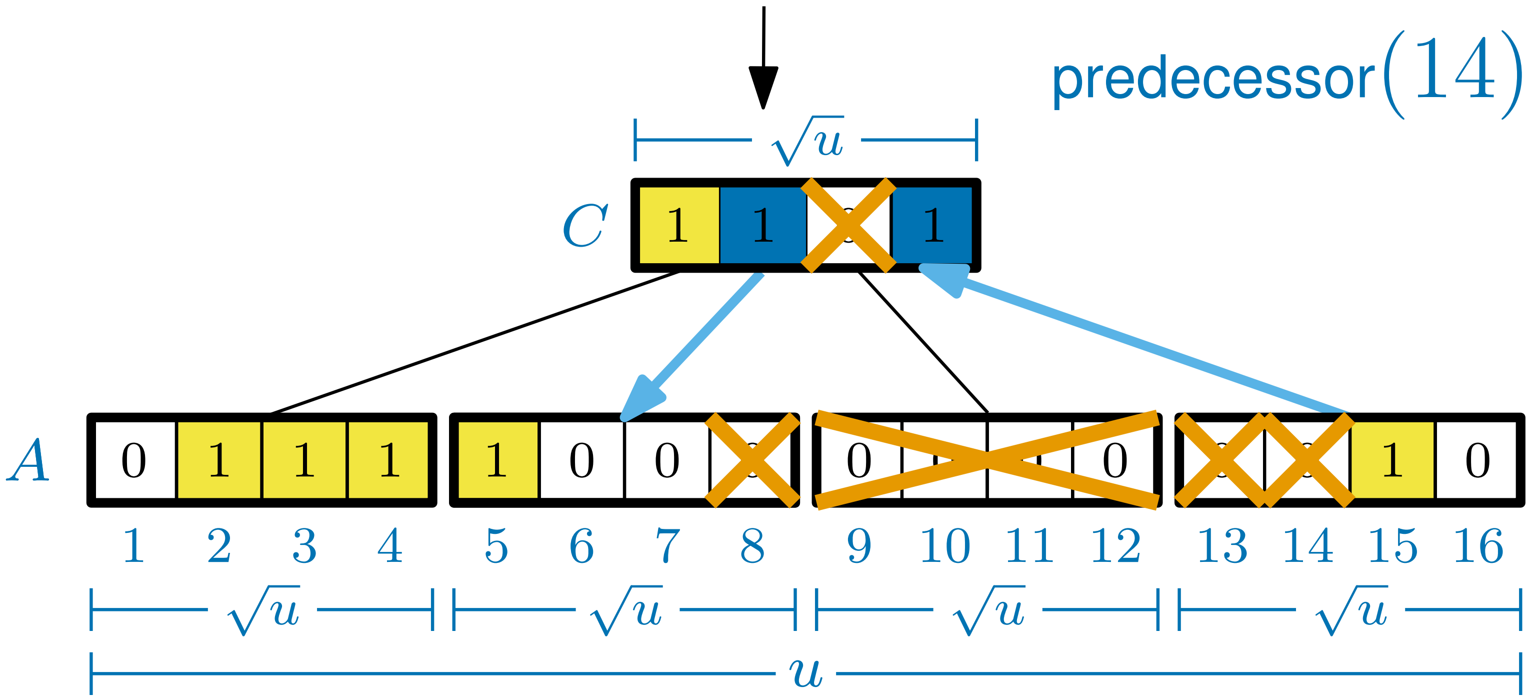

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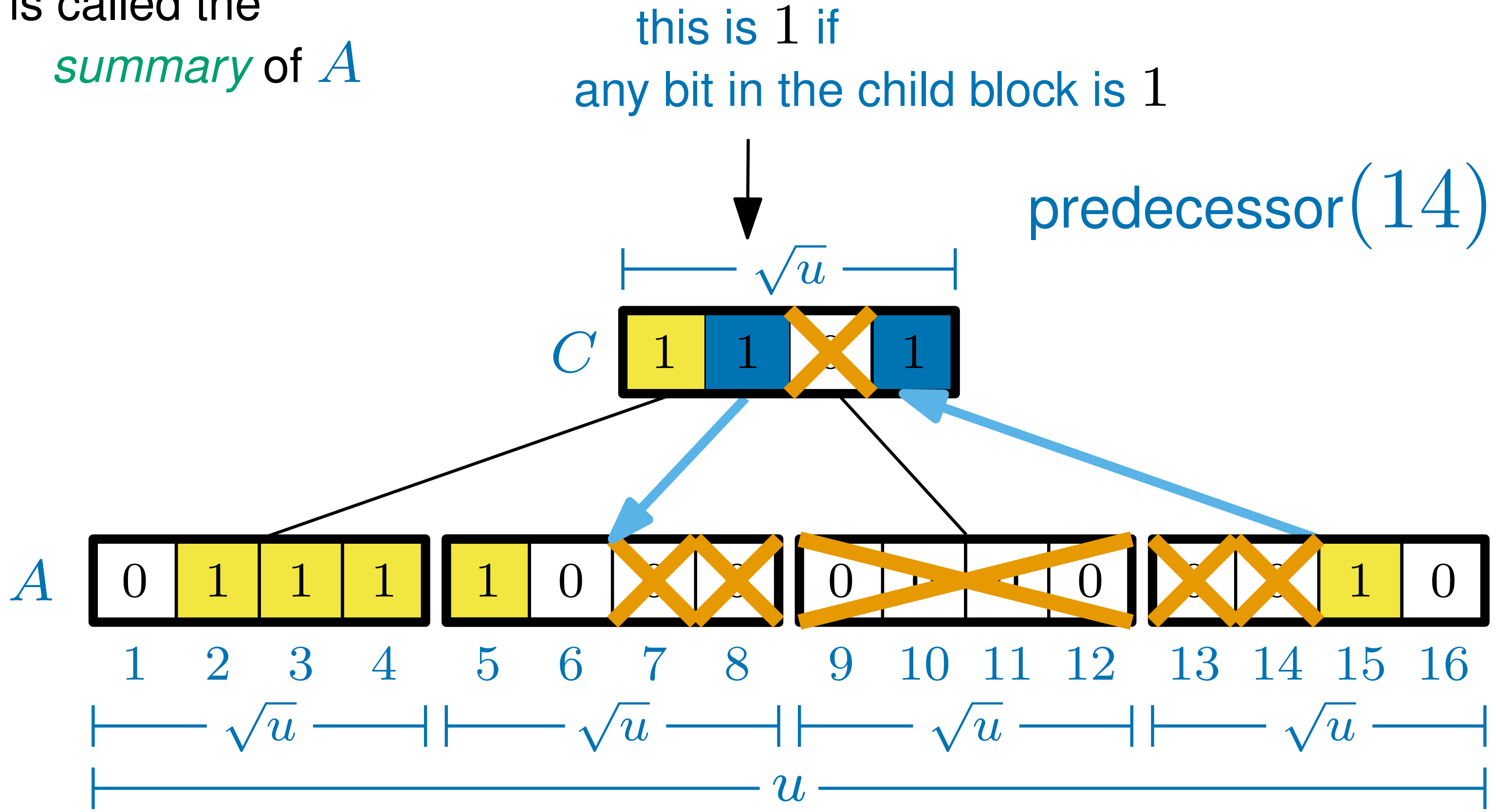

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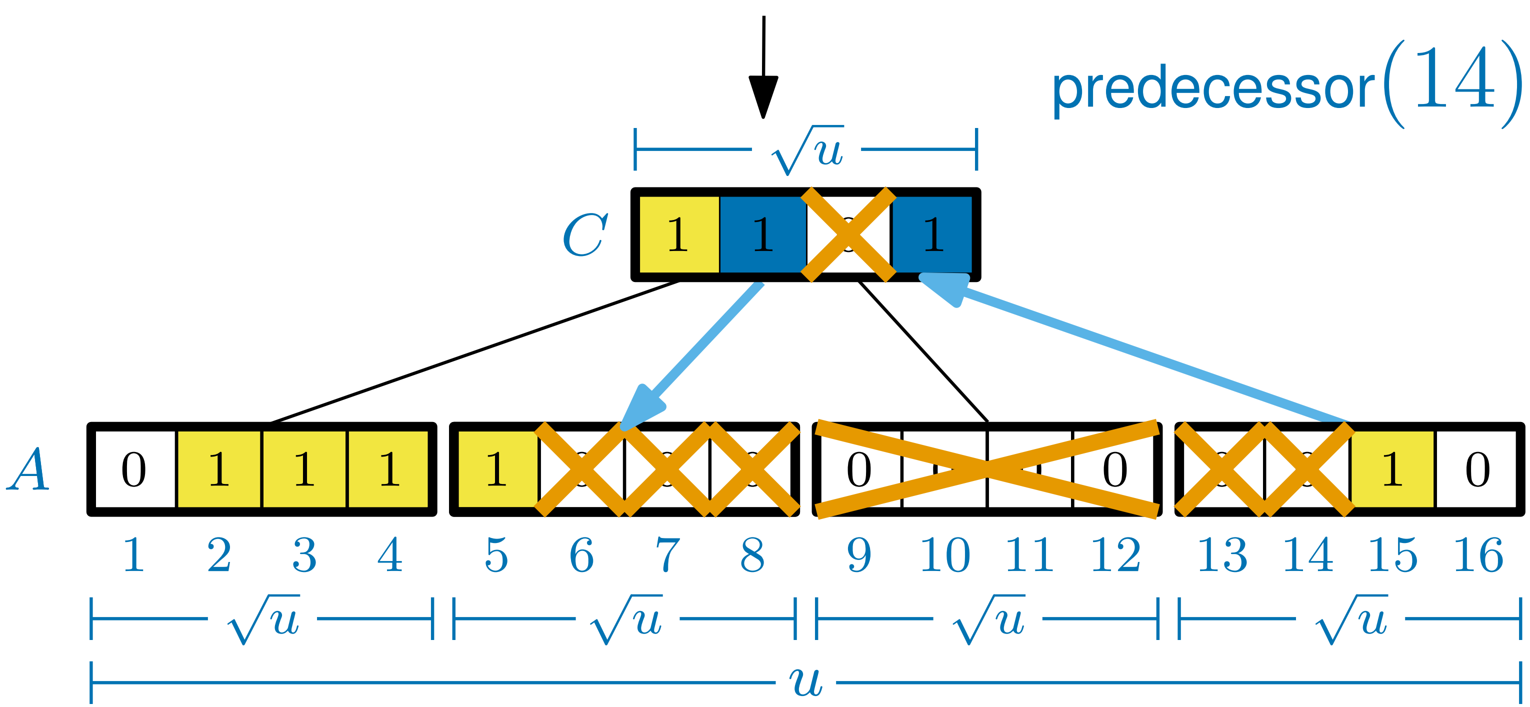

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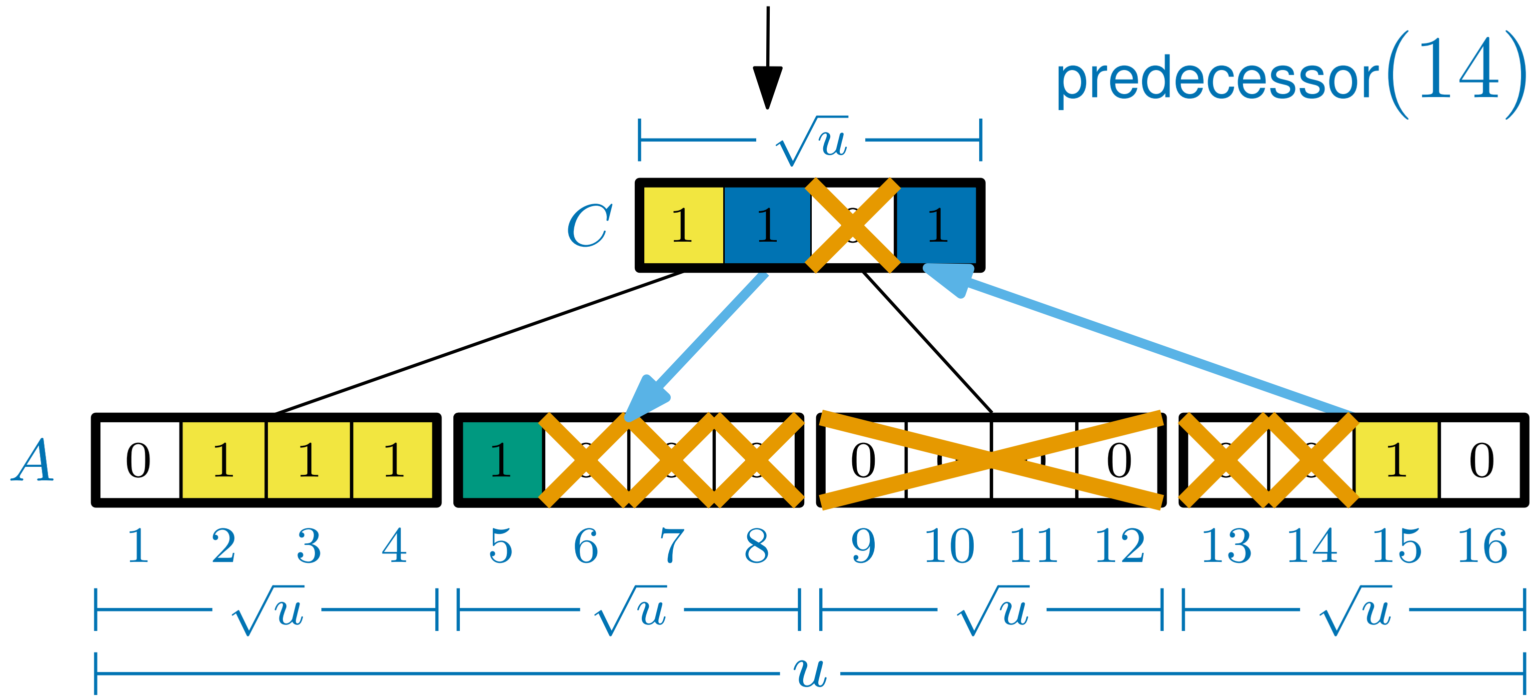

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Attempt 2: a constant height tree

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C is called the *summary* of A

In the worst case we look at all of C and all of two blocks

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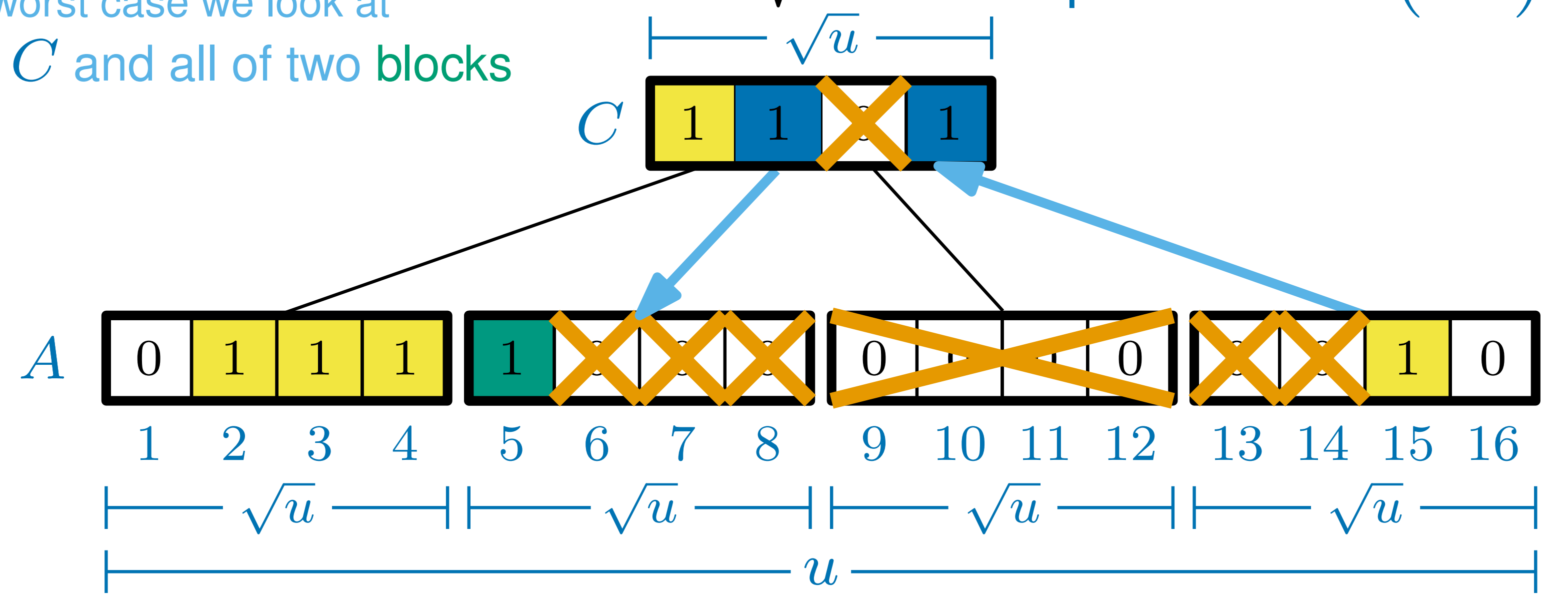

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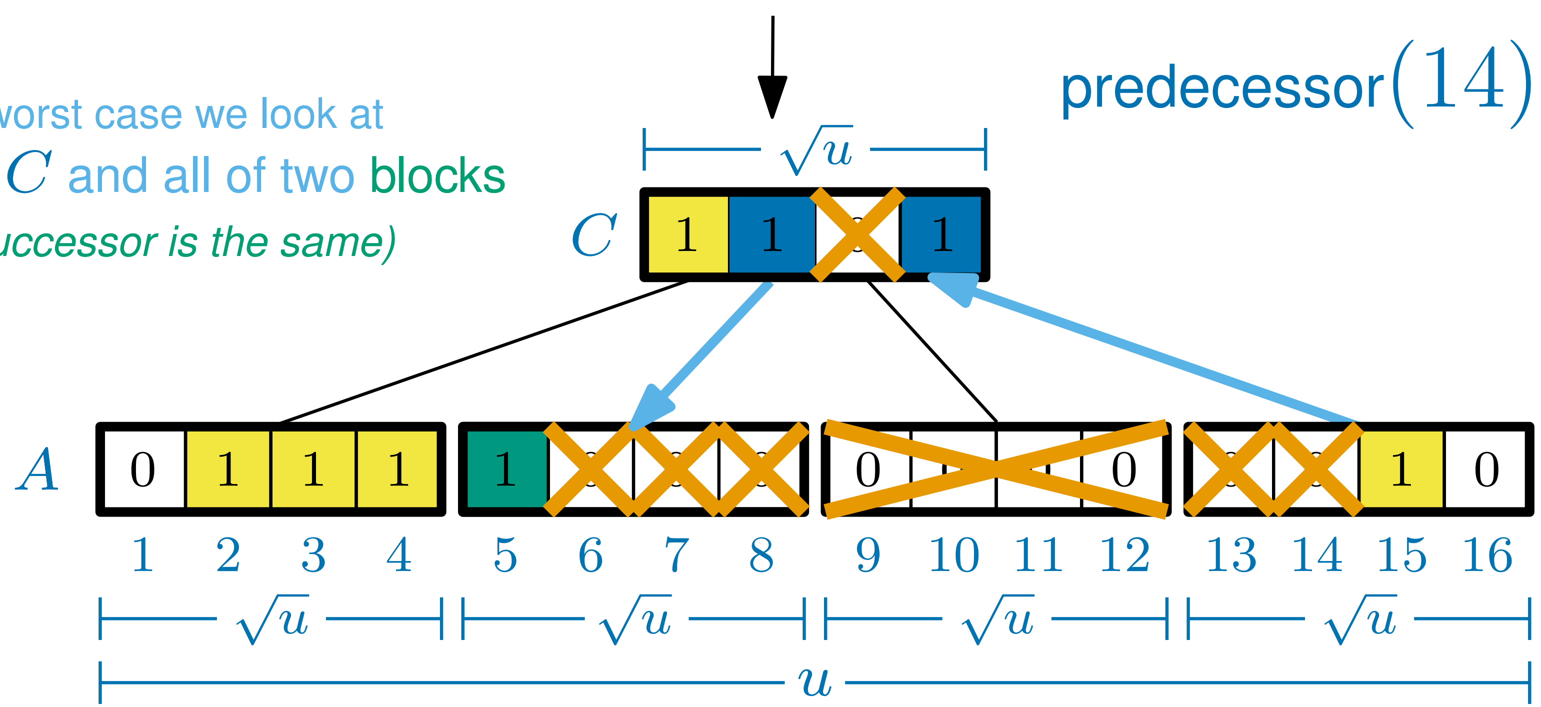
Attempt 2: a constant height tree

(on top of a big array)

C is called the *summary* of A

In the worst case we look at all of C and all of two blocks *(successor is the same)*



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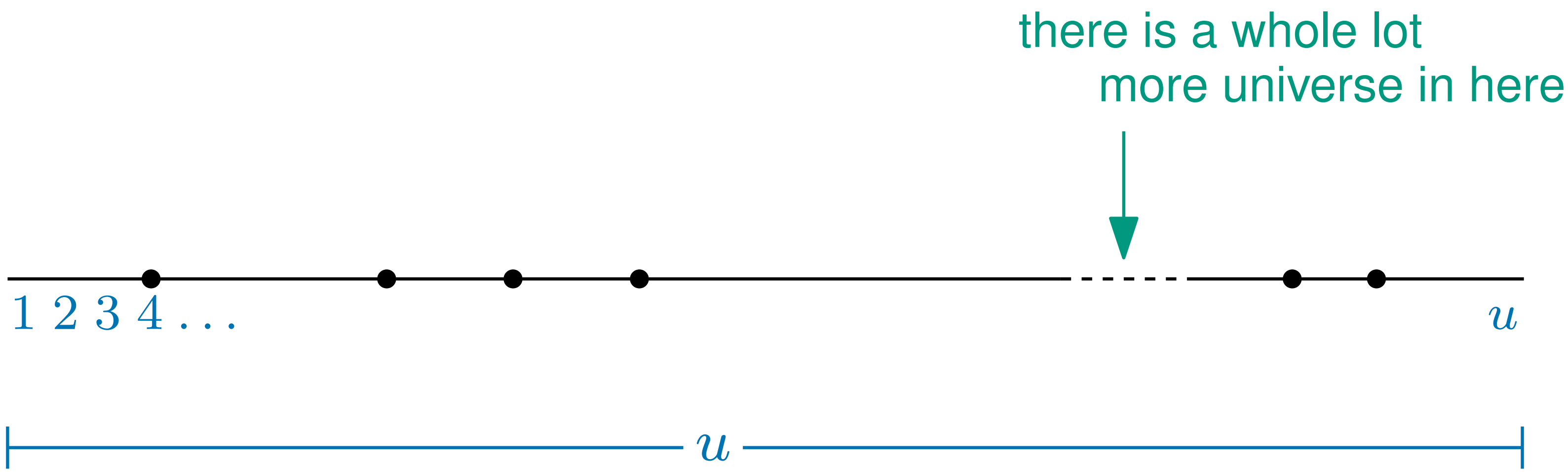
An abstract view

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements



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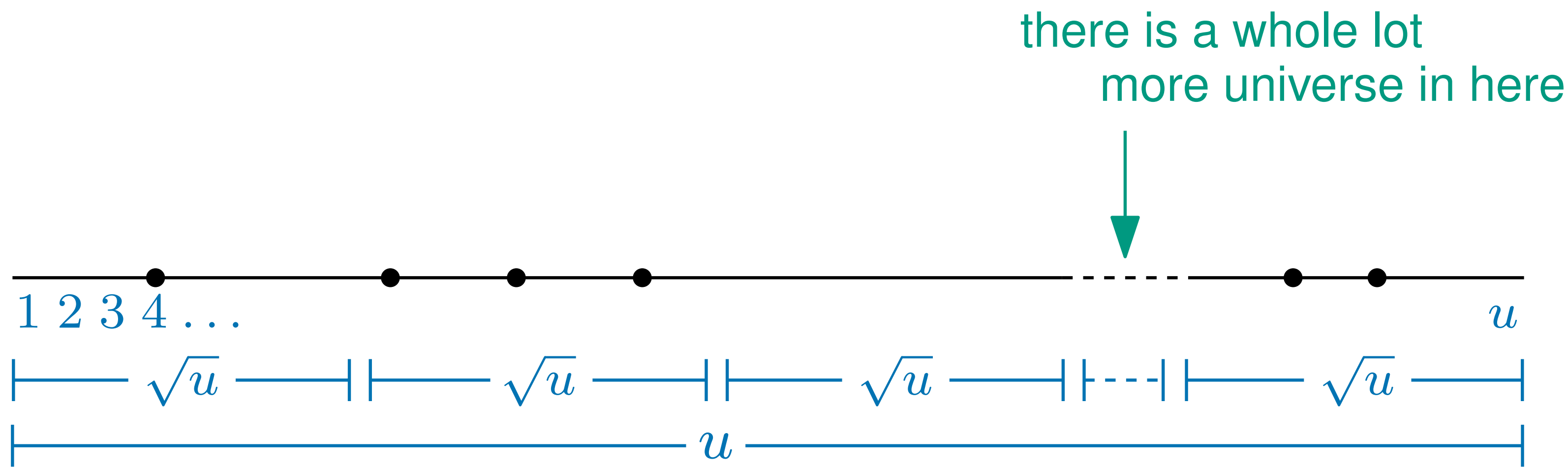

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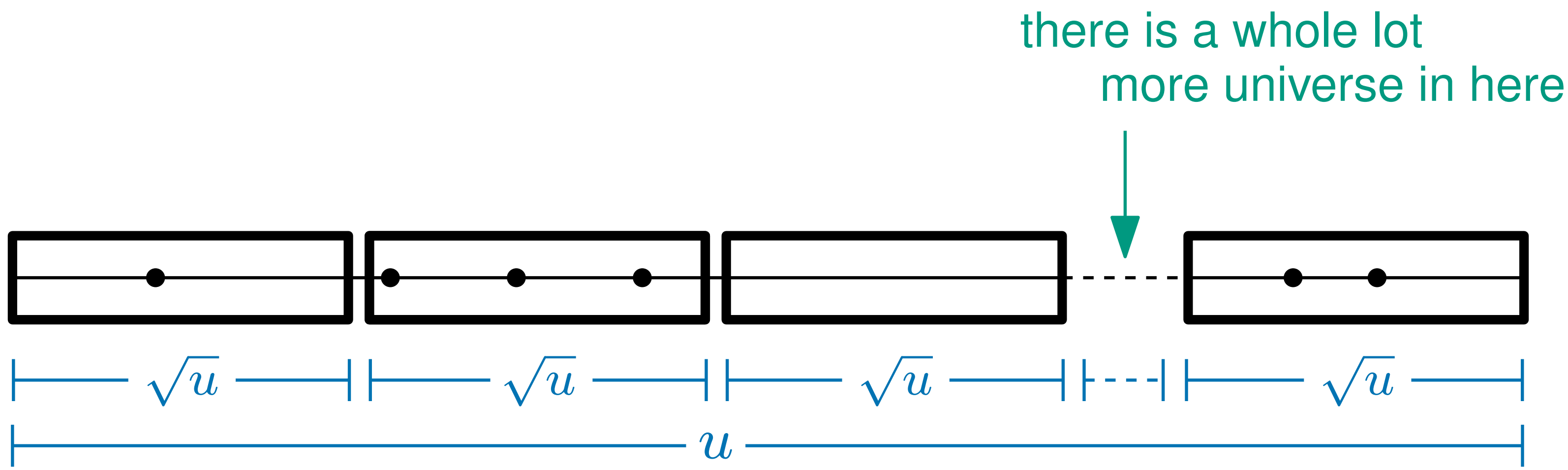
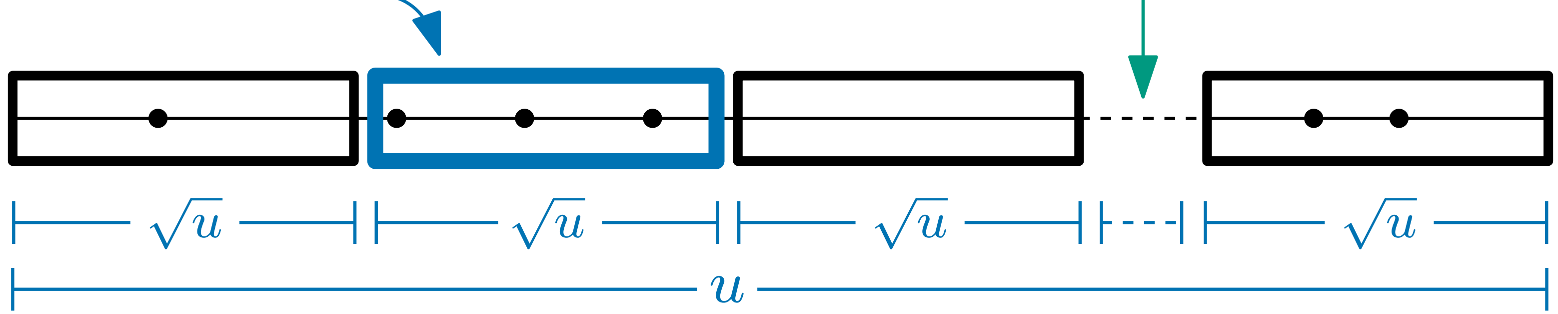


An abstract view

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements

we can think of each block
as a *'little'* universe of size $\sqrt{u}$

there is a whole lot
more universe in here



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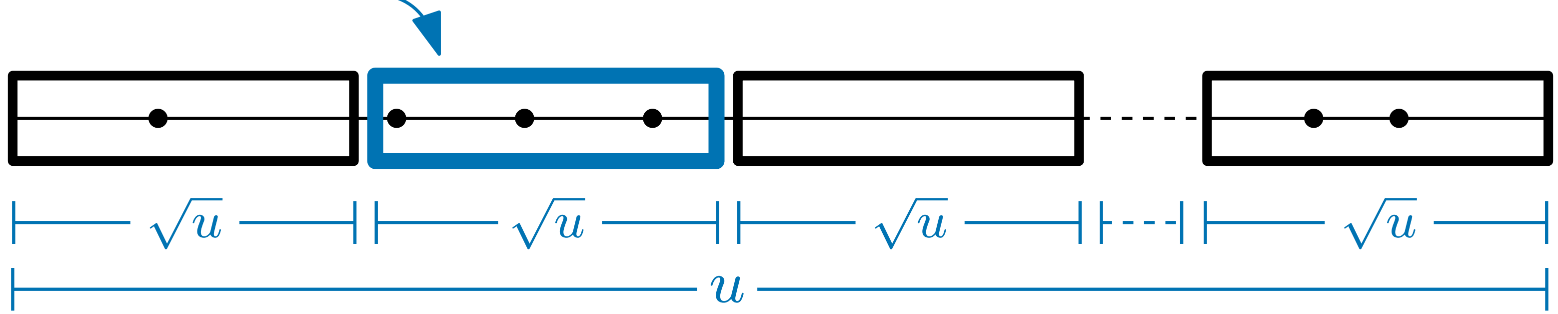

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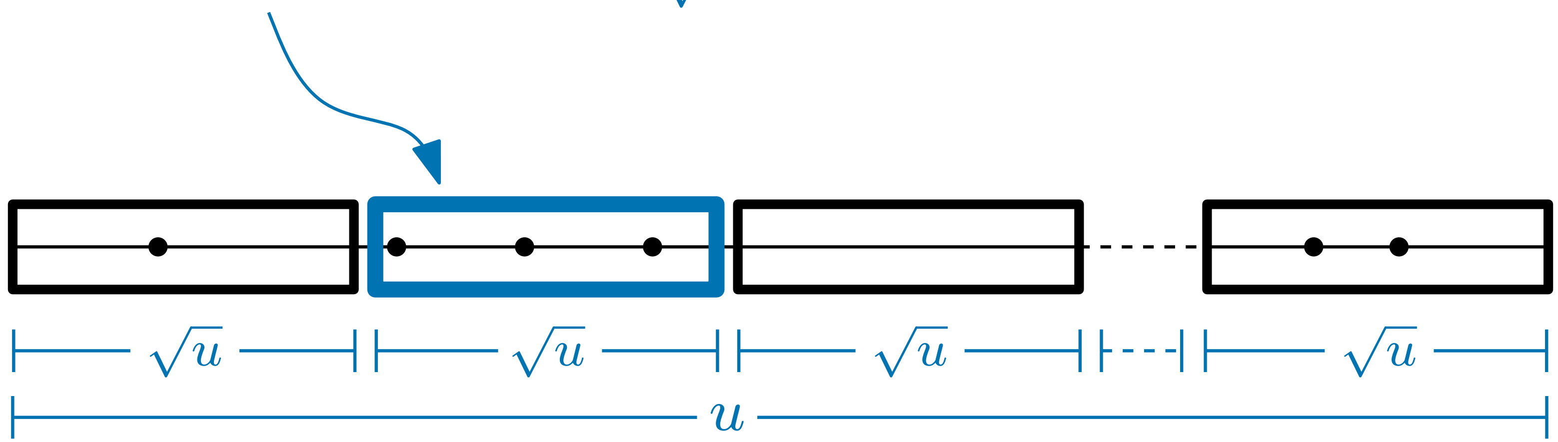

For block i, we build a data structure $B[i]$

which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

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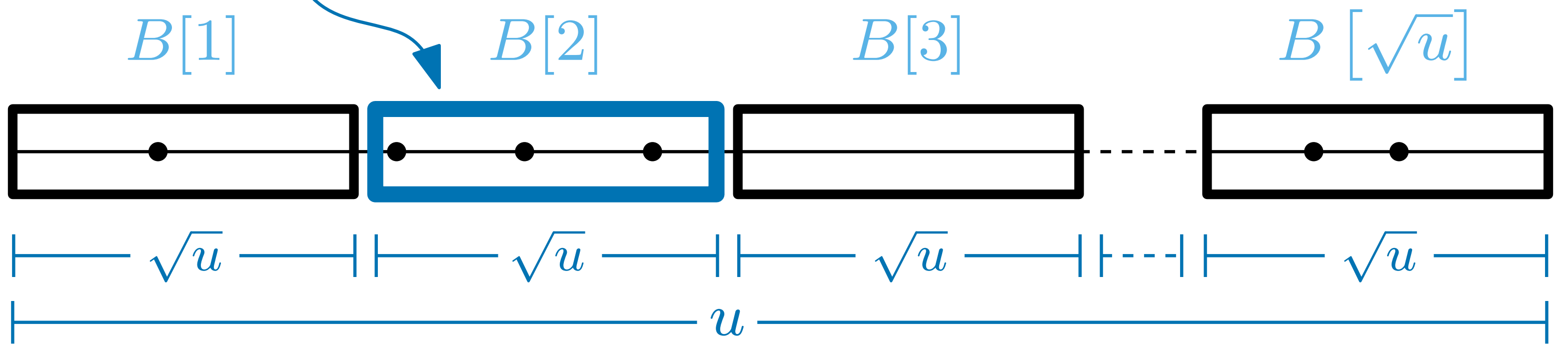

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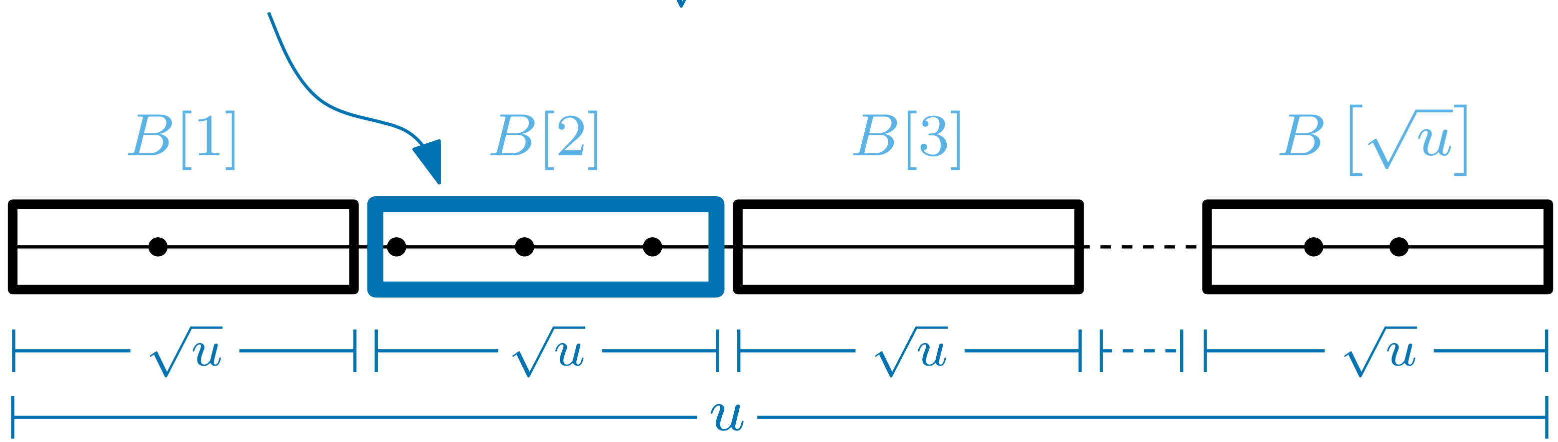

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x is stored in $B[i]$ iff $\left(x + (i-1)\sqrt{u}\right) \in S$

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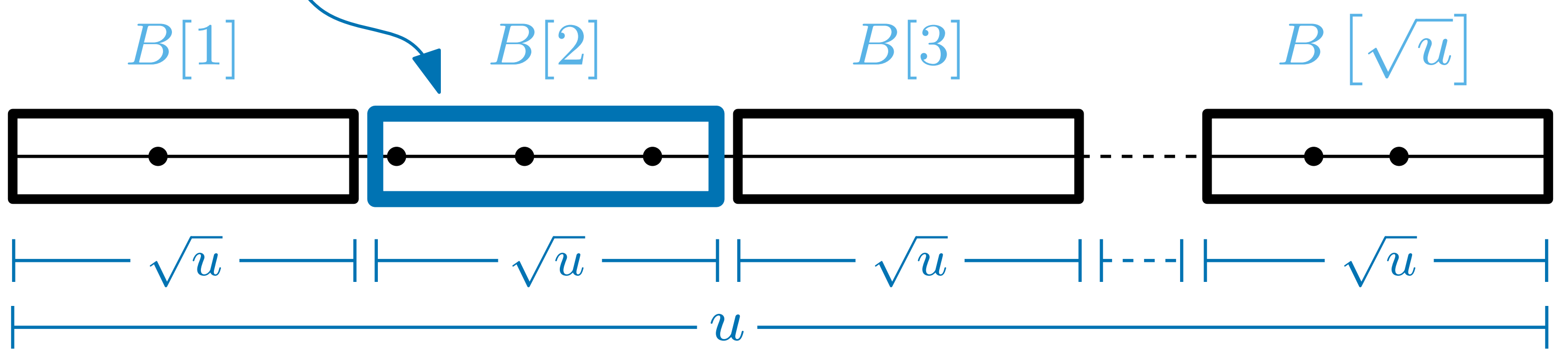

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(this is just to deal with the offset from the start of the real universe)

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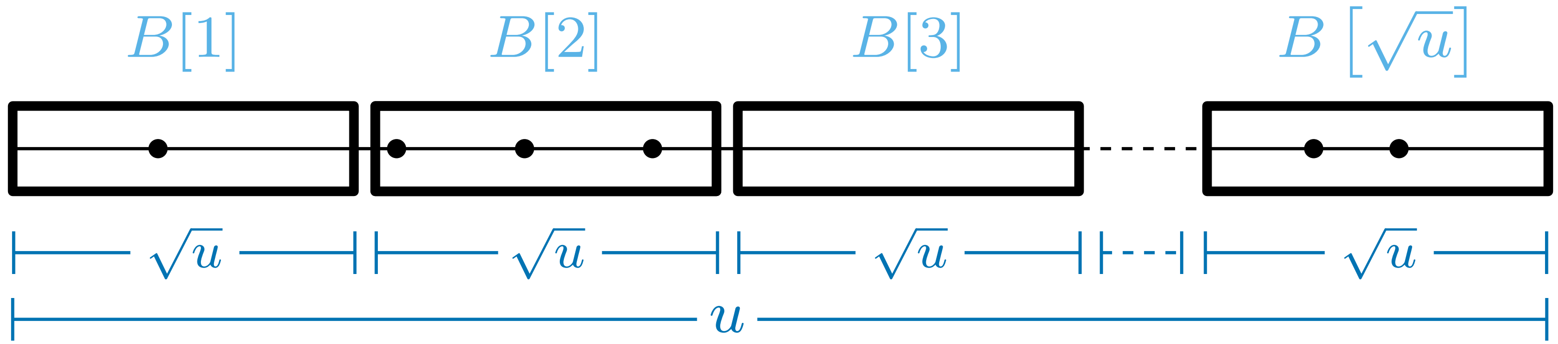

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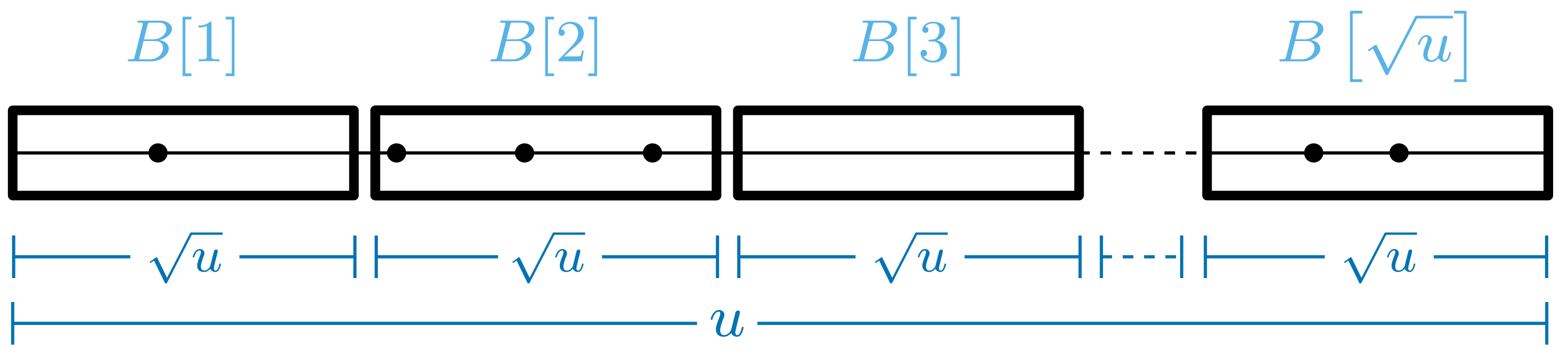

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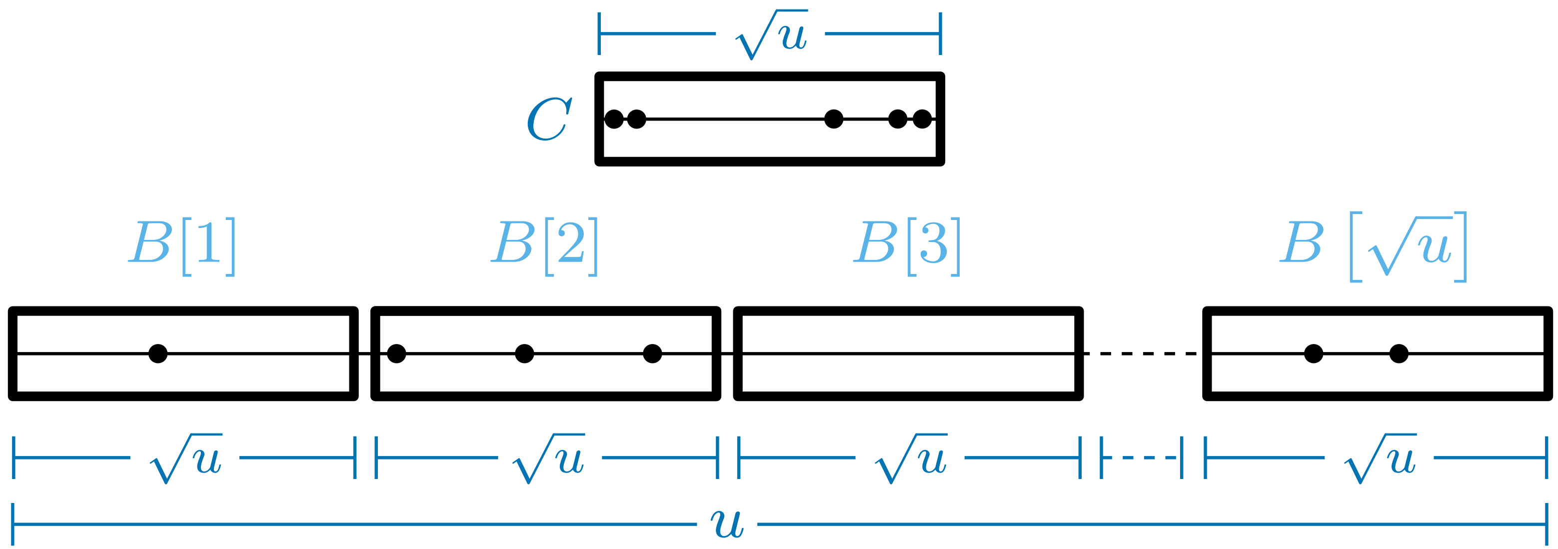

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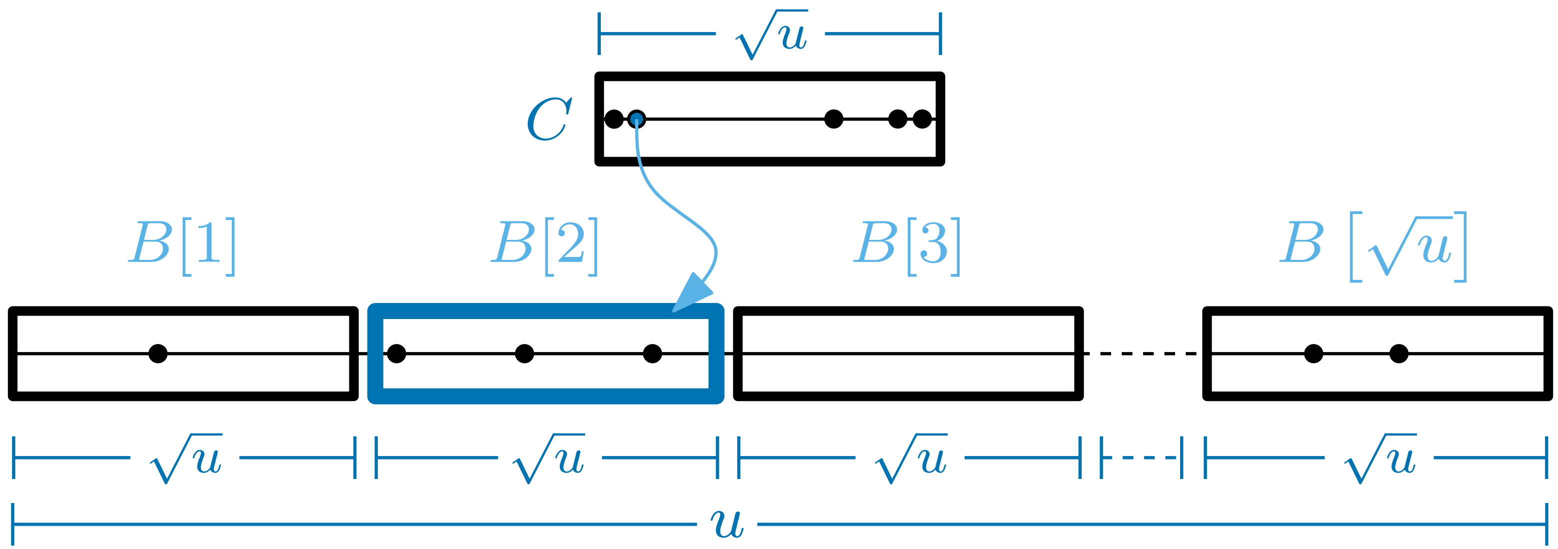
We also build a summary data structure C

which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

i is stored in C iff $B[i]$ is non-empty

An abstract view

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements



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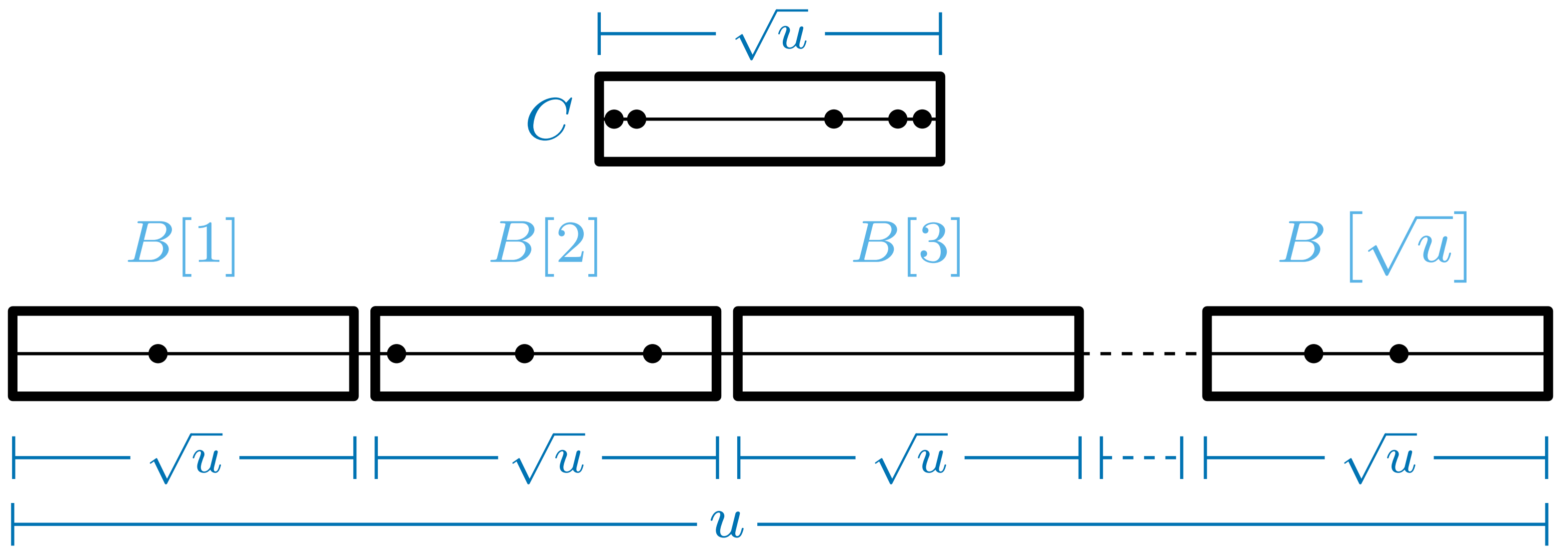
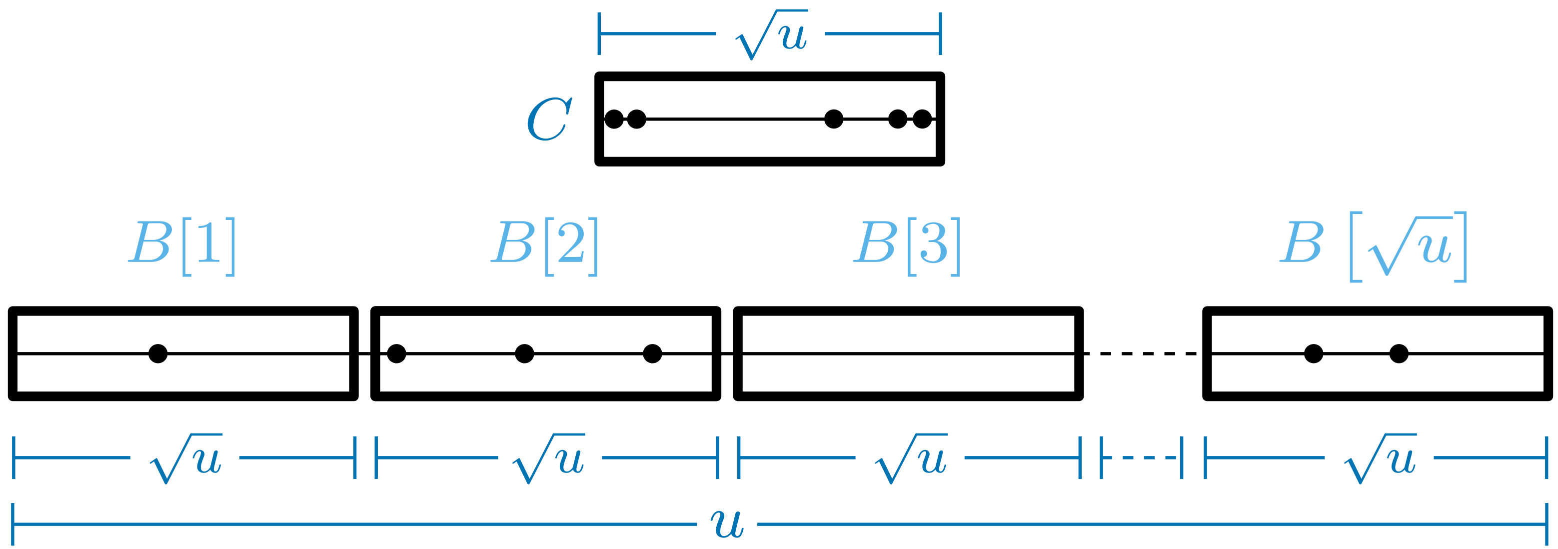


How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and C?

An abstract view

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements



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Recursion!

An abstract view

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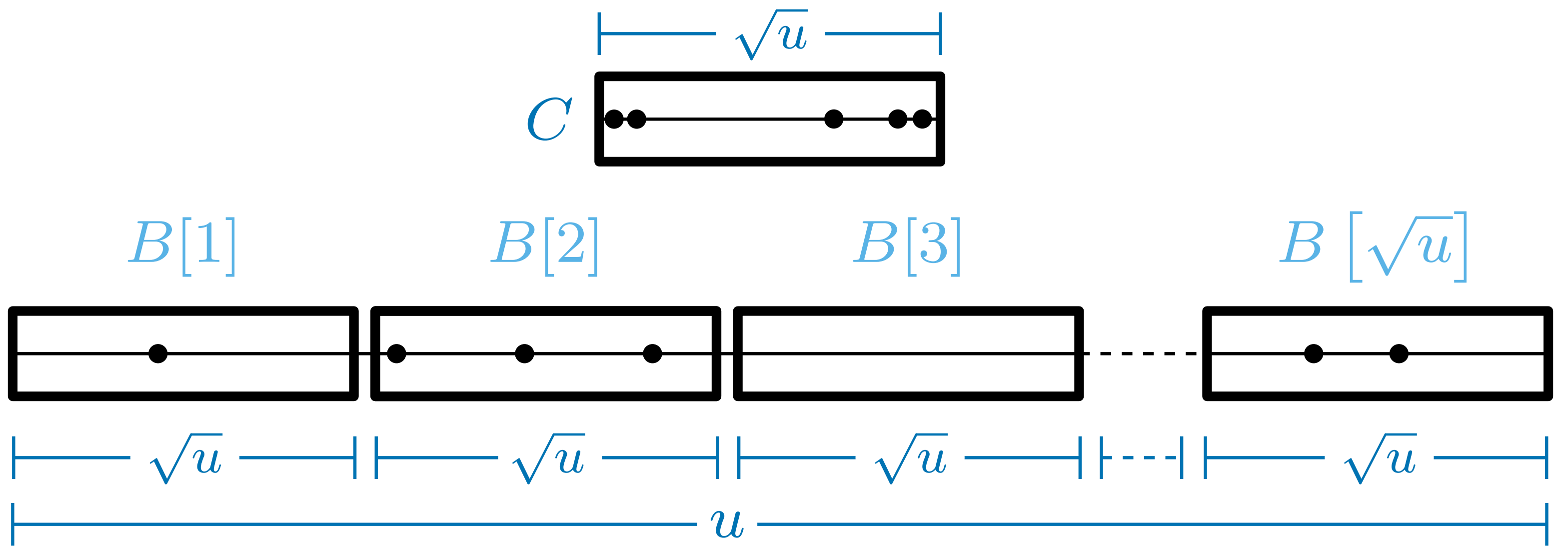

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and C?

Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$

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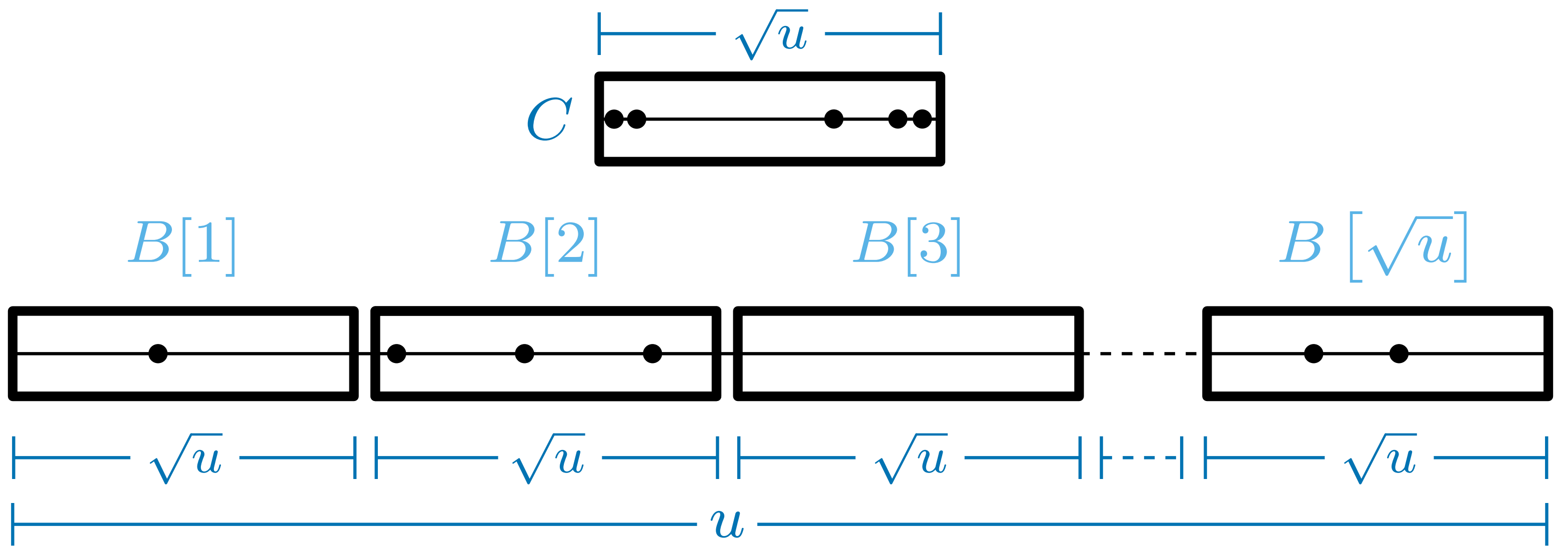


How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and C?

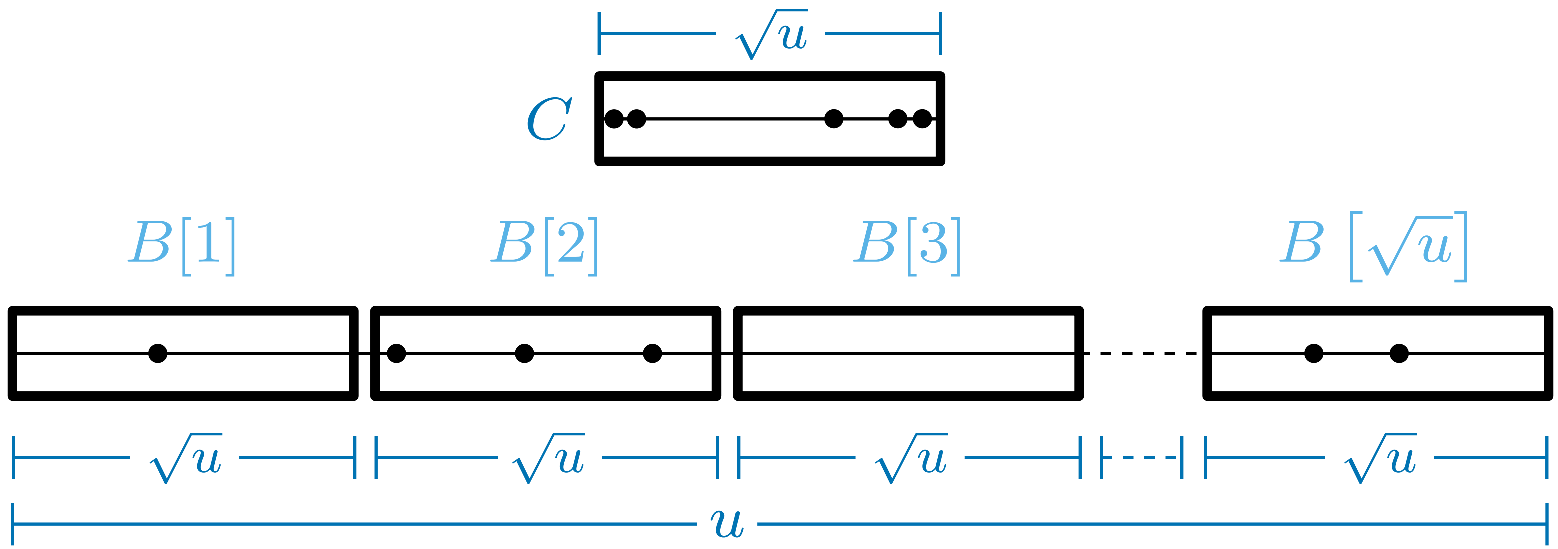
Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$

We recursively split this into $\sqrt[4]{u}$ blocks each associated with $\sqrt[4]{u}$ elements...

An abstract view

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements



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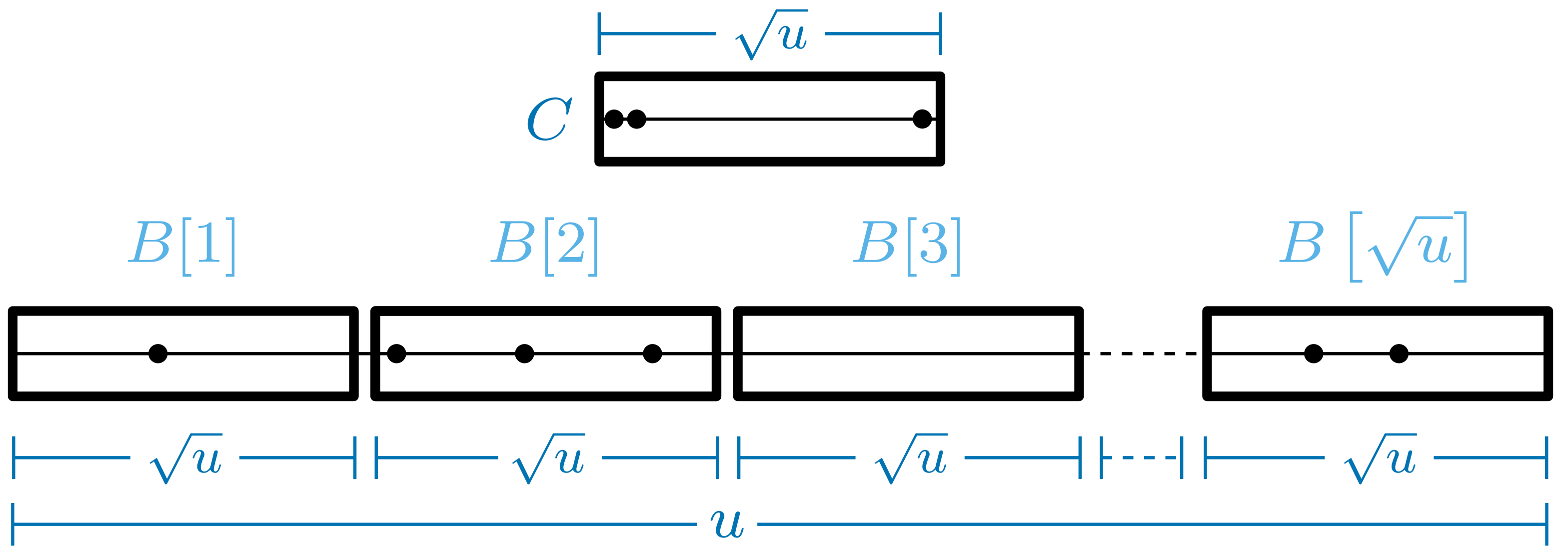
We recursively split this into $\sqrt[4]{u}$ blocks each associated with $\sqrt[4]{u}$ elements...

eventually (after some more work), this will lead to an $O(\log \log u)$ time solution

Attempt 3: Recursion

Split the universe U into $\sqrt{u}$ *blocks* each associated with $\sqrt{u}$ elements



Attempt 3: Recursion

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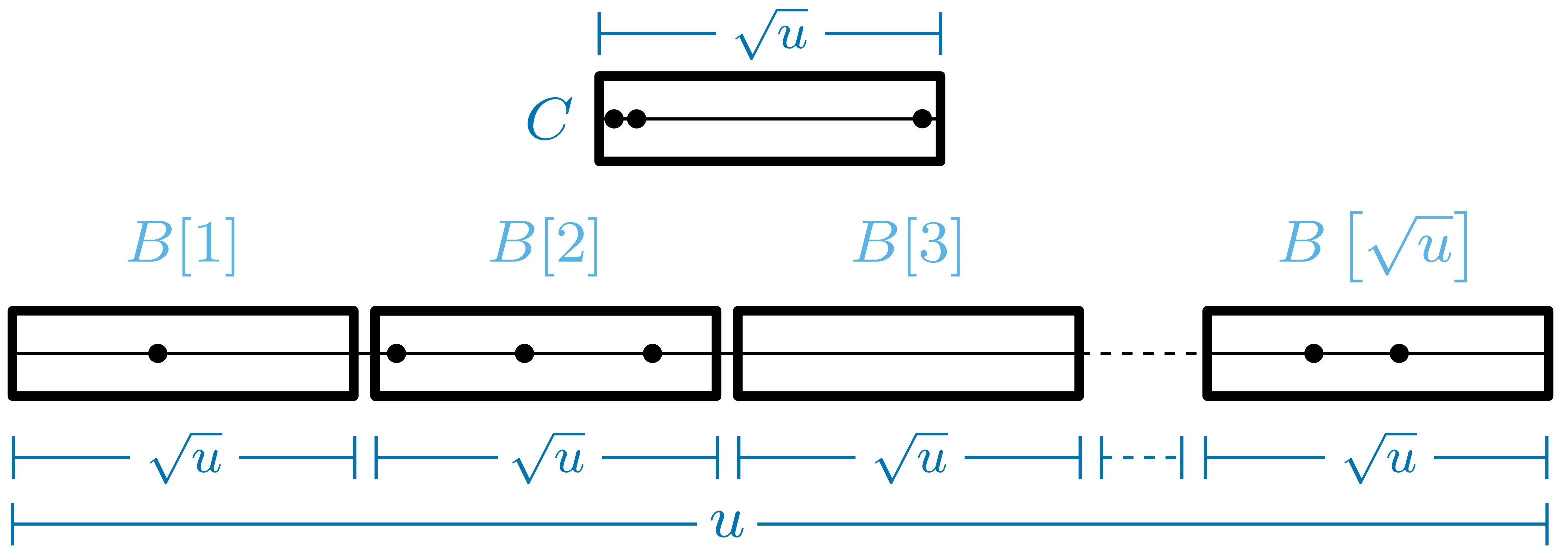

How do we perform the operations?

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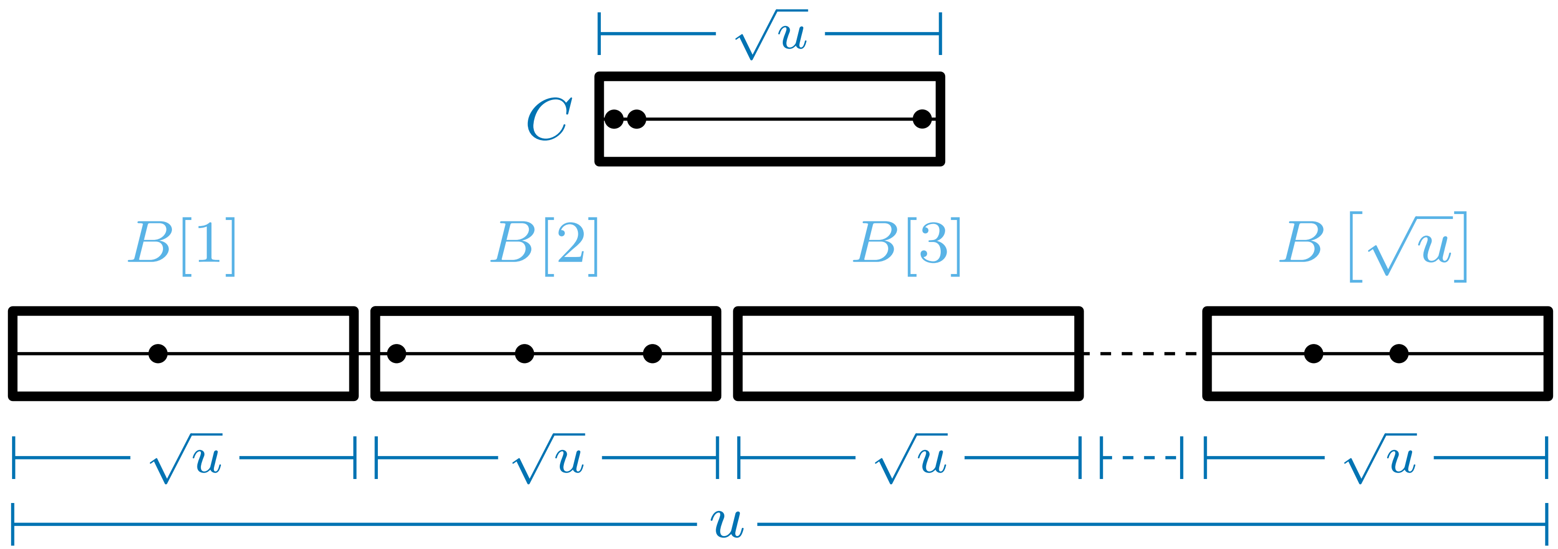

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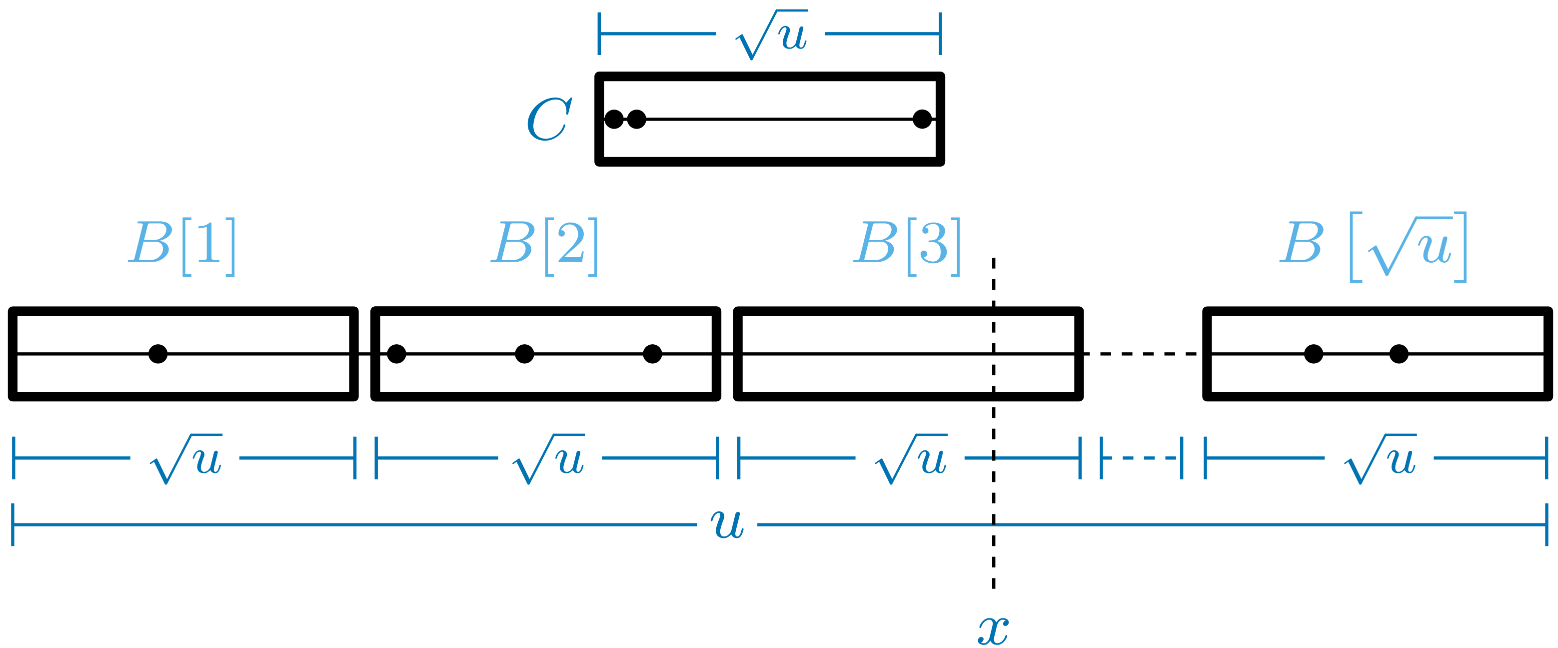

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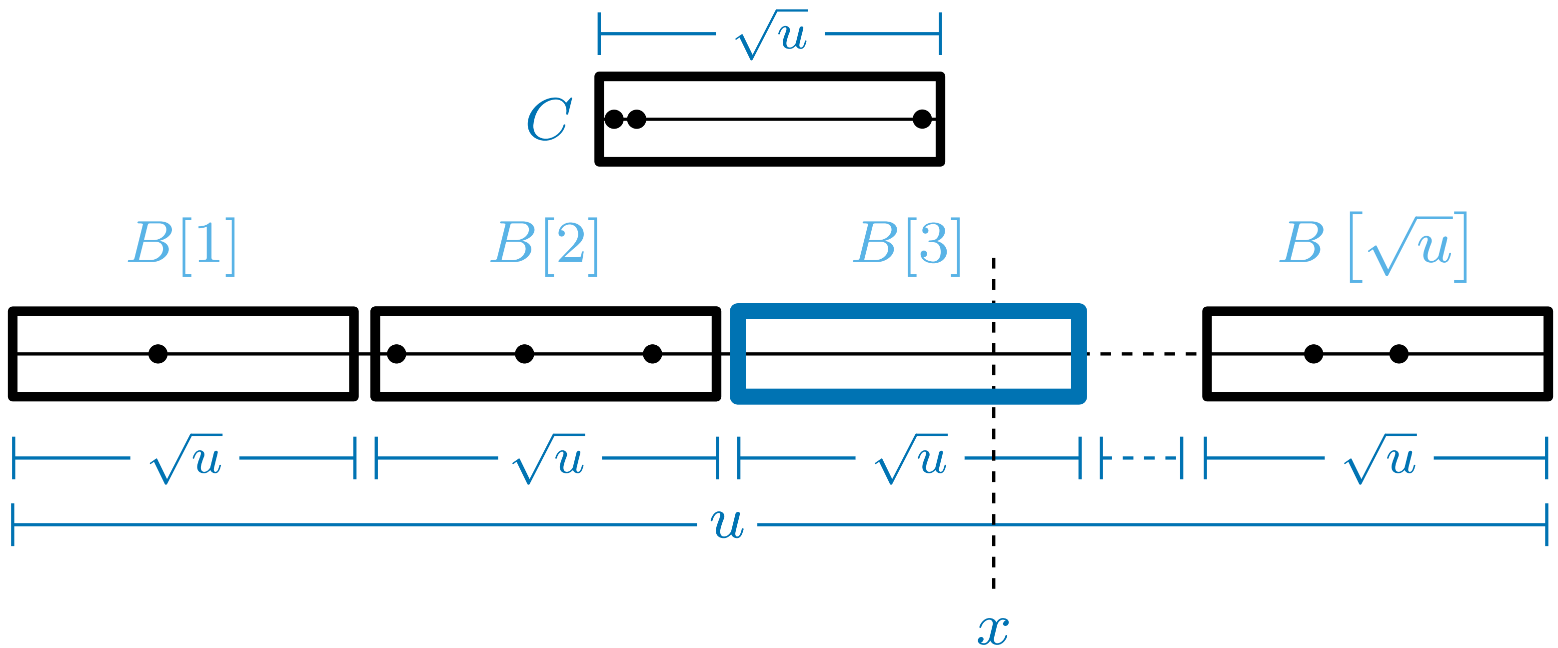

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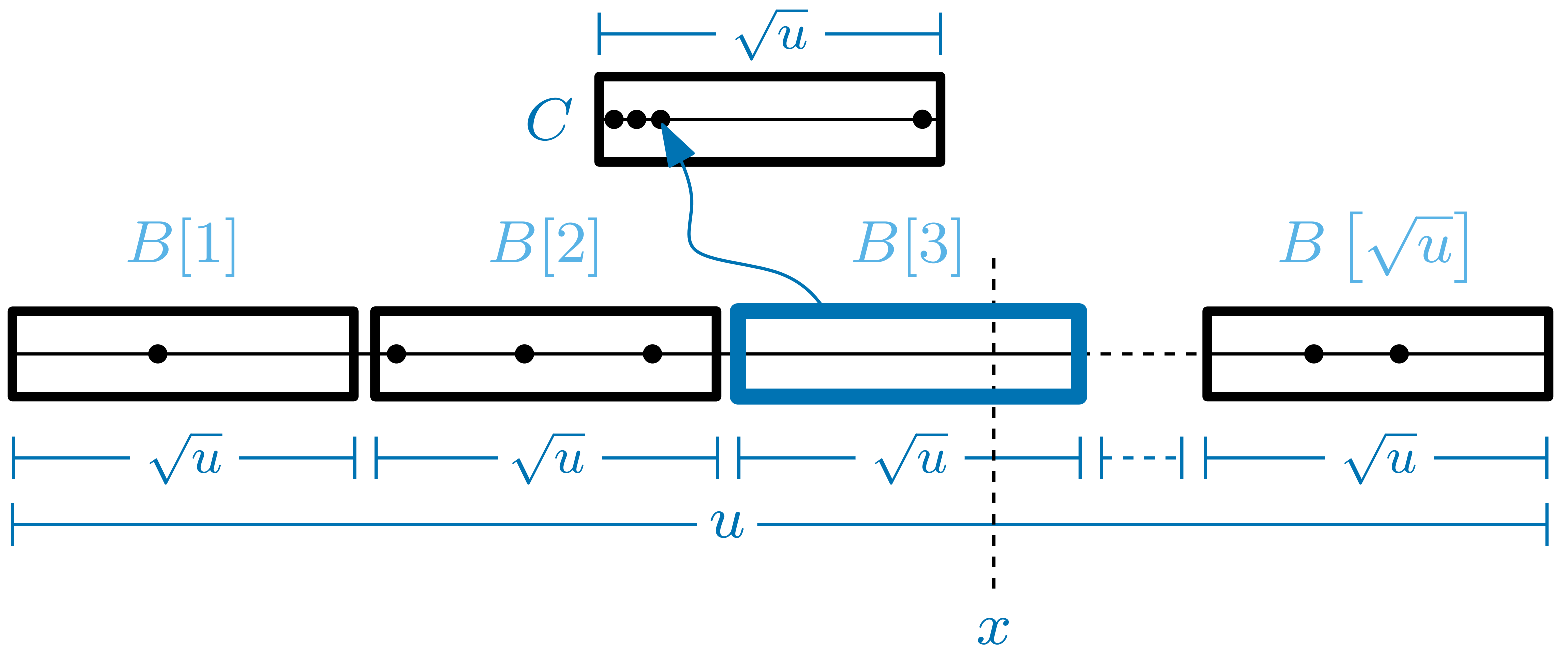

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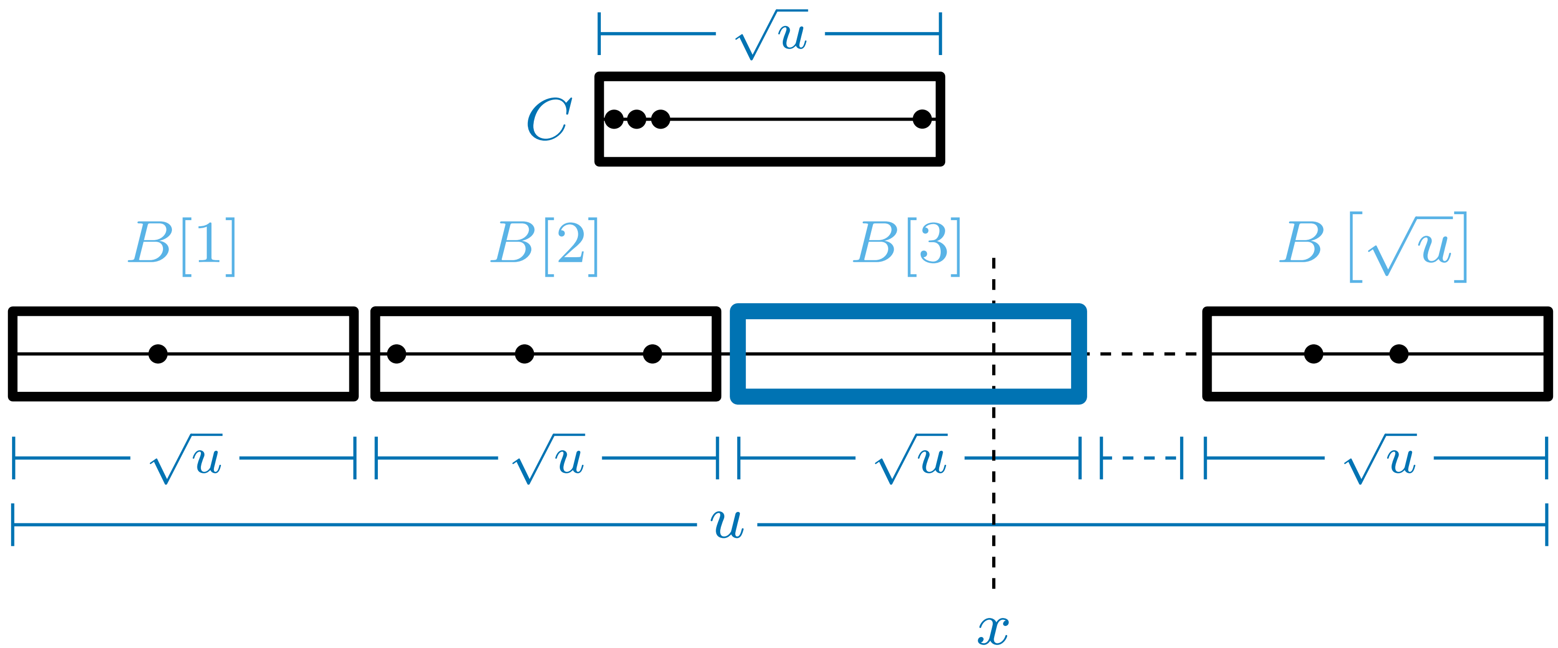

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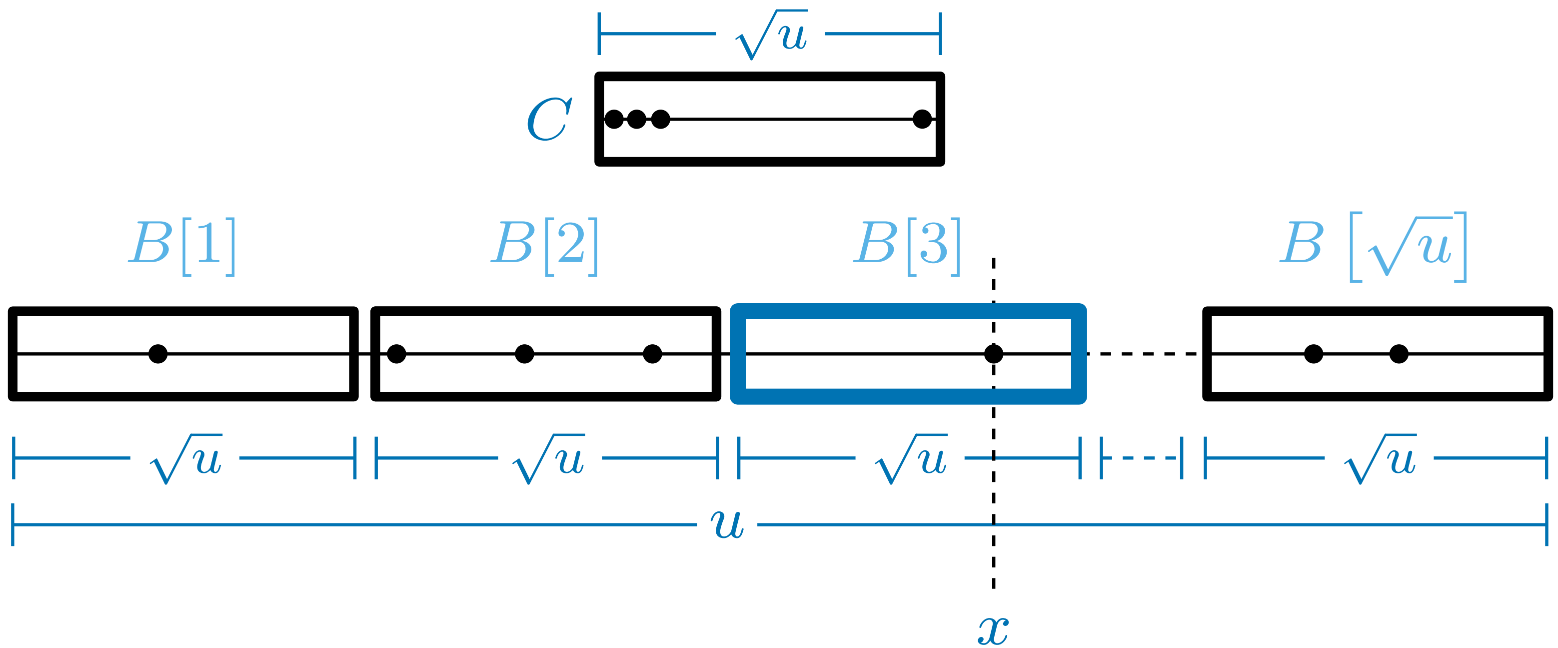

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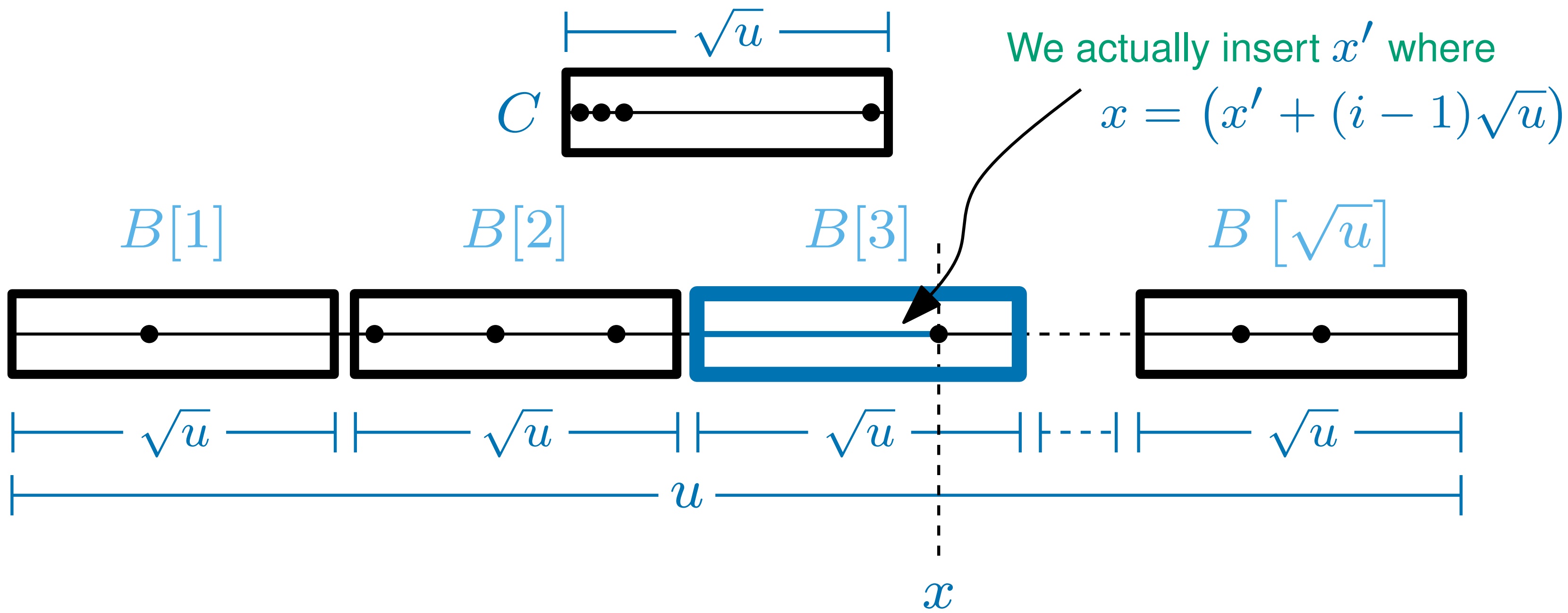

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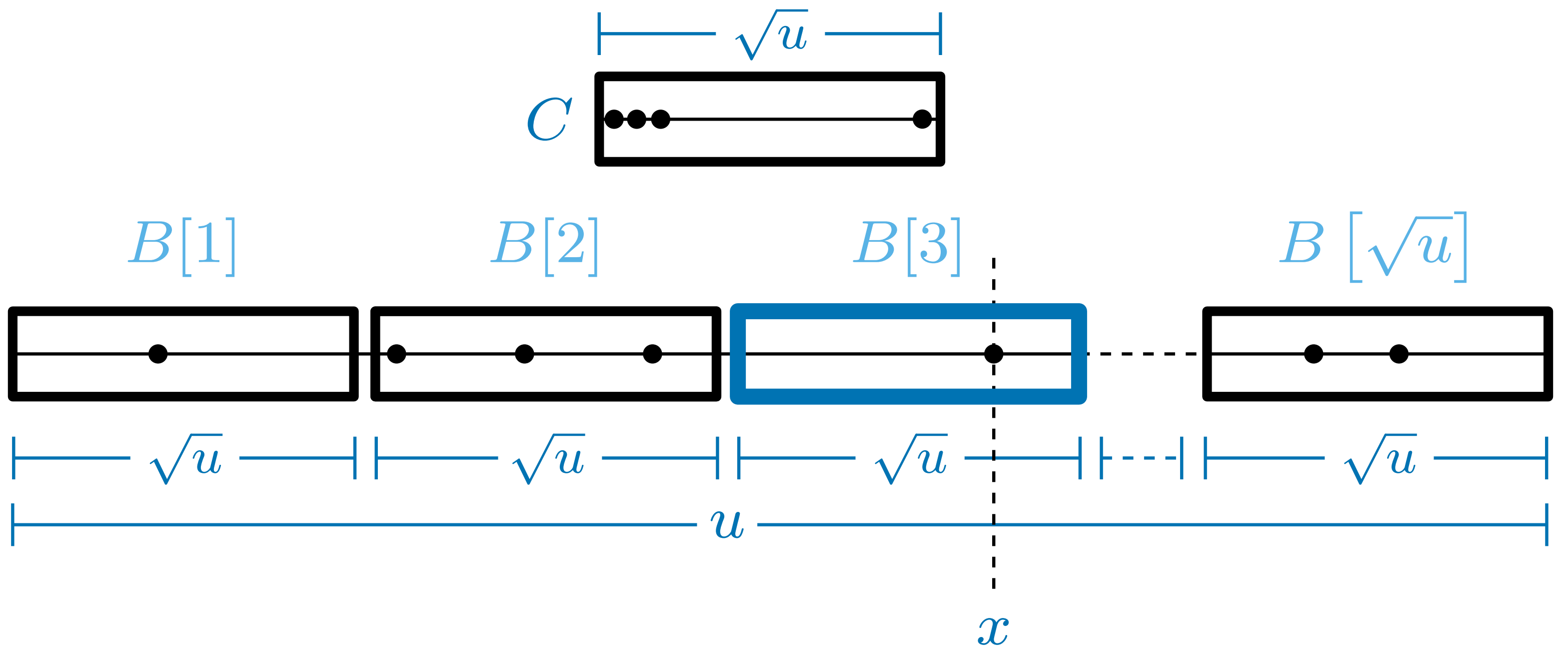

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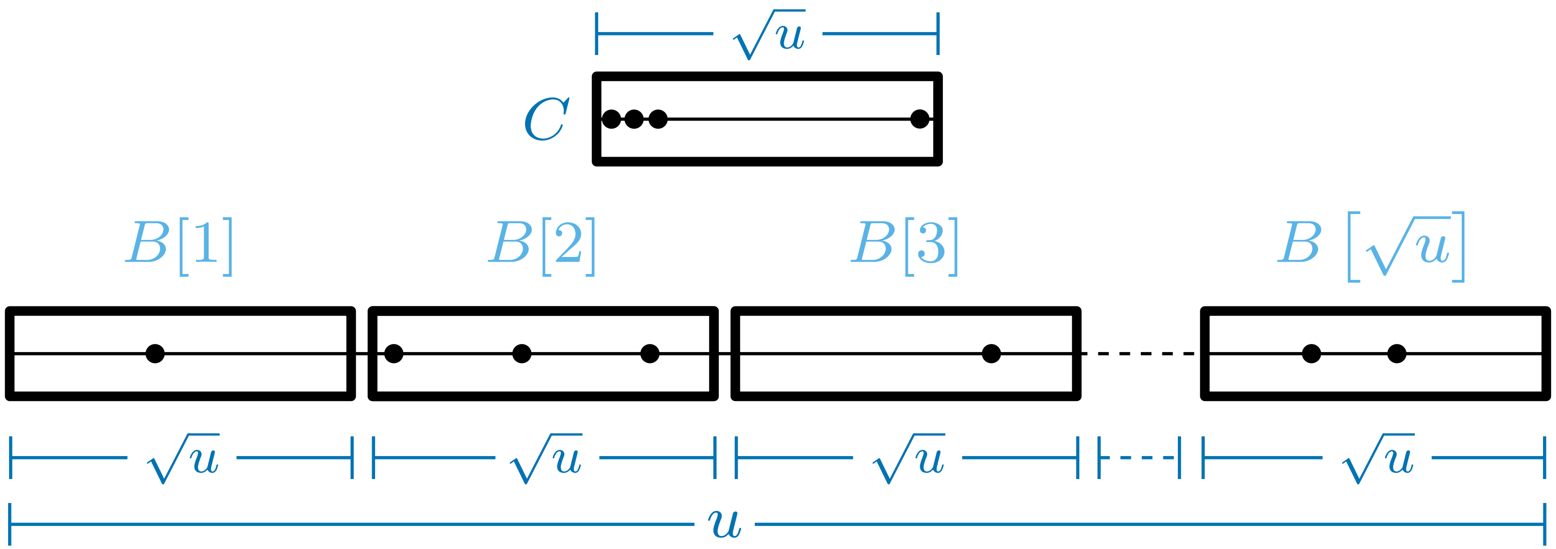

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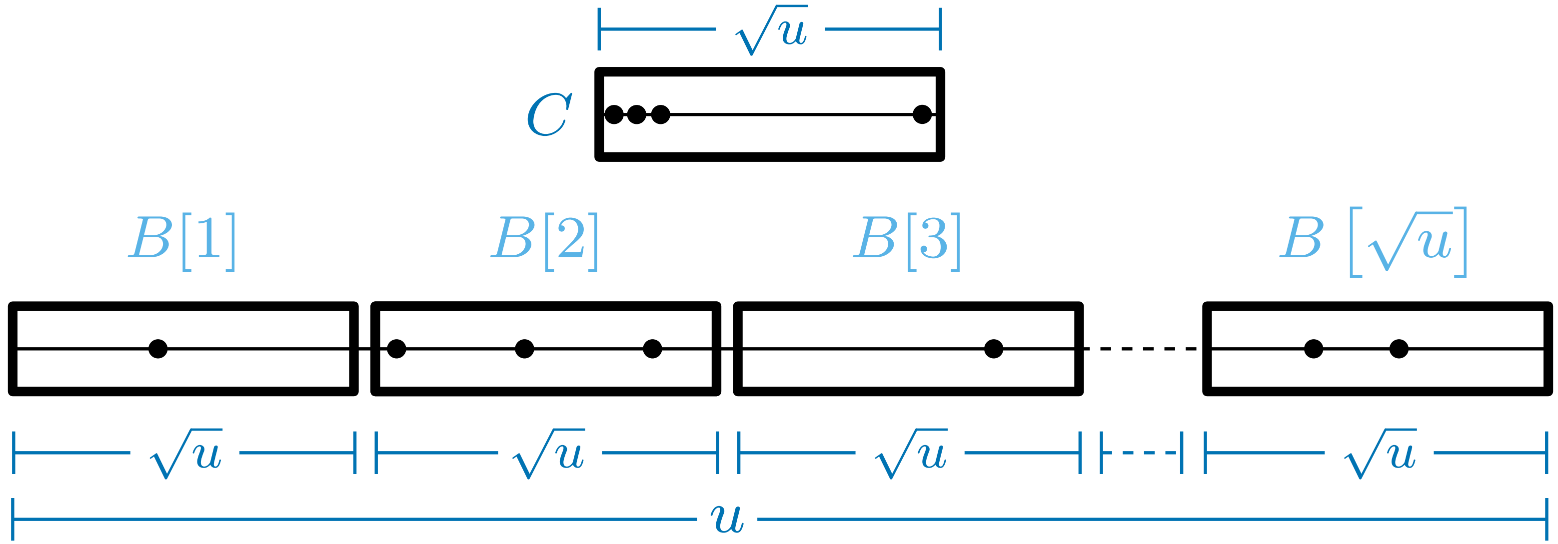

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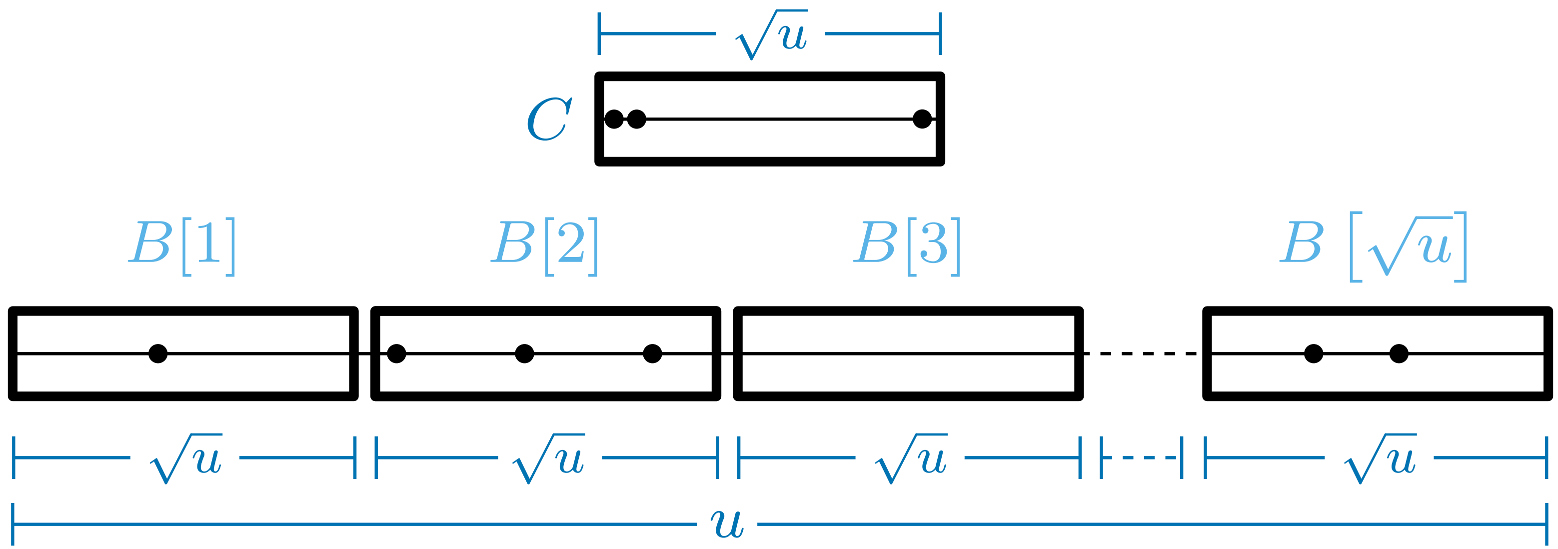

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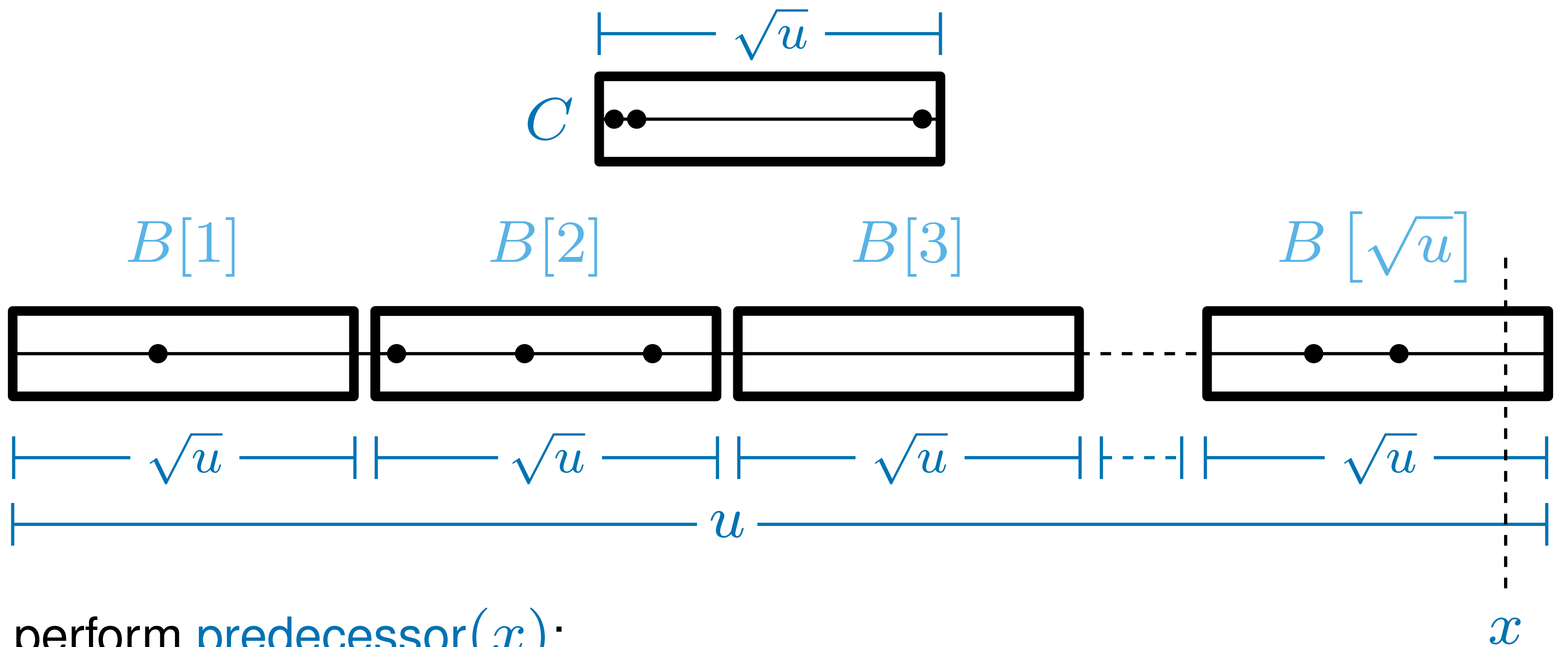

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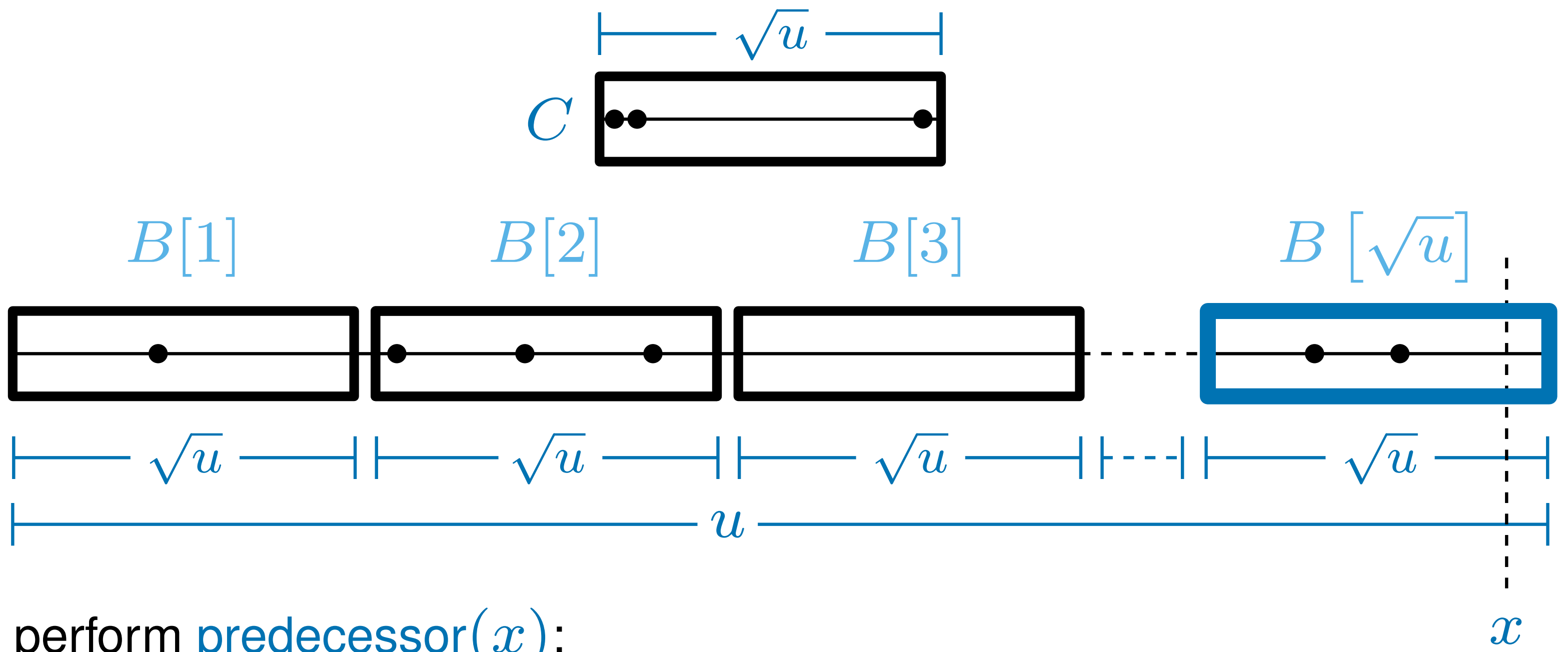

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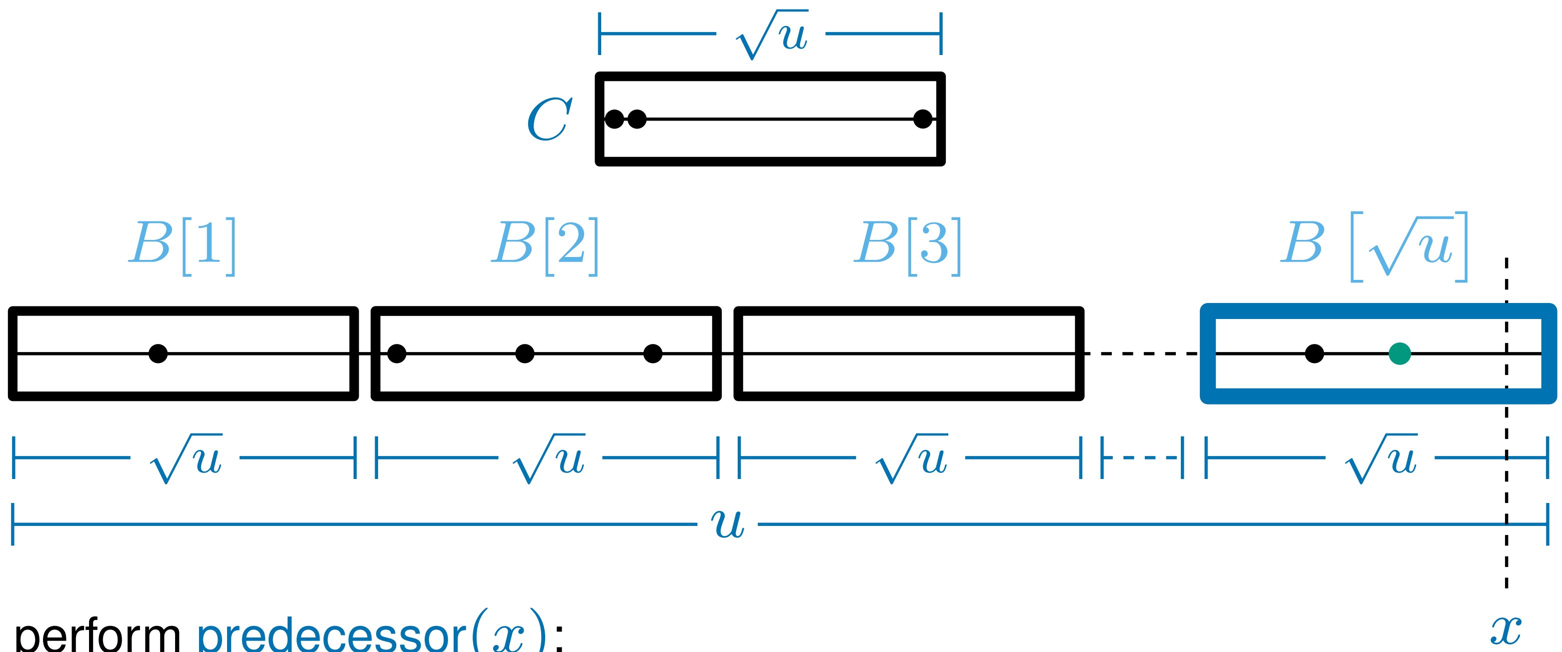

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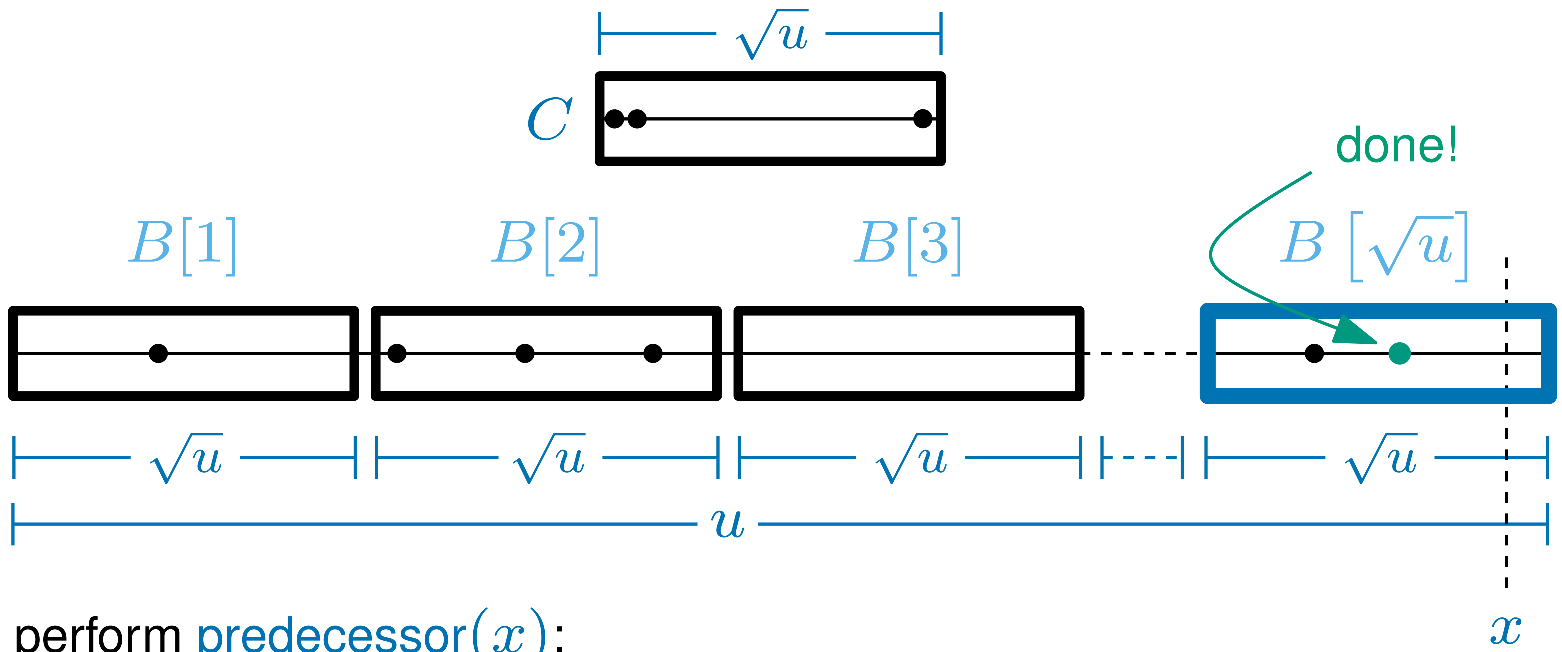

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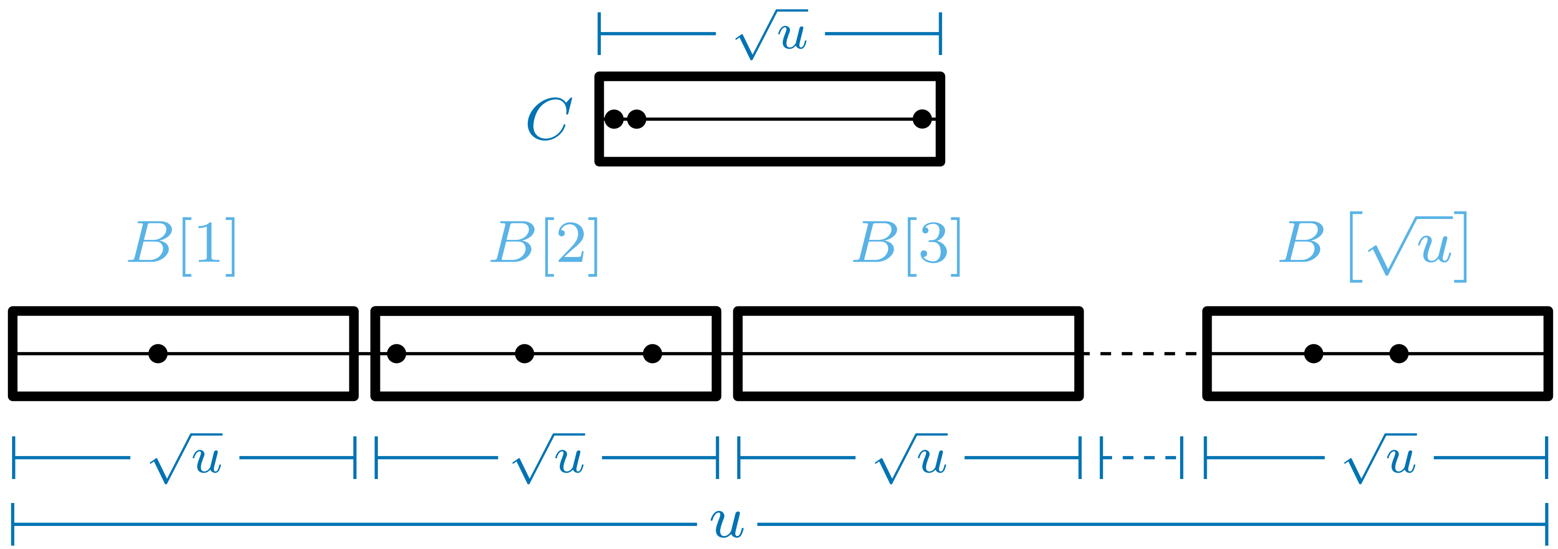

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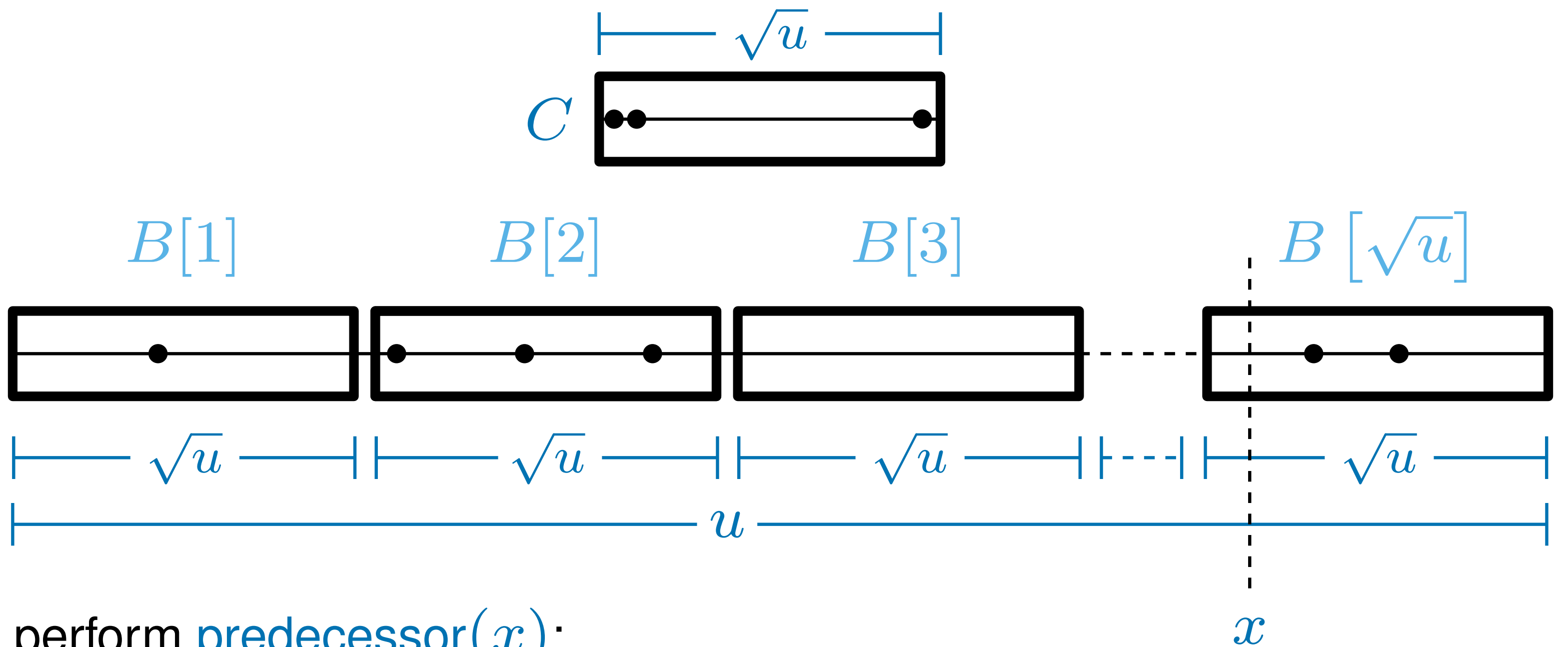

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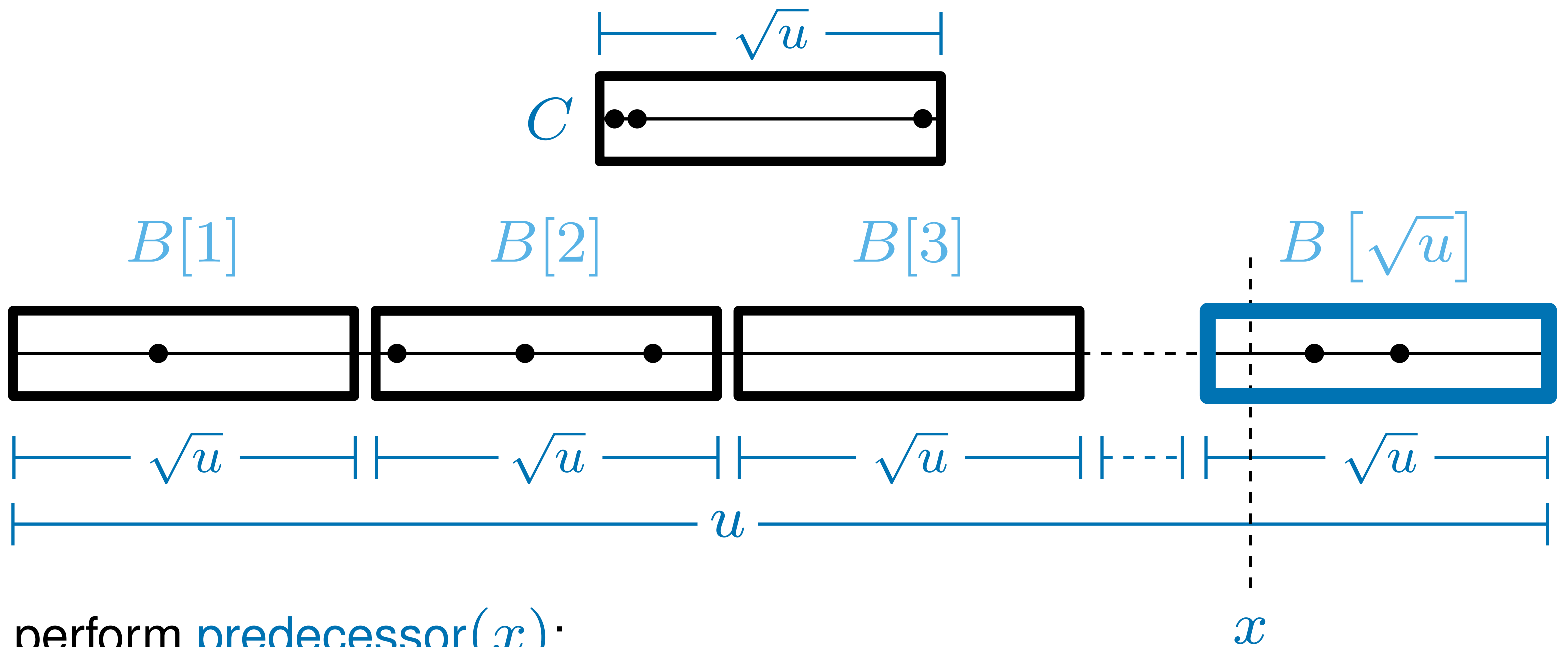

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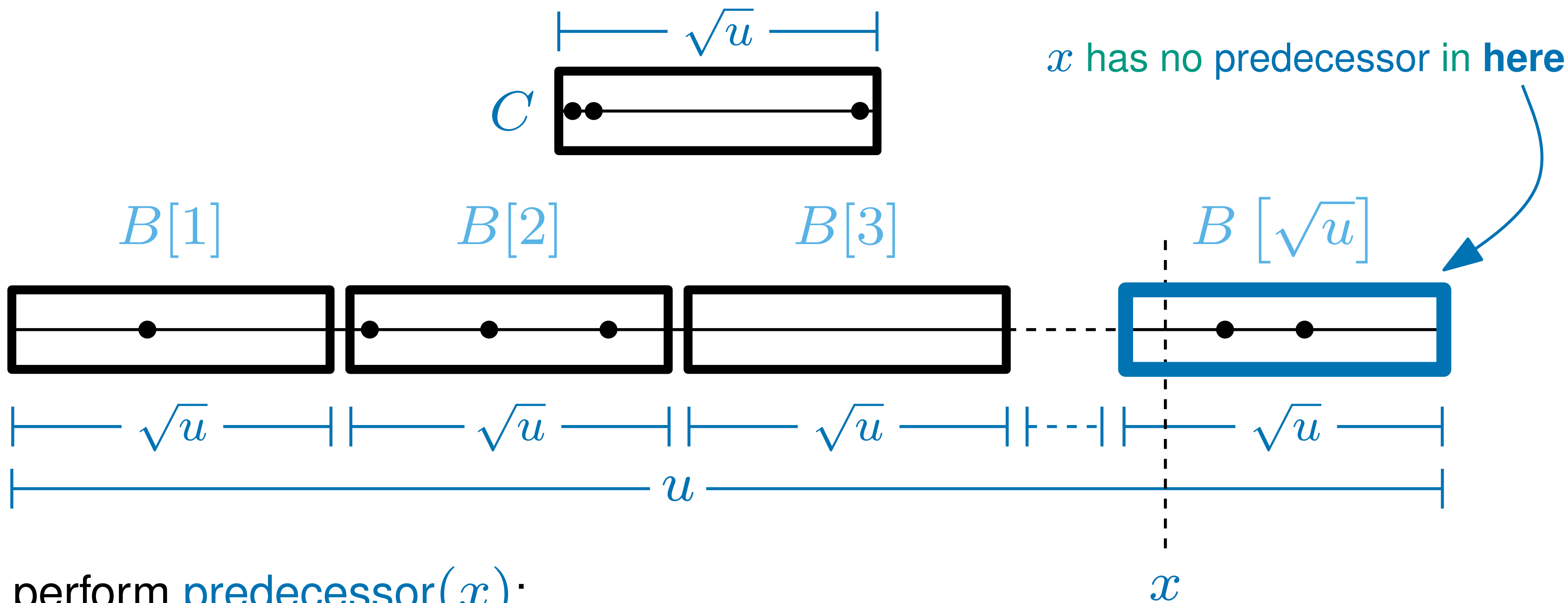

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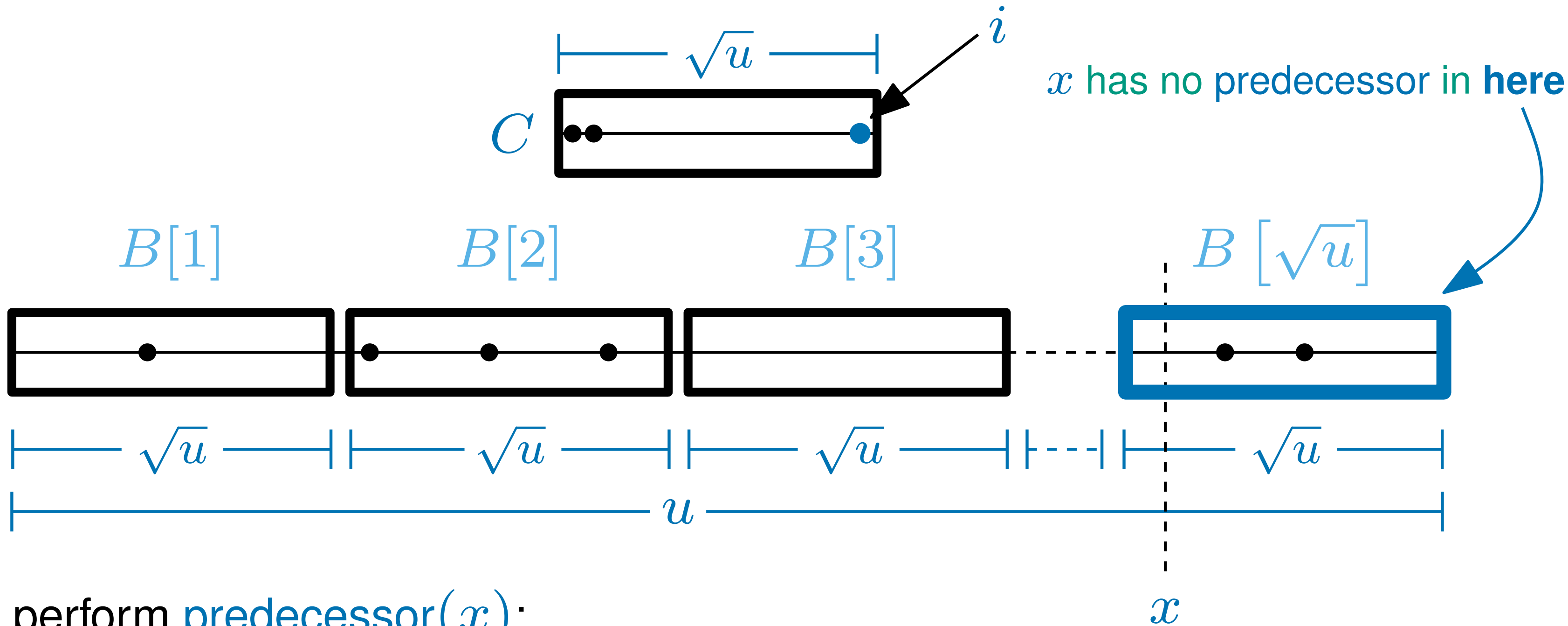

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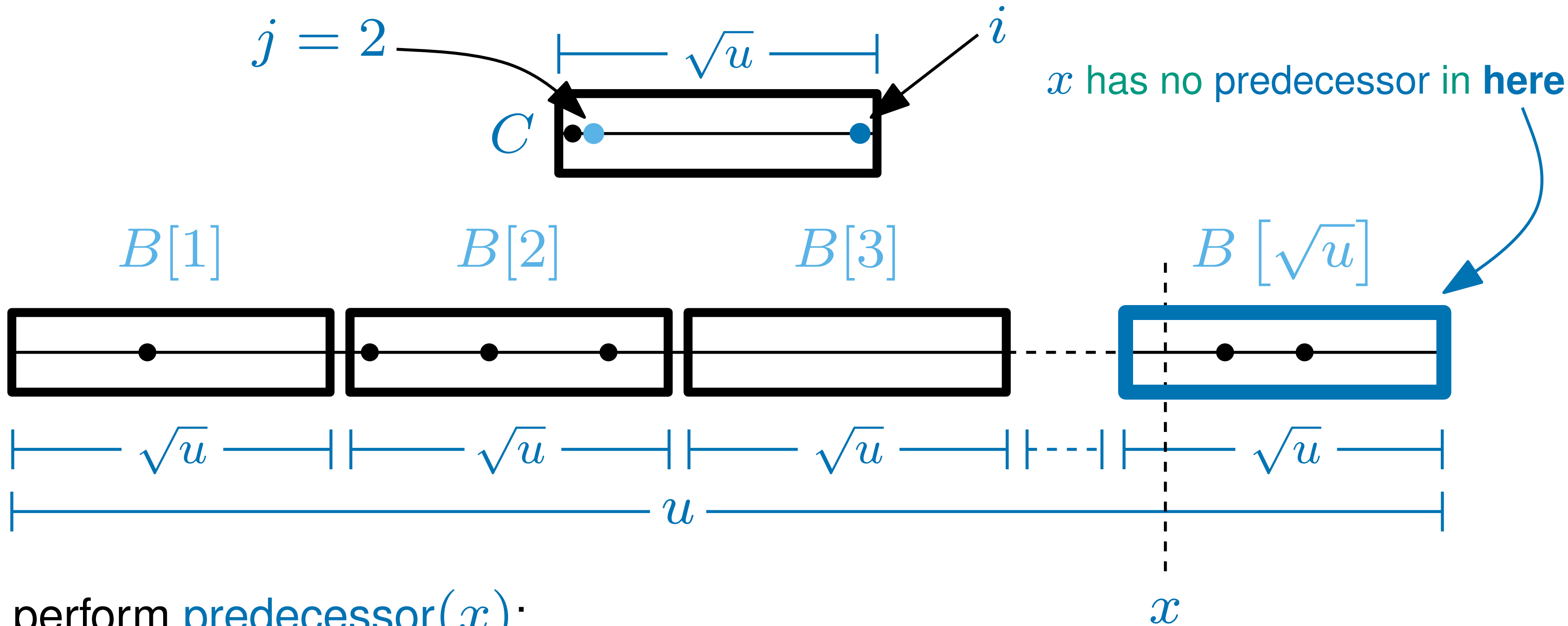

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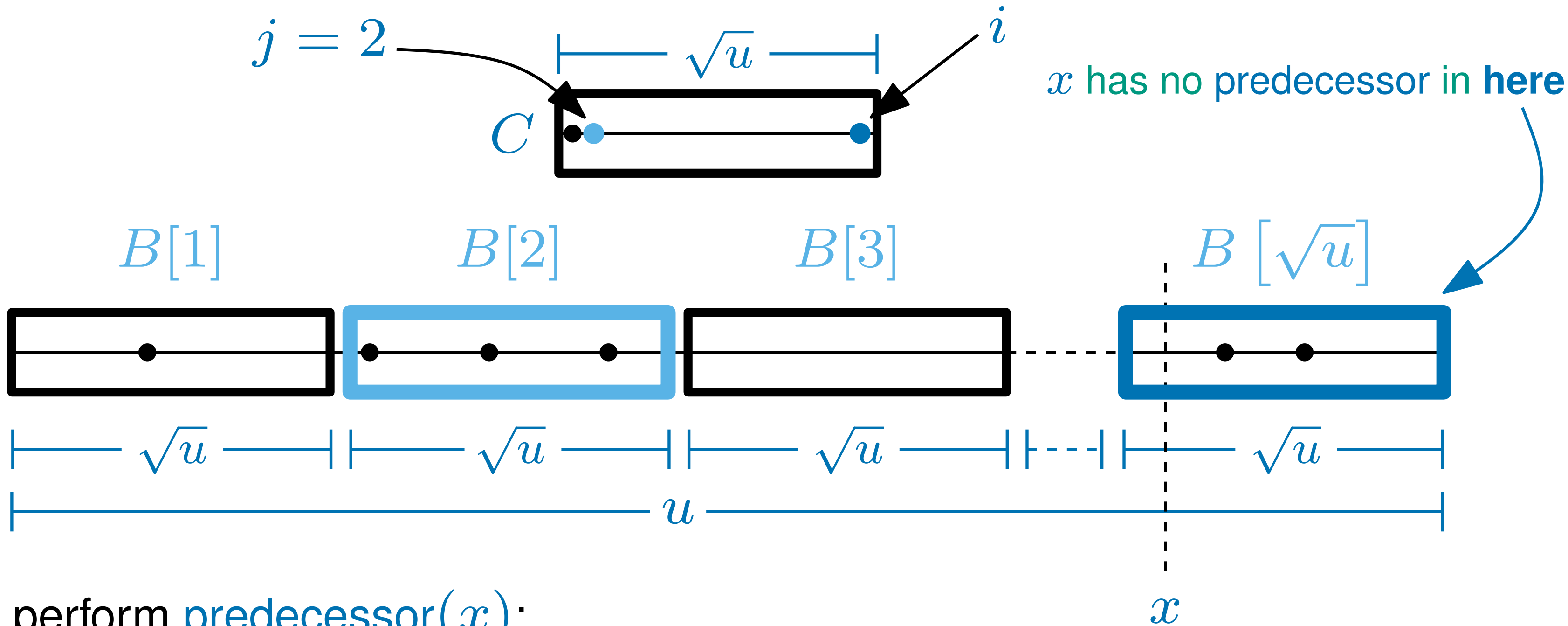

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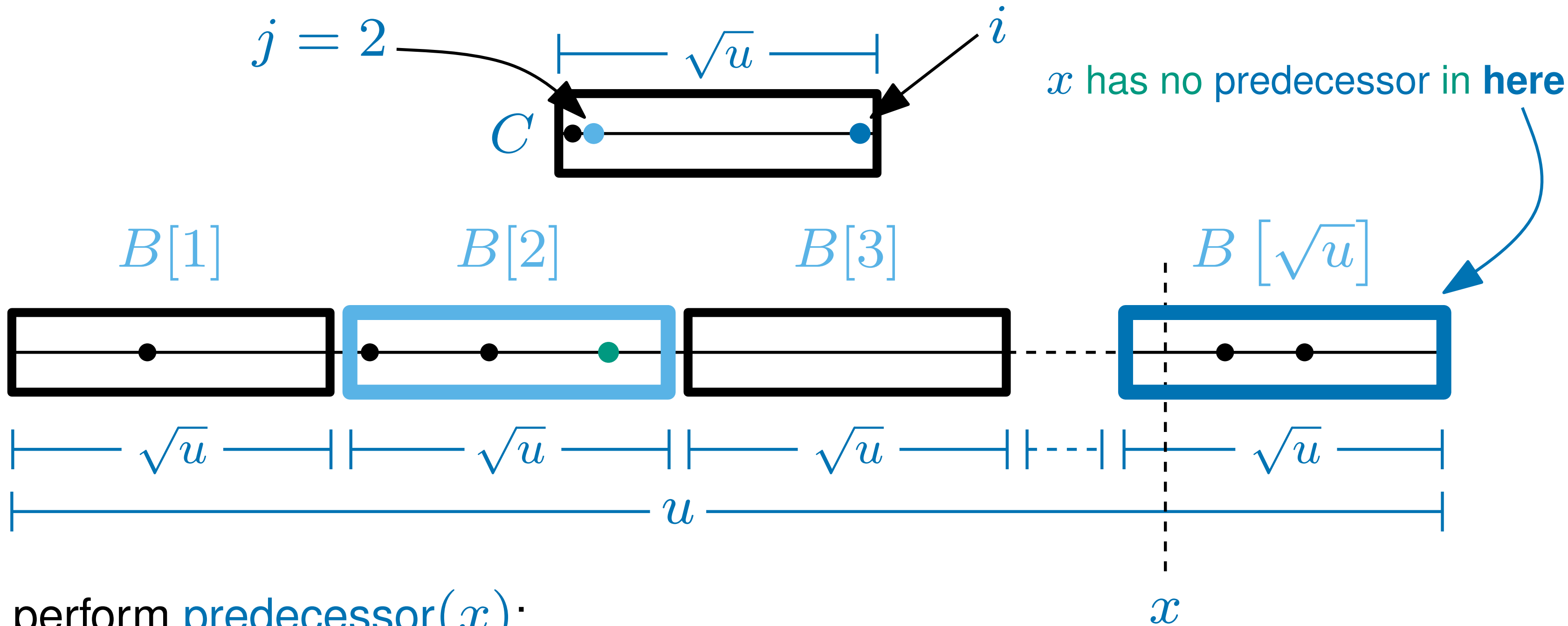

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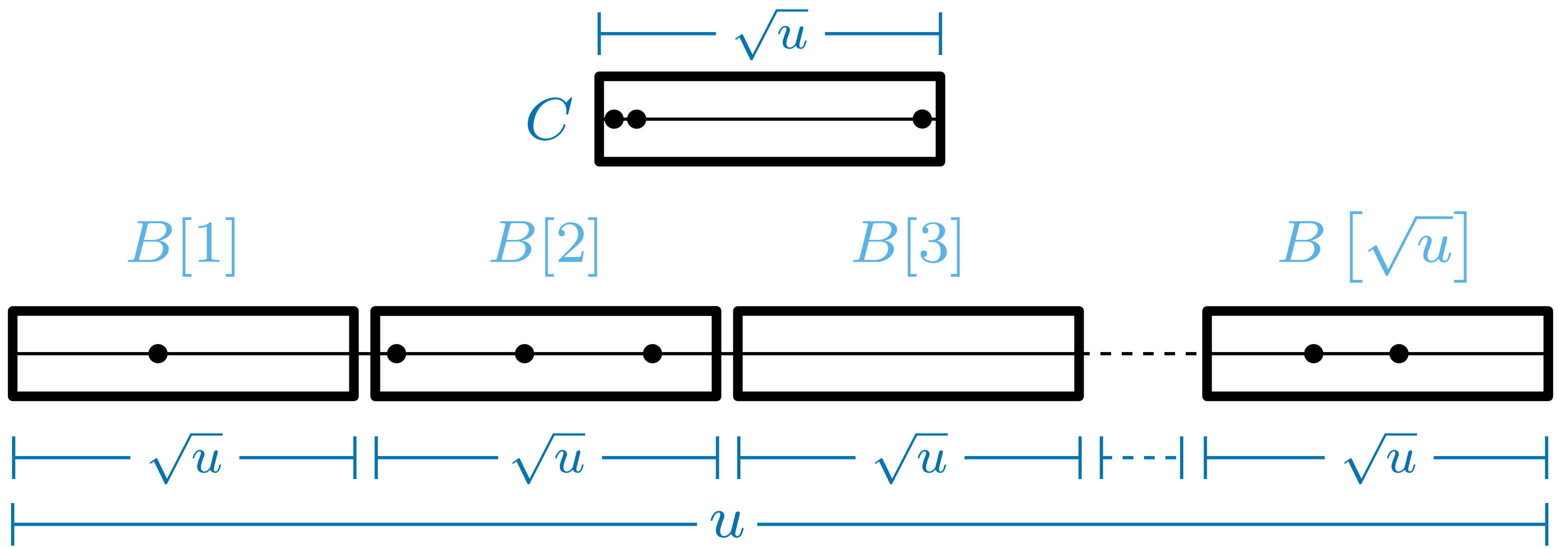

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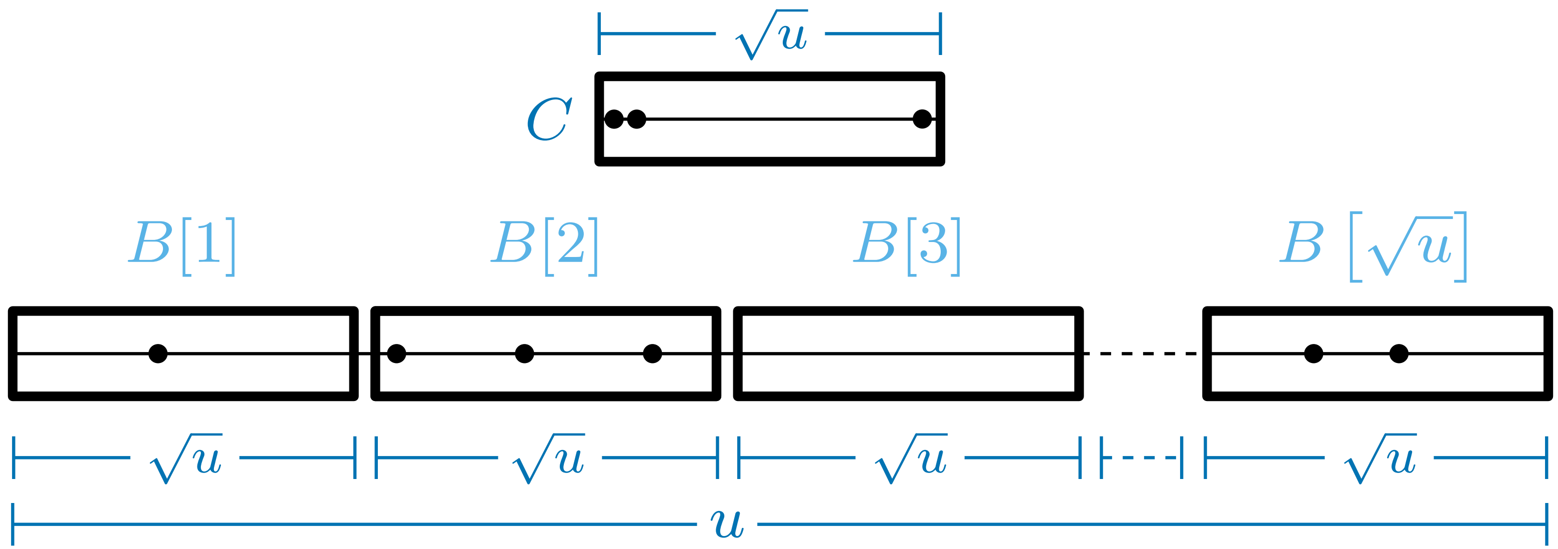

The operations lookup, delete and successor can all also be defined in a similar, recursive manner

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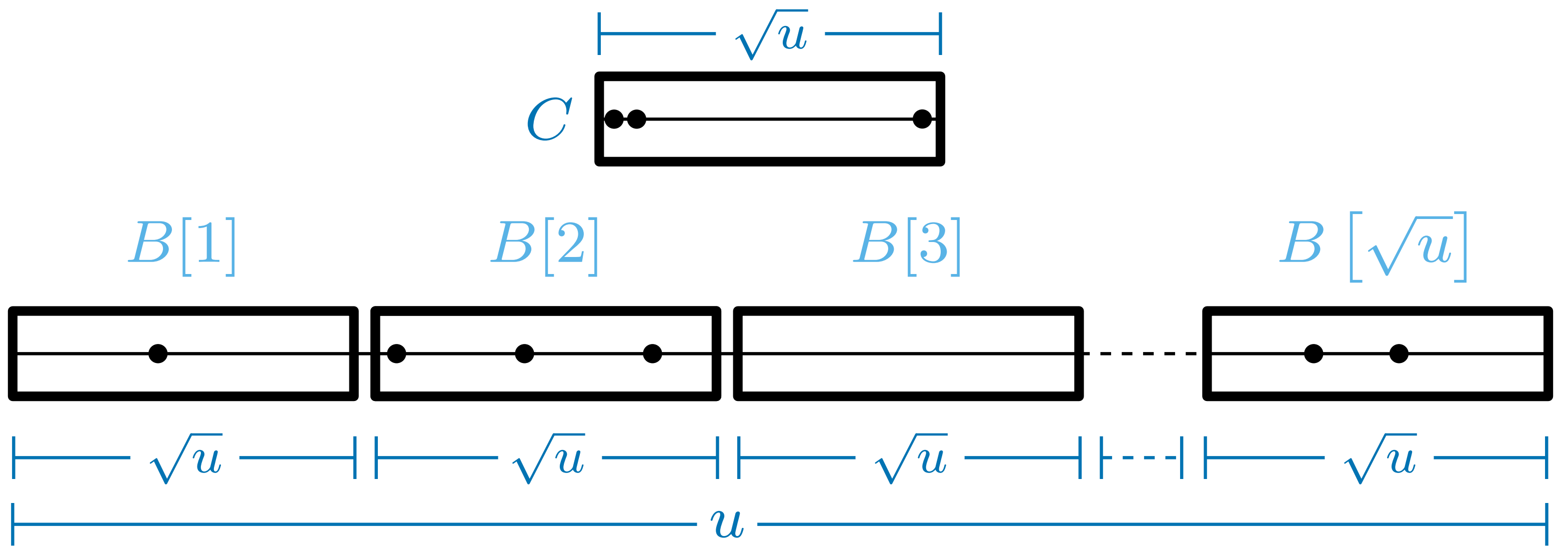

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How efficient are the operations?

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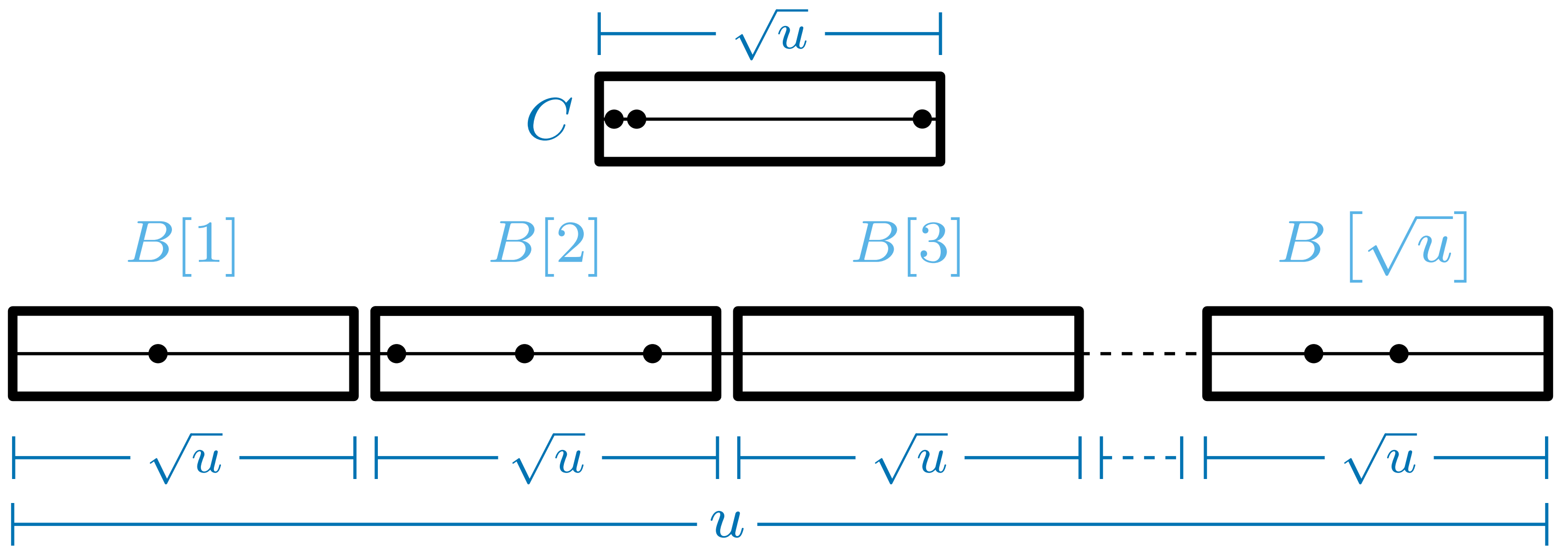

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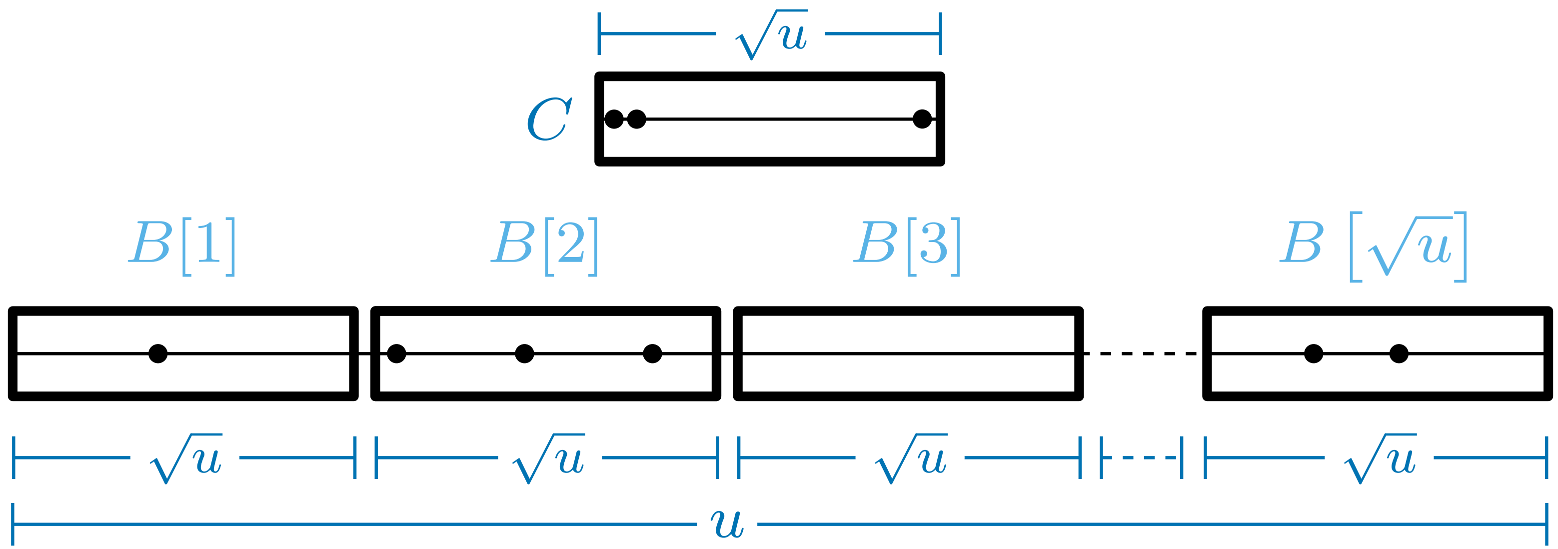

add makes up to two recursive calls

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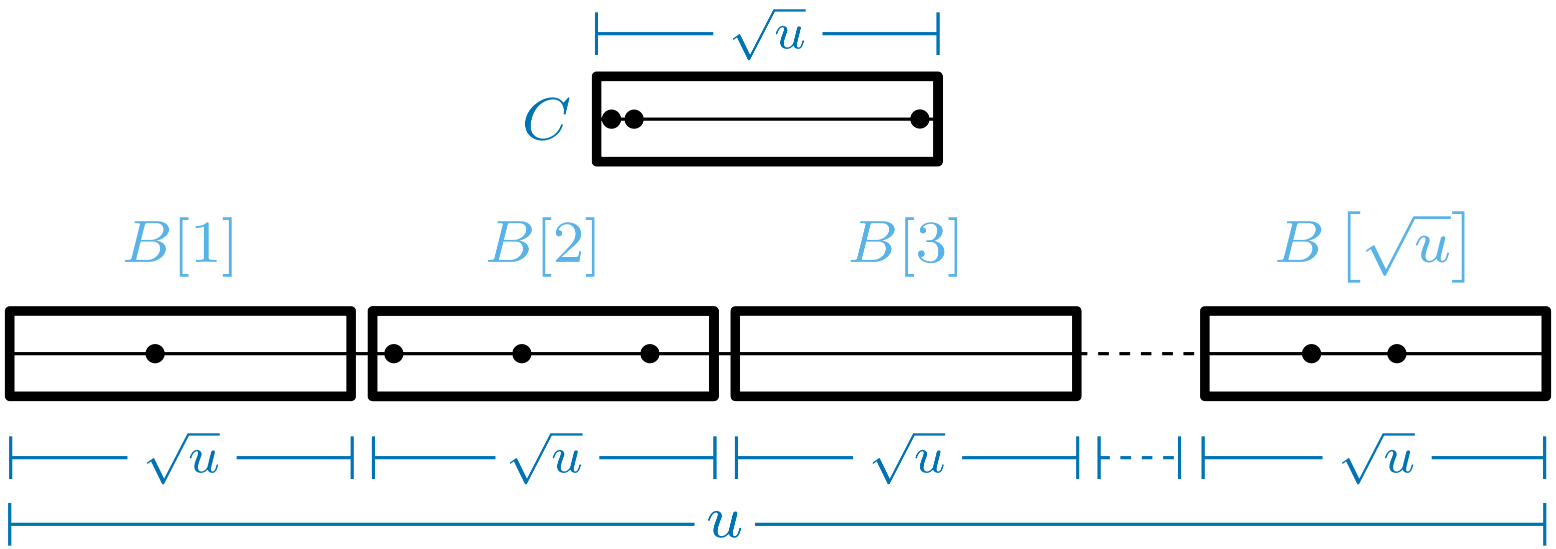

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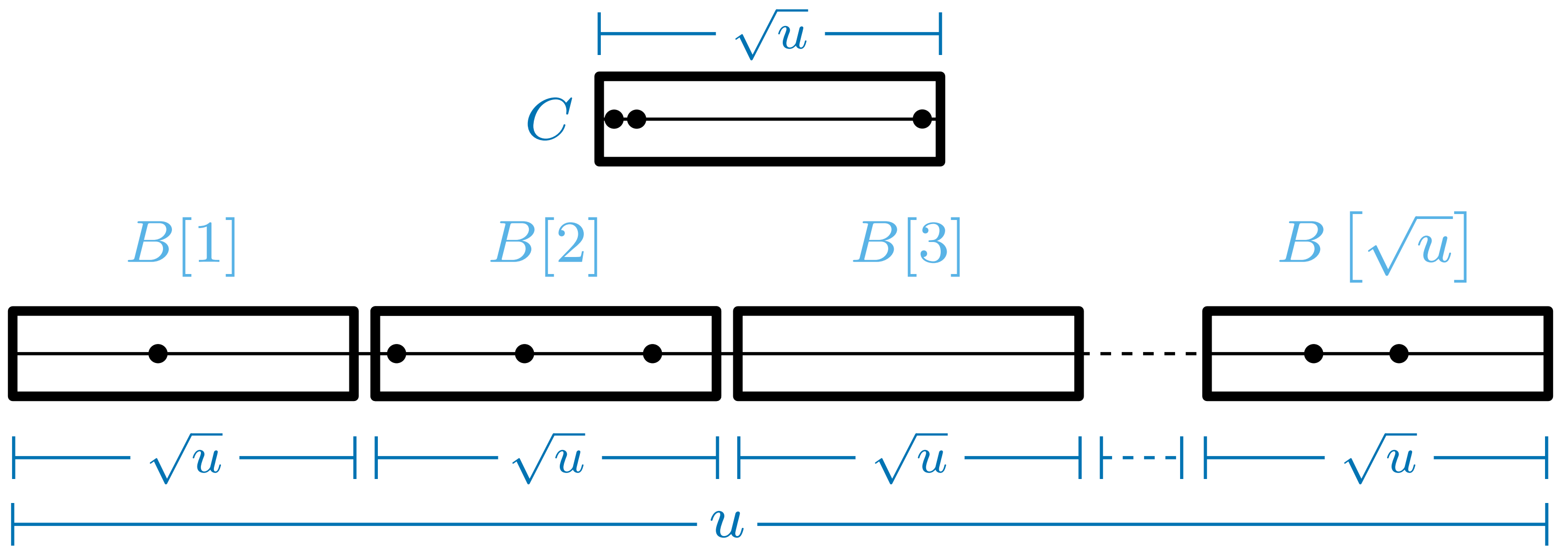

To perform predecessor(x):

predecessor makes up to **three** recursive calls!

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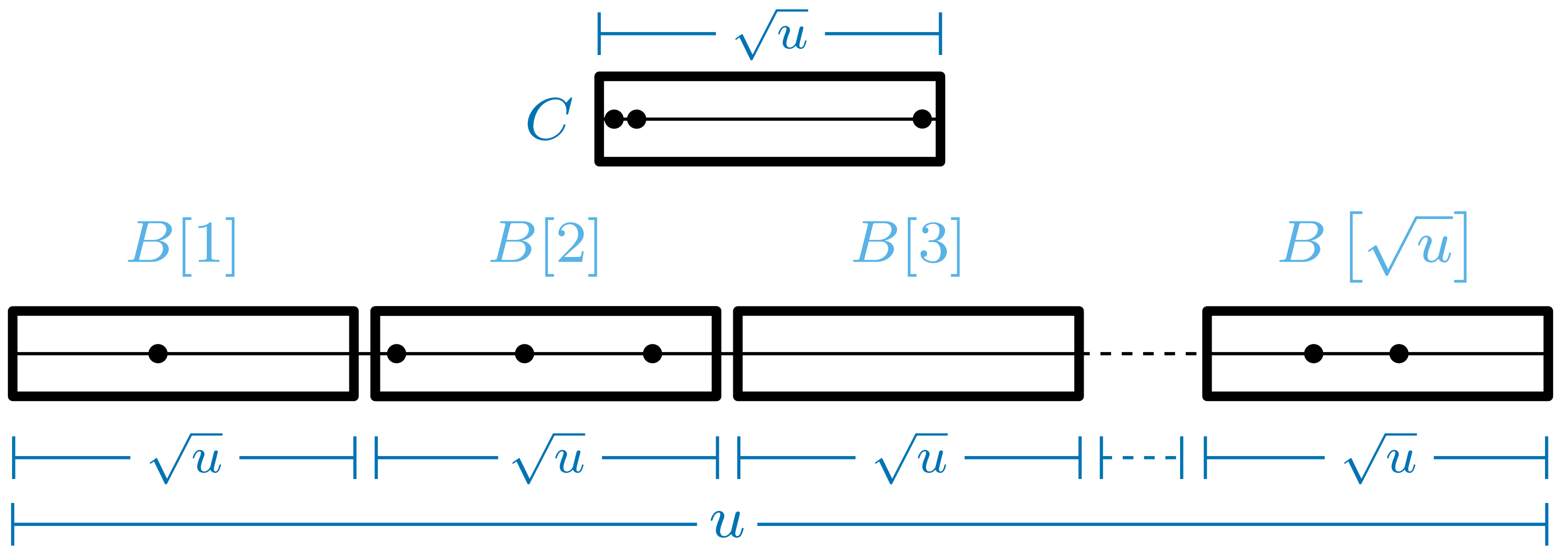

The operations lookup, delete and successor can all also be defined in a similar, recursive manner

How efficient are the operations?

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Attempt 3: Recursion

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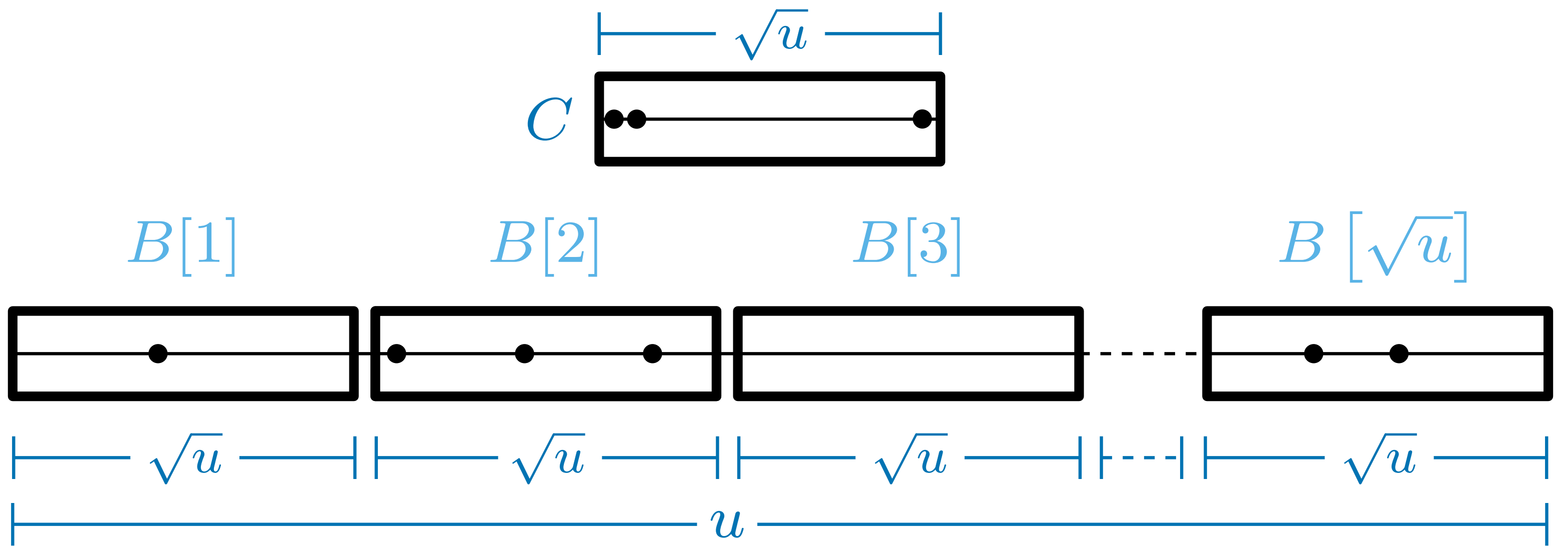

The operations lookup, delete and successor can all also be defined in a similar, recursive manner

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The add operation makes up to two recursive calls and the predecessor operation makes up to three

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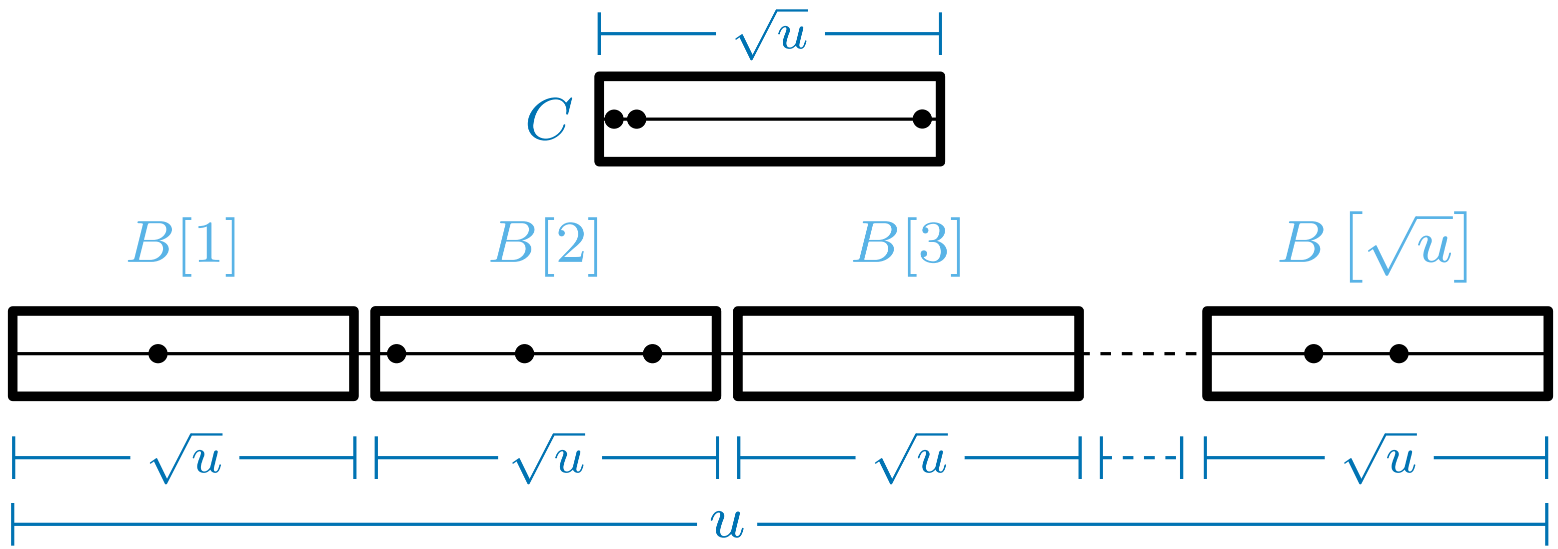

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Each recursive call could in turn make multiple recursive calls. . .

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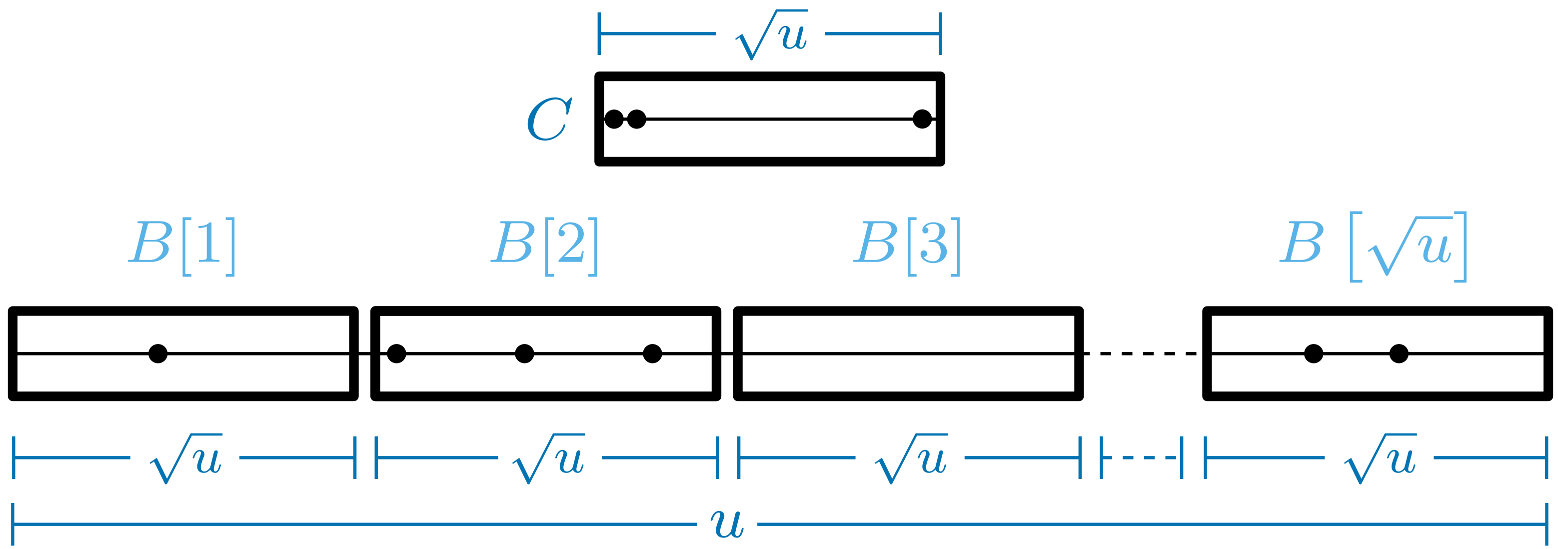

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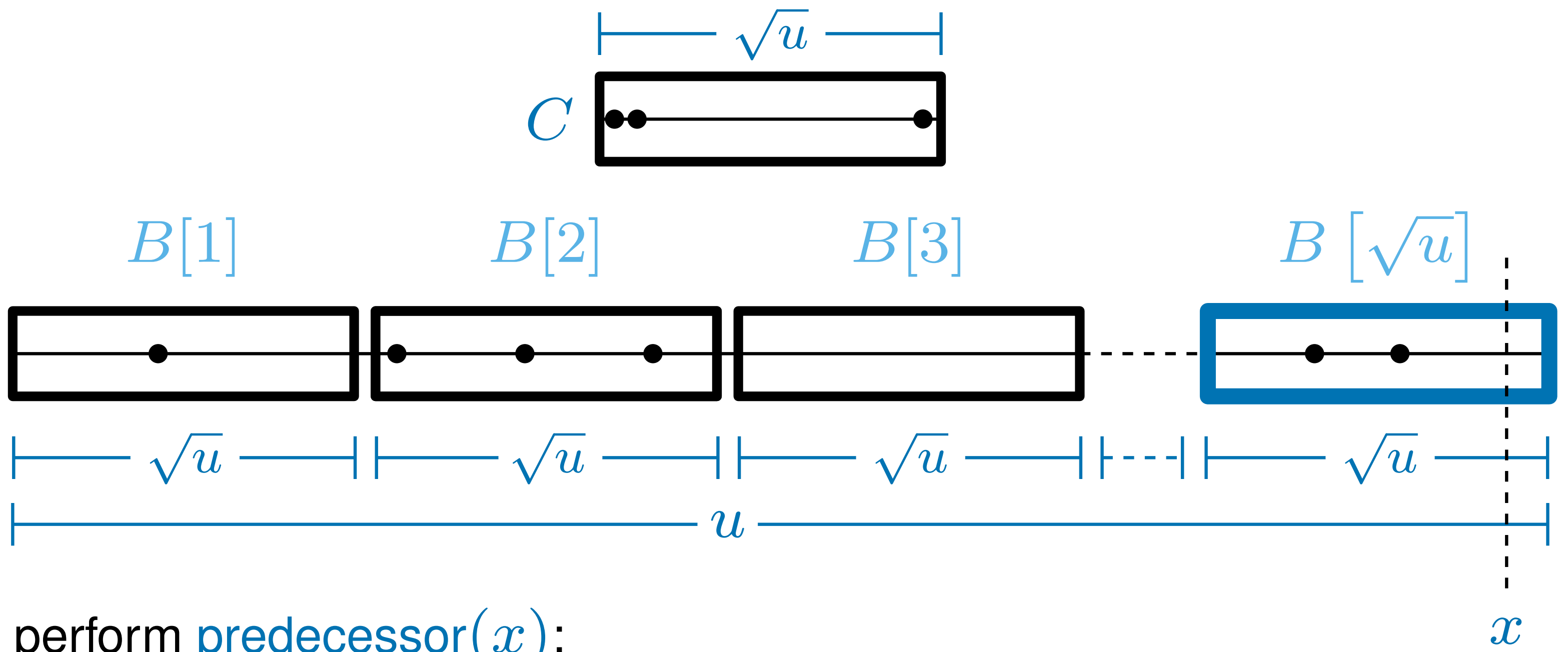
this could get out of hand!

A closer look at predecessor



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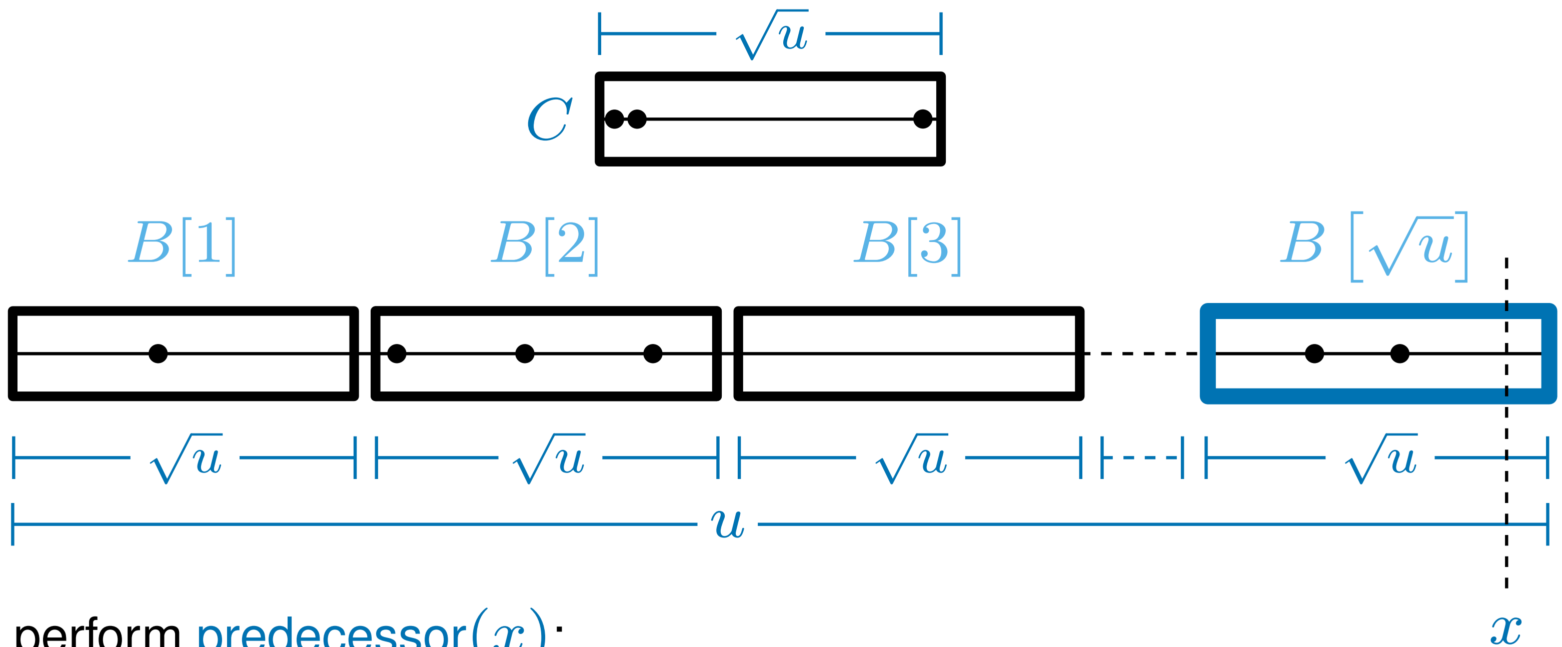
Observation 1: if x has a predecessor in $B[i]$ we only make one recursive call



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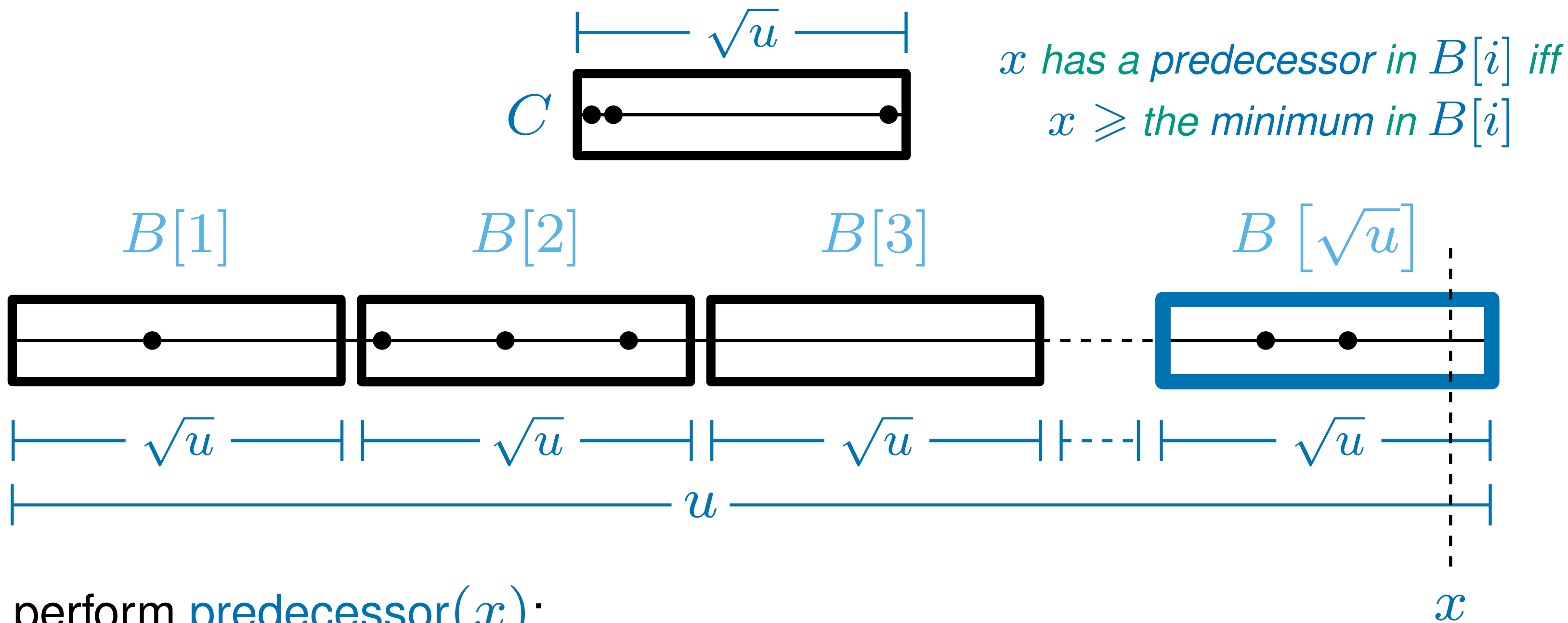

To perform predecessor(x):

Step 1 Determine which $B[i]$ the element x belongs in

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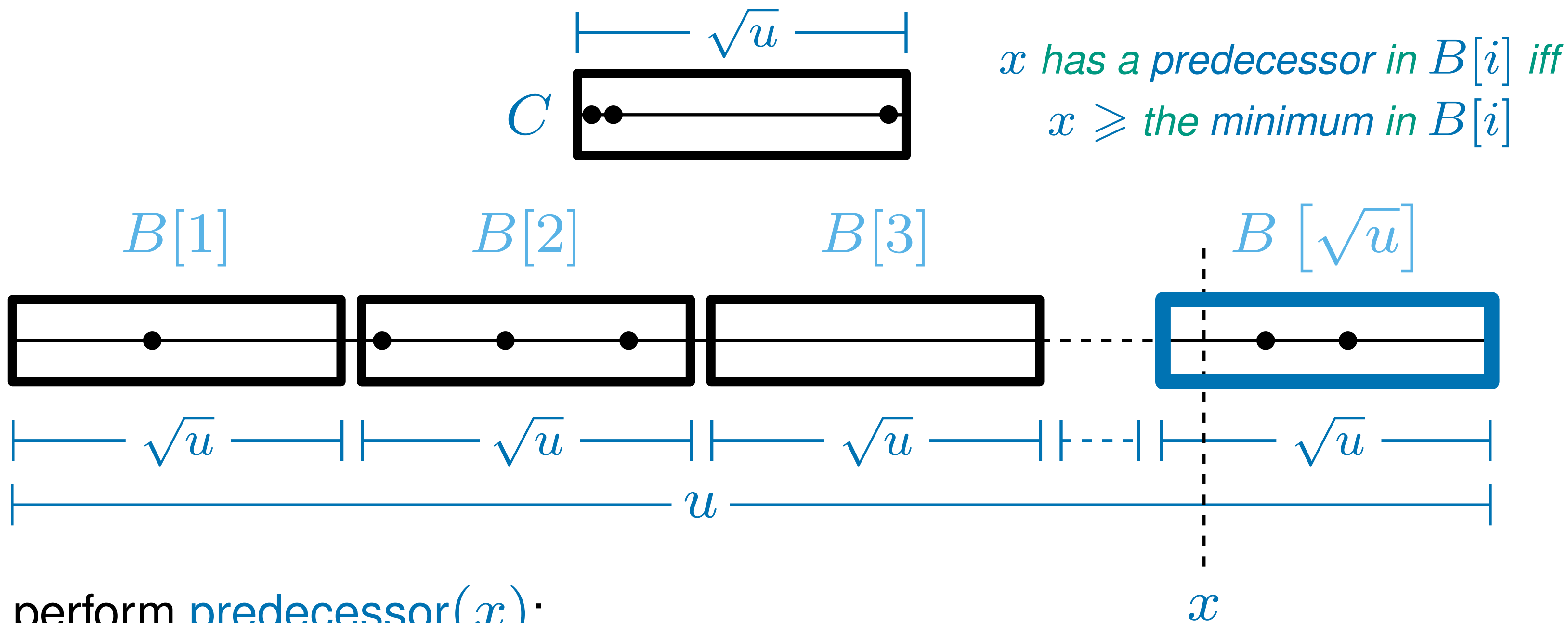

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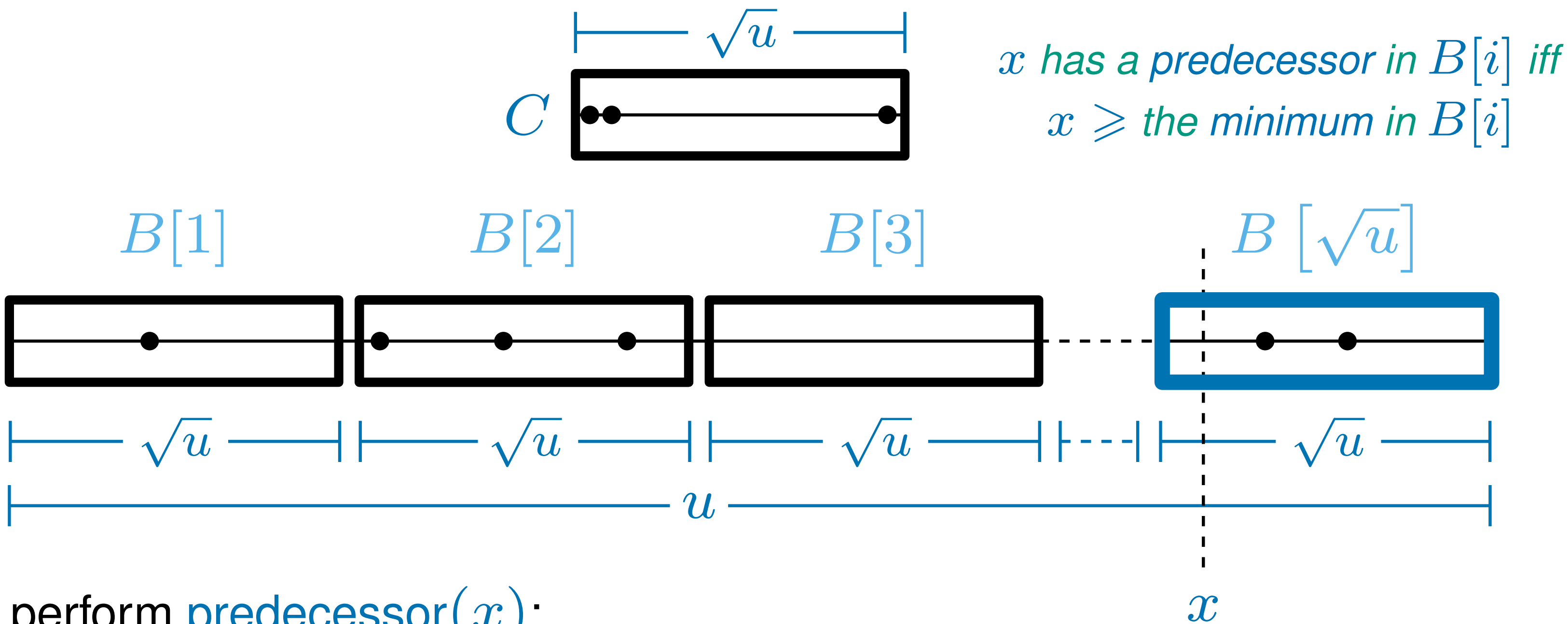

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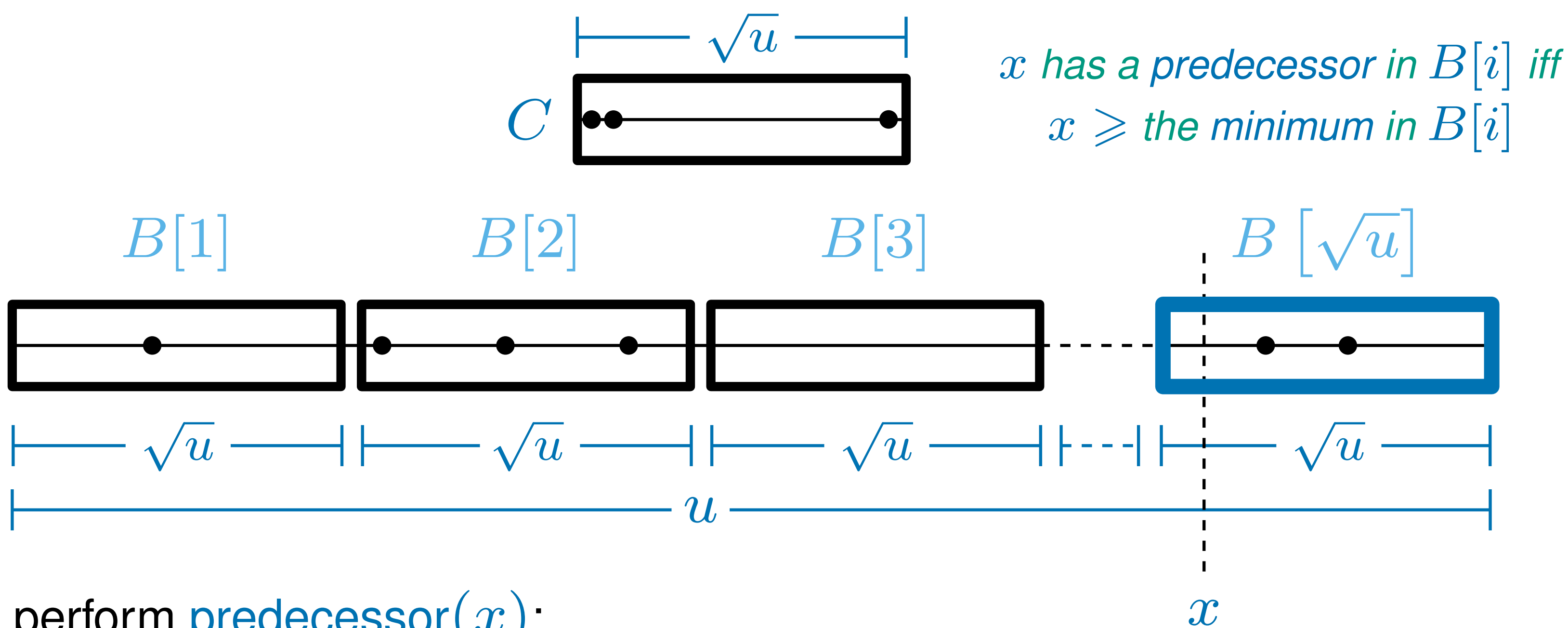

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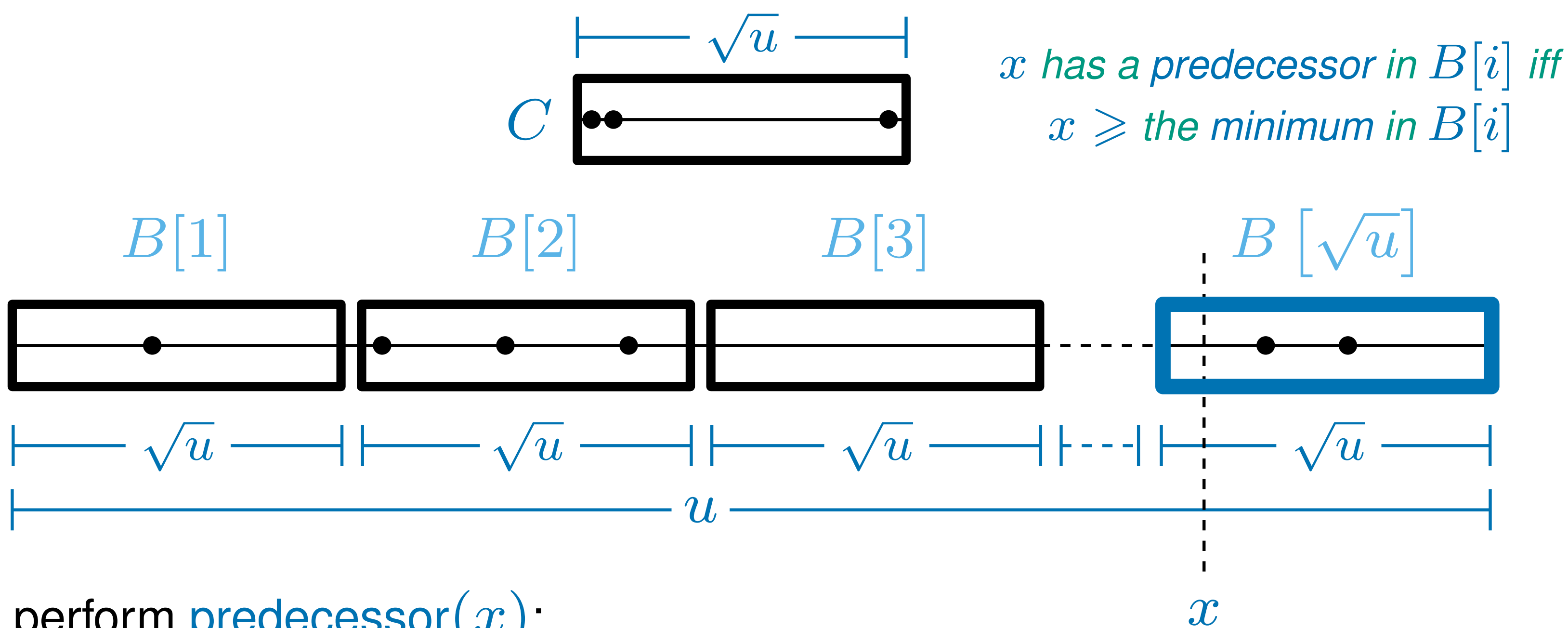

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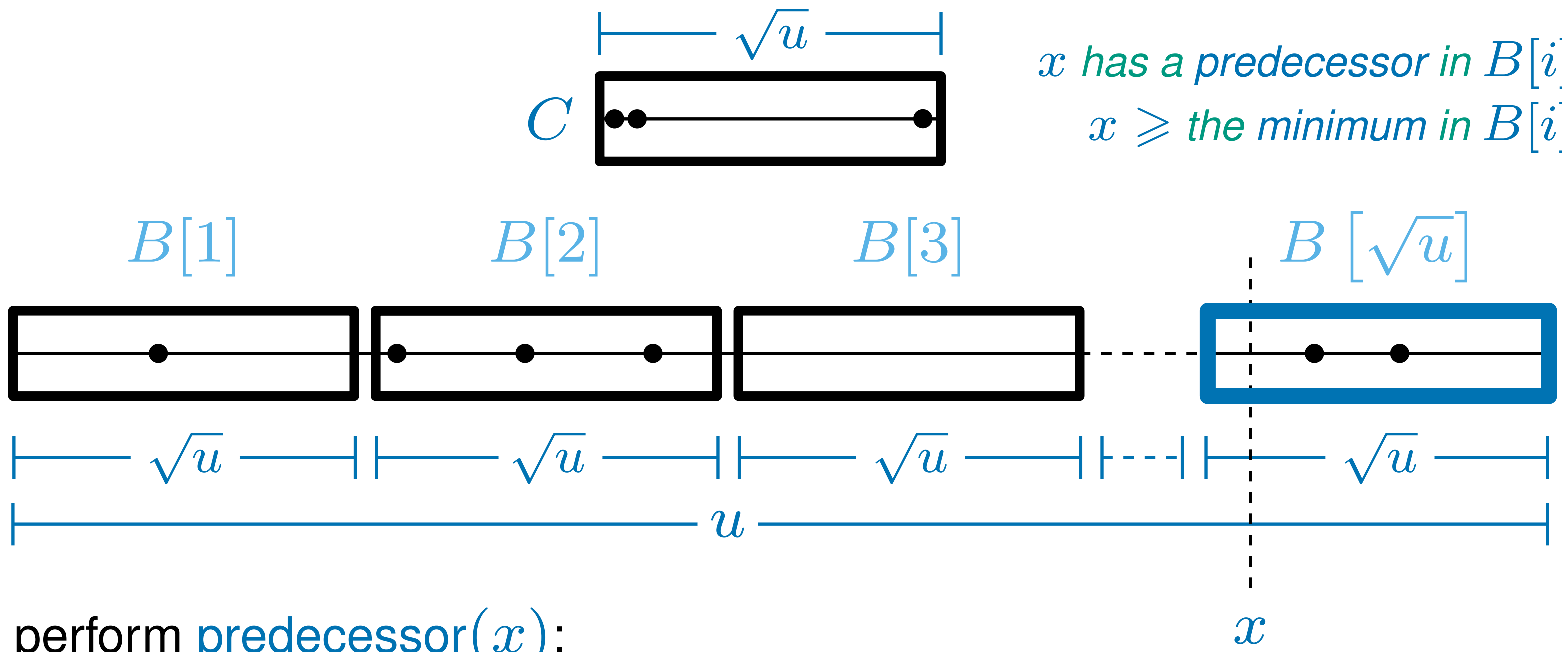

x has a predecessor in $B[i]$ iff $x \geqslant$ the minimum in $B[i]$

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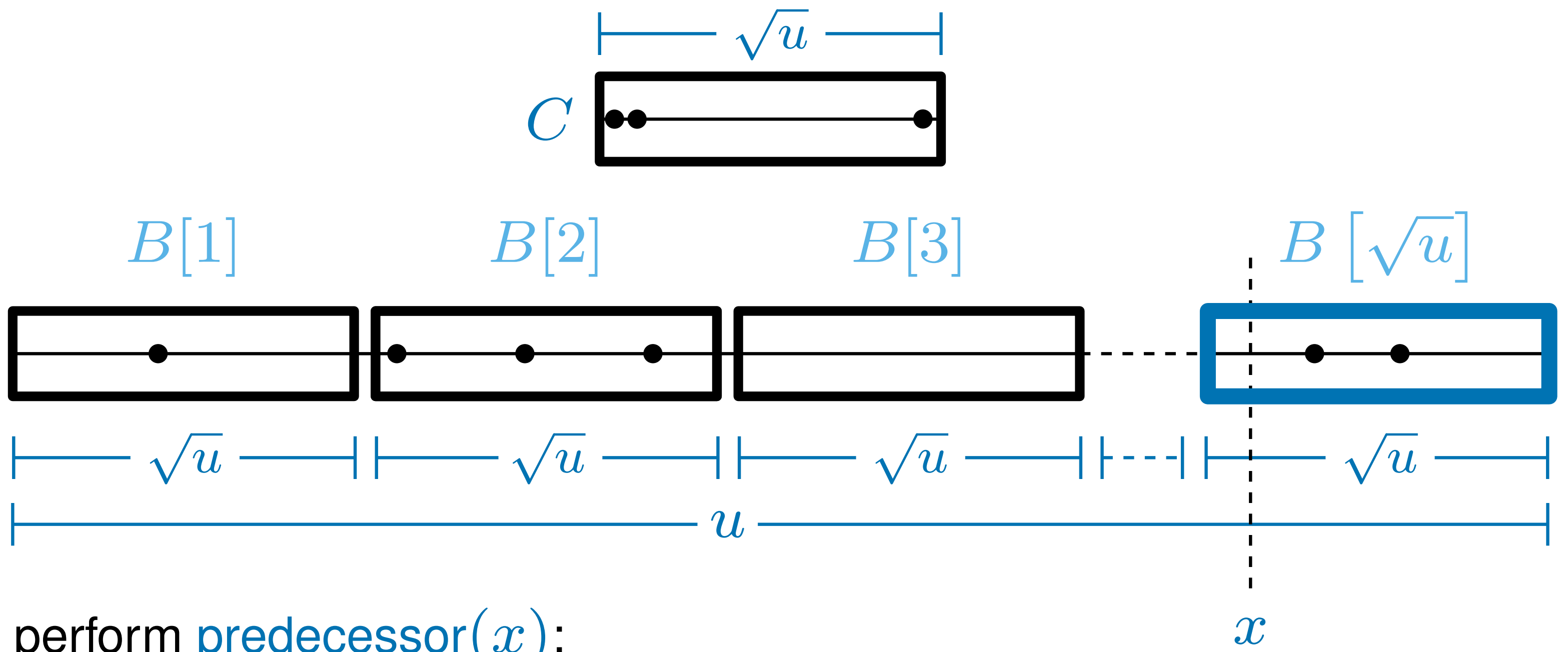

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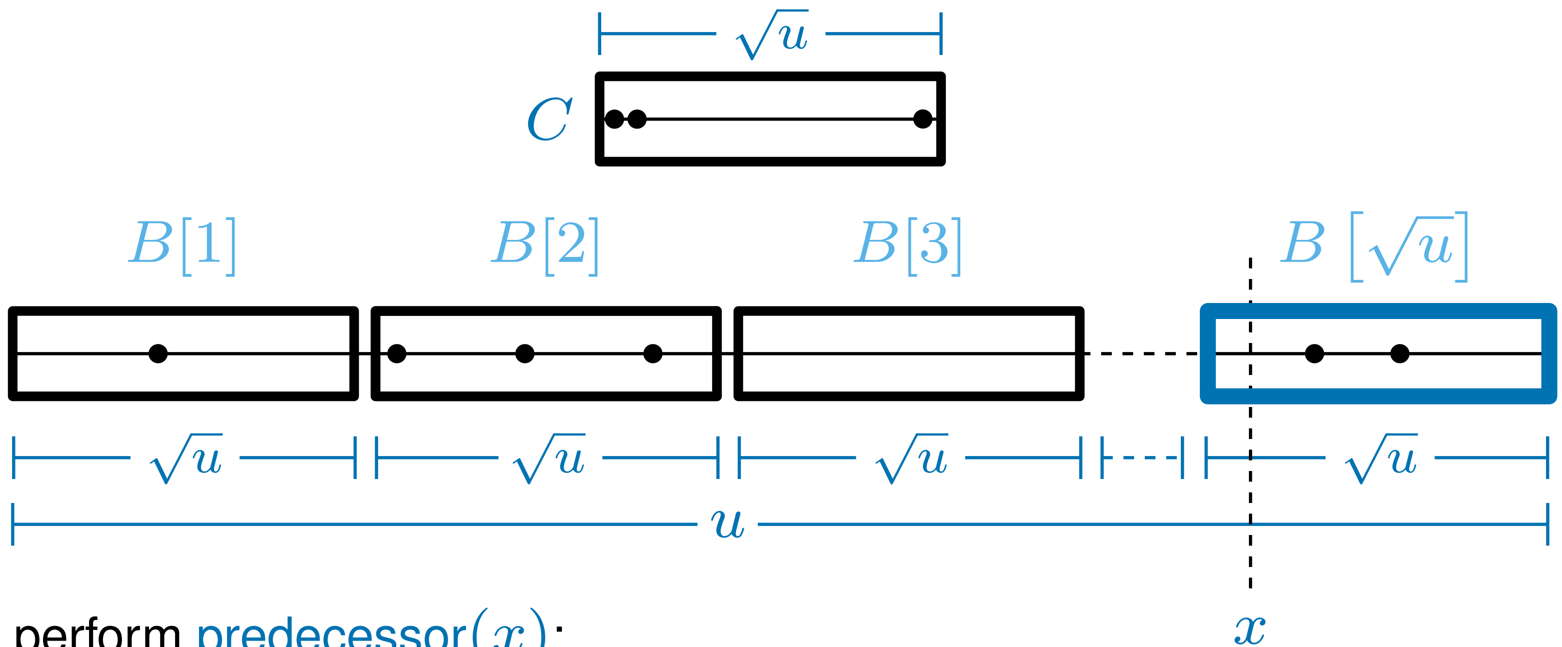

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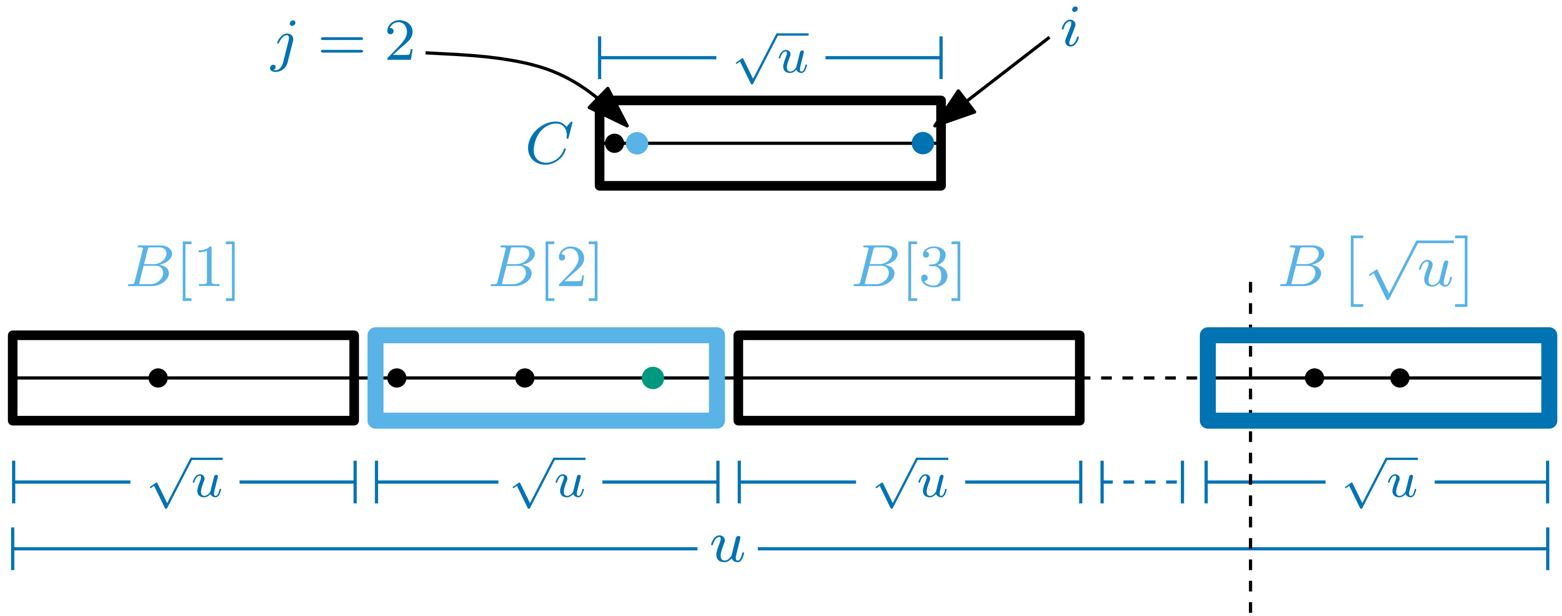

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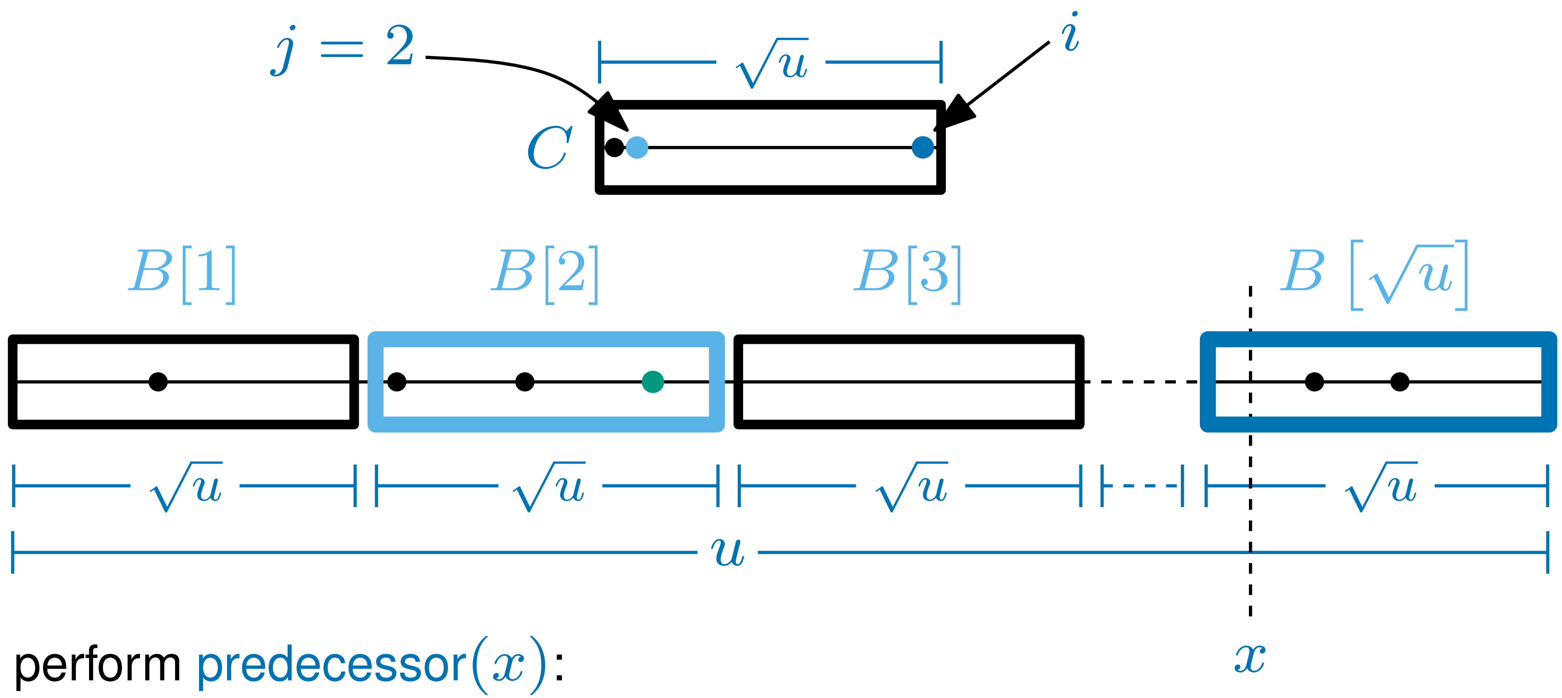

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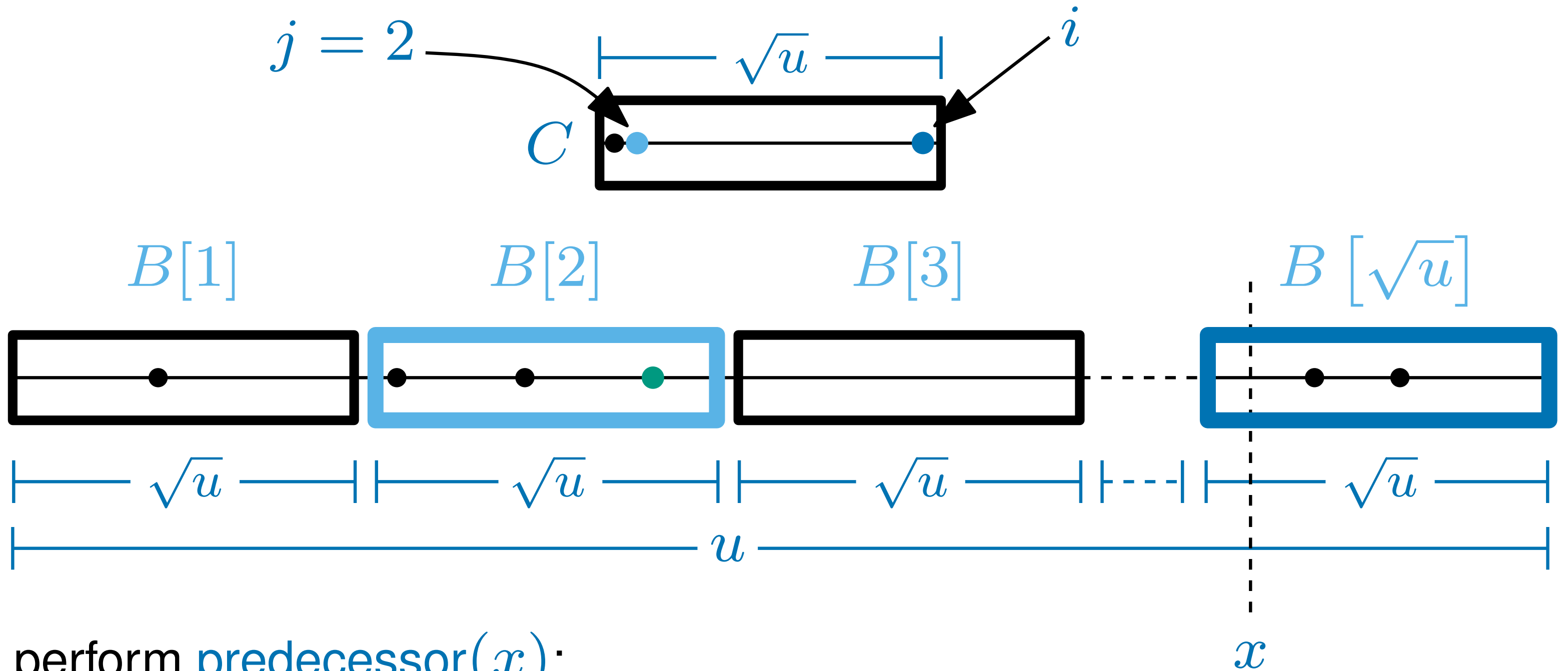

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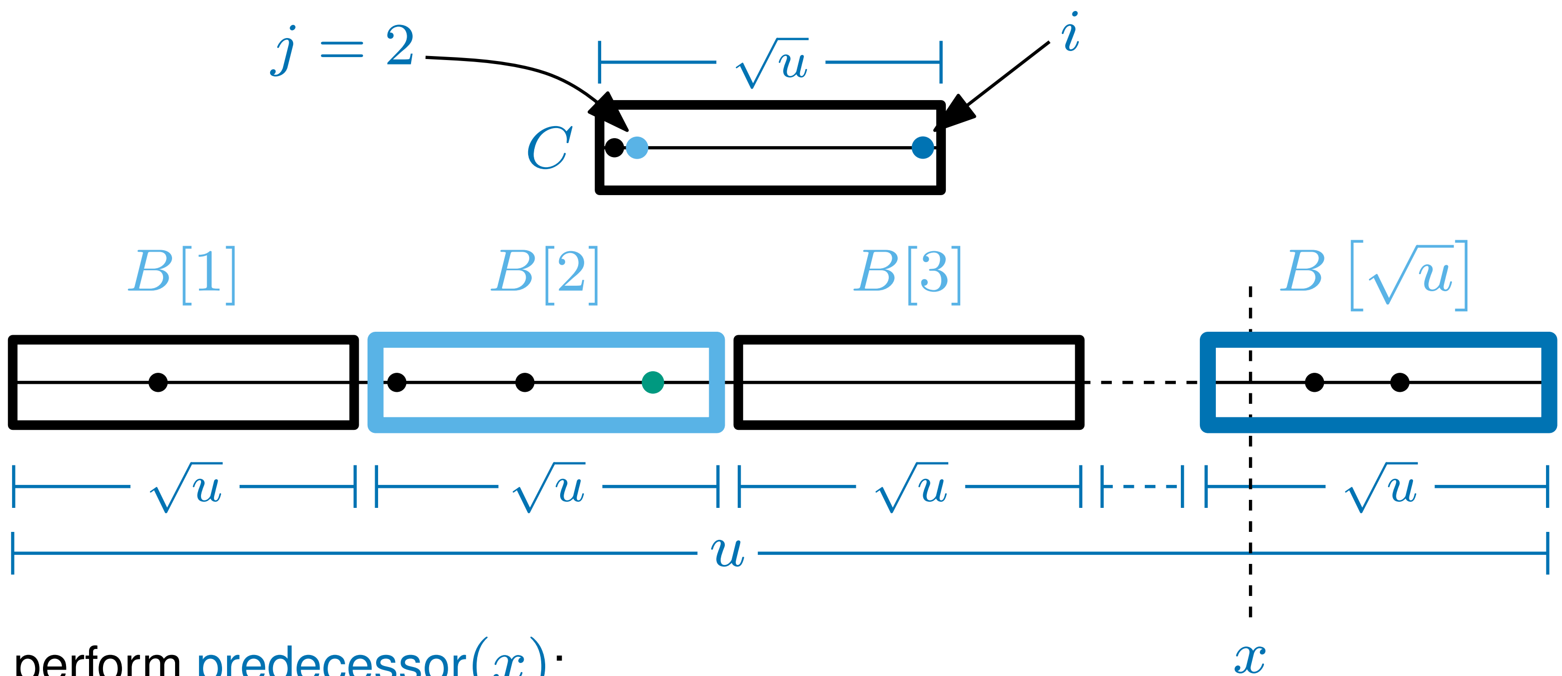

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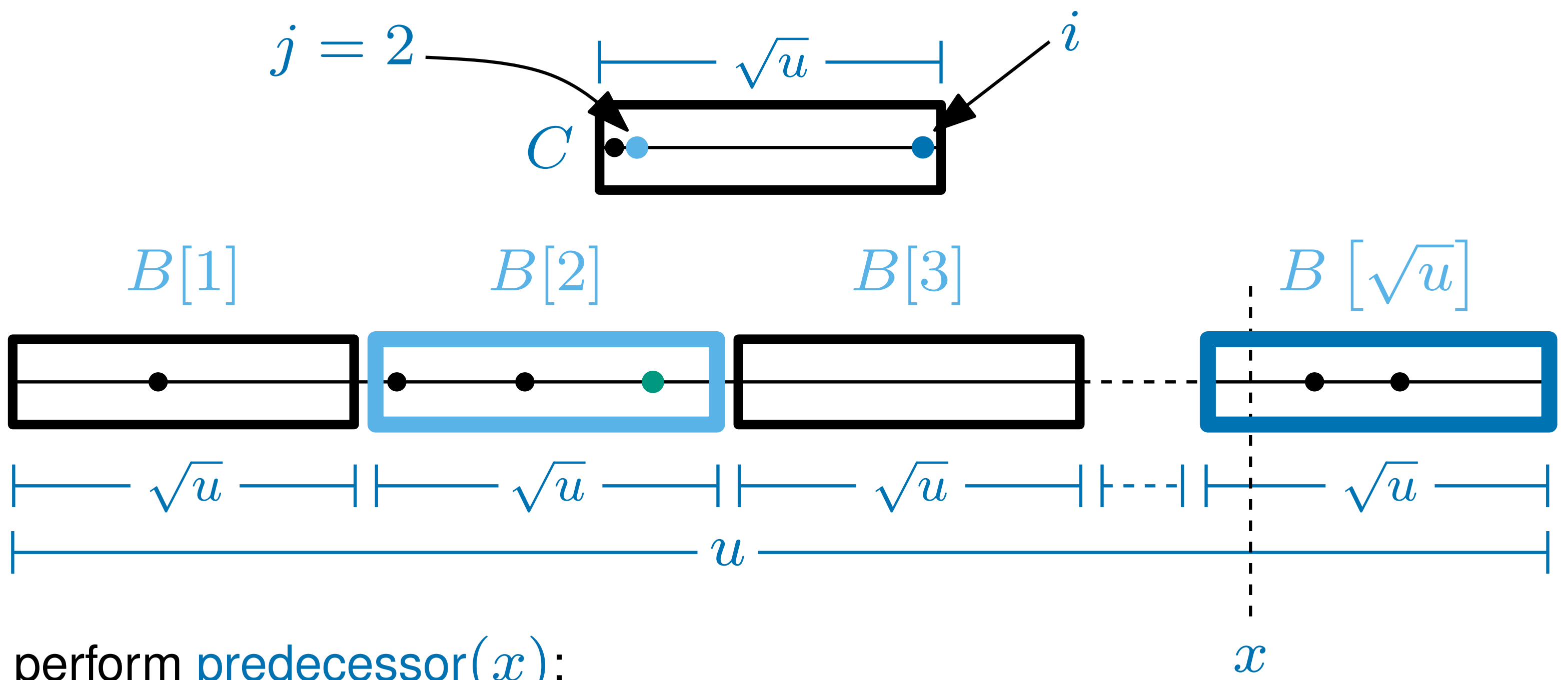

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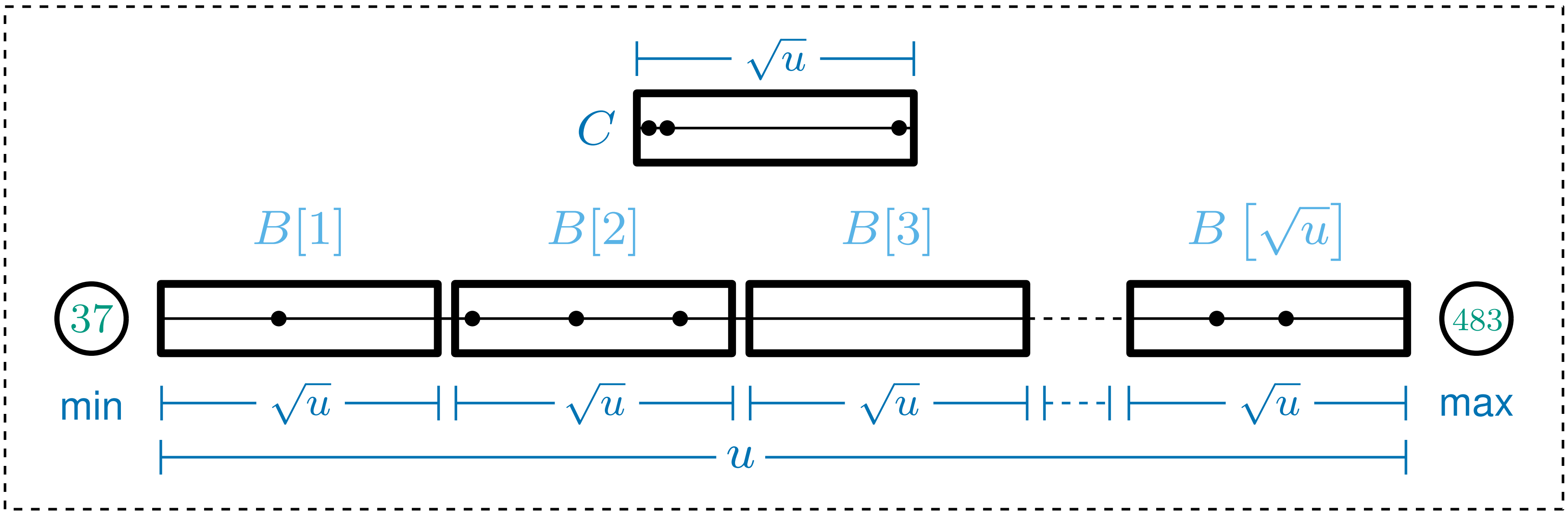
Now we make exactly one recursive call

(ignoring finding the min/max)

Finally: van Emde Boas Trees

So that we can find the min/max quickly we store them seperately...



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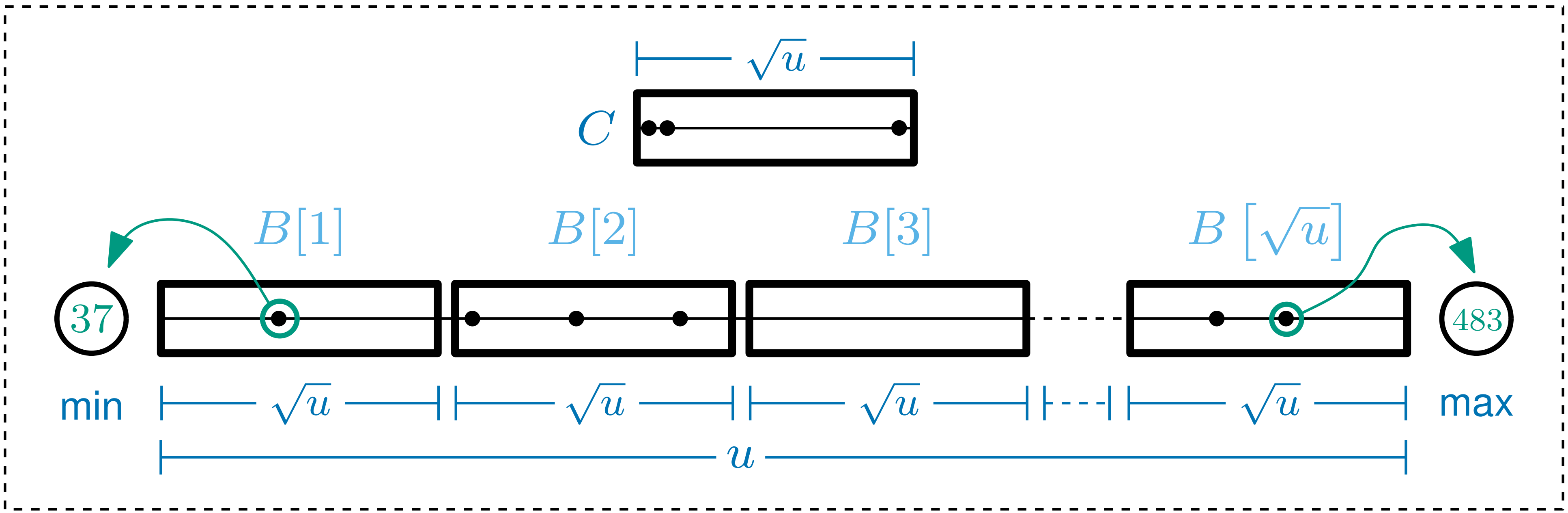

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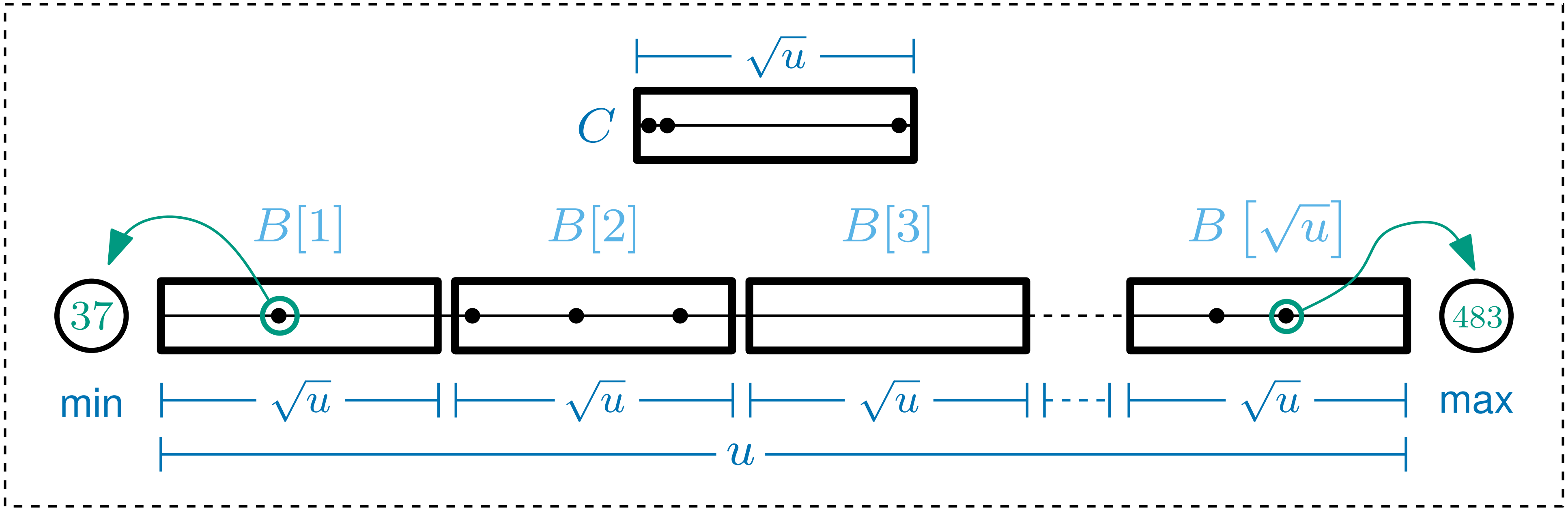

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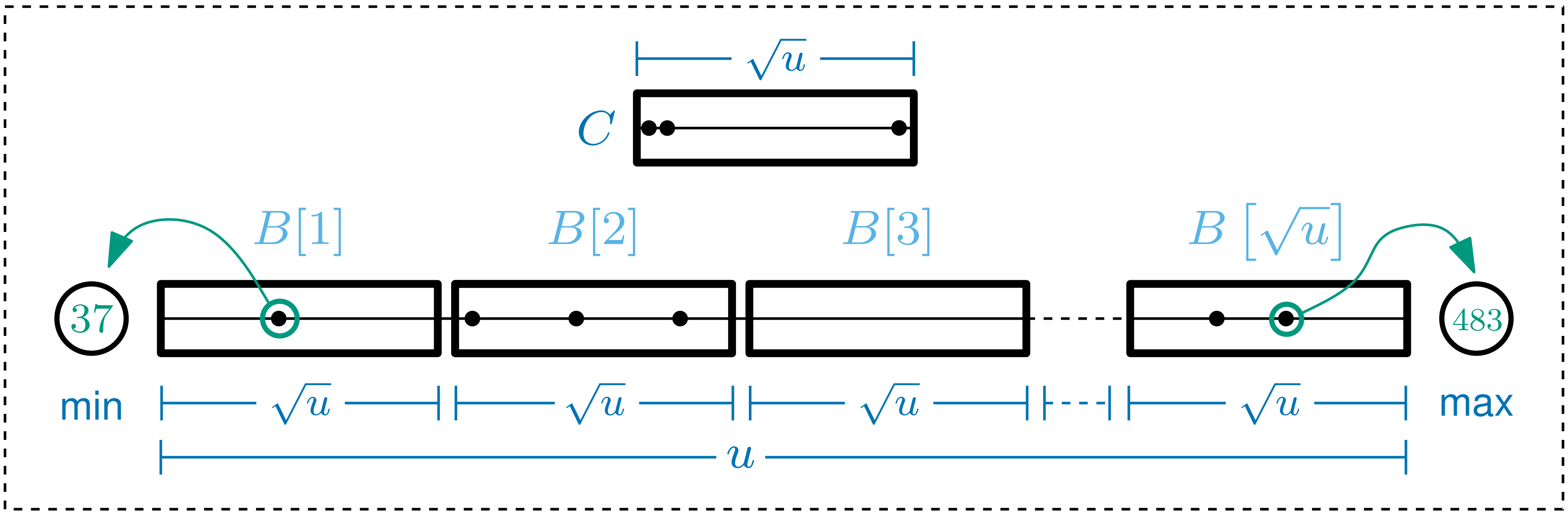

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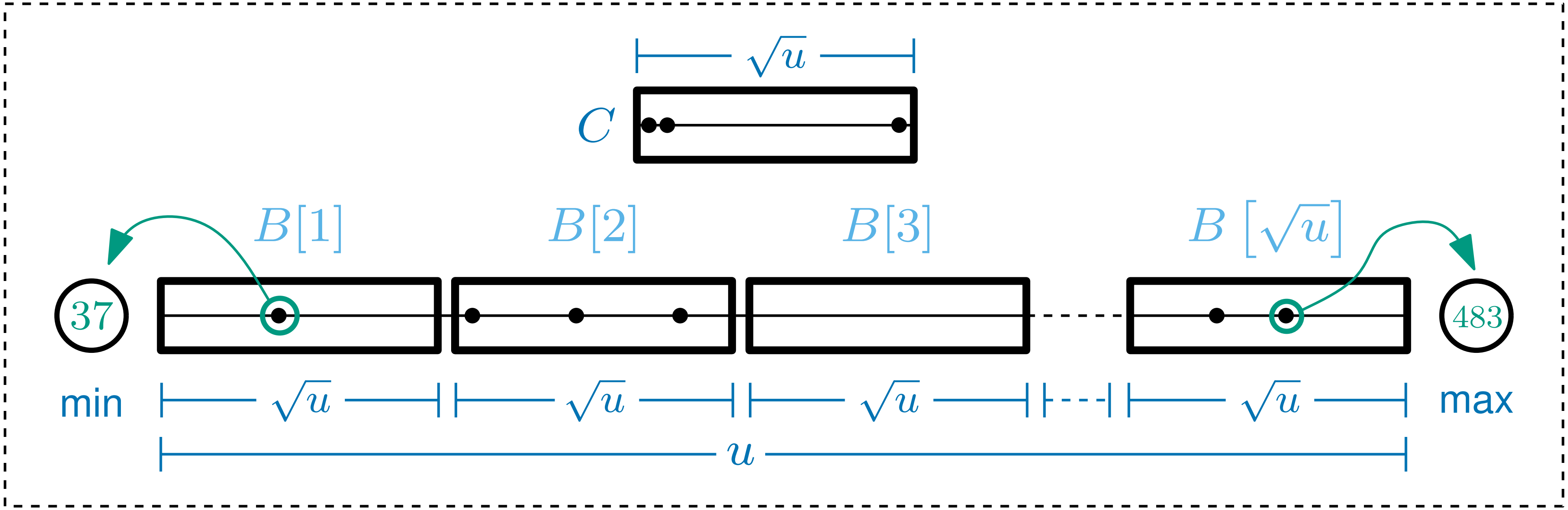

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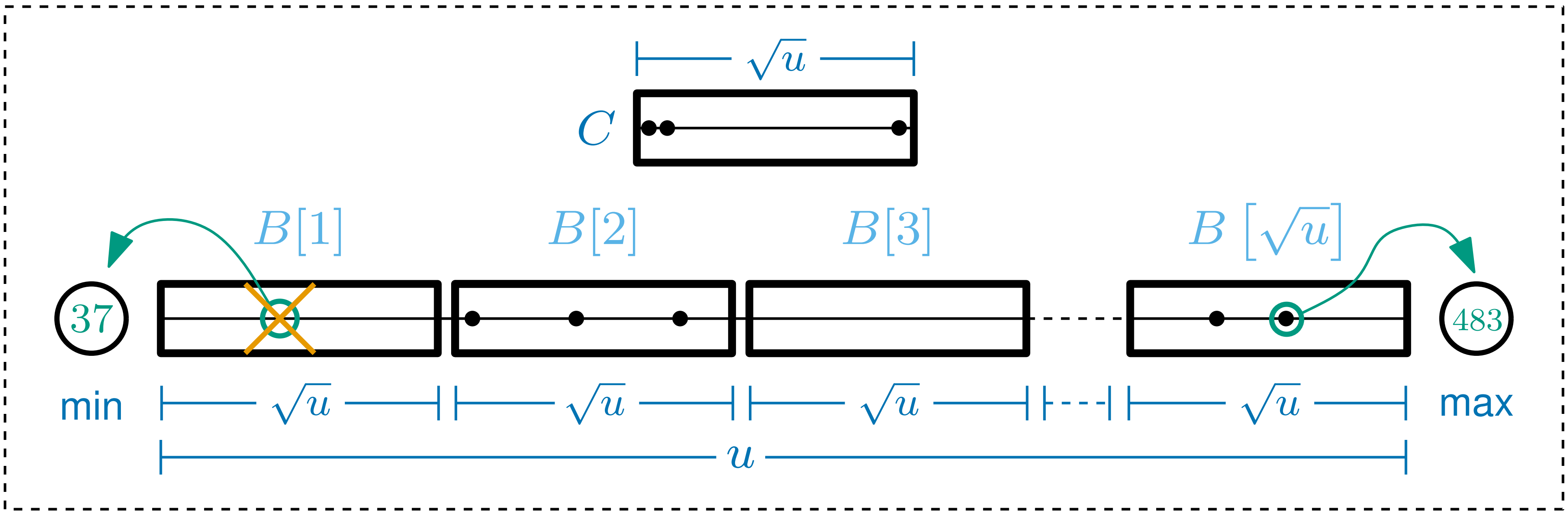

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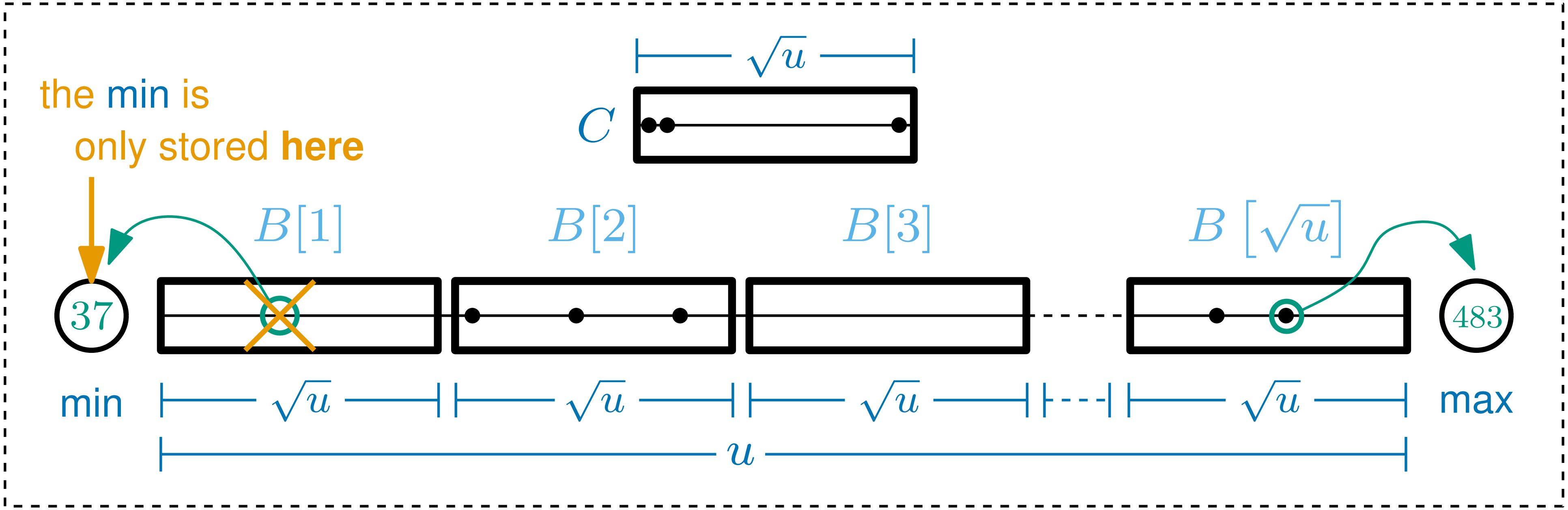

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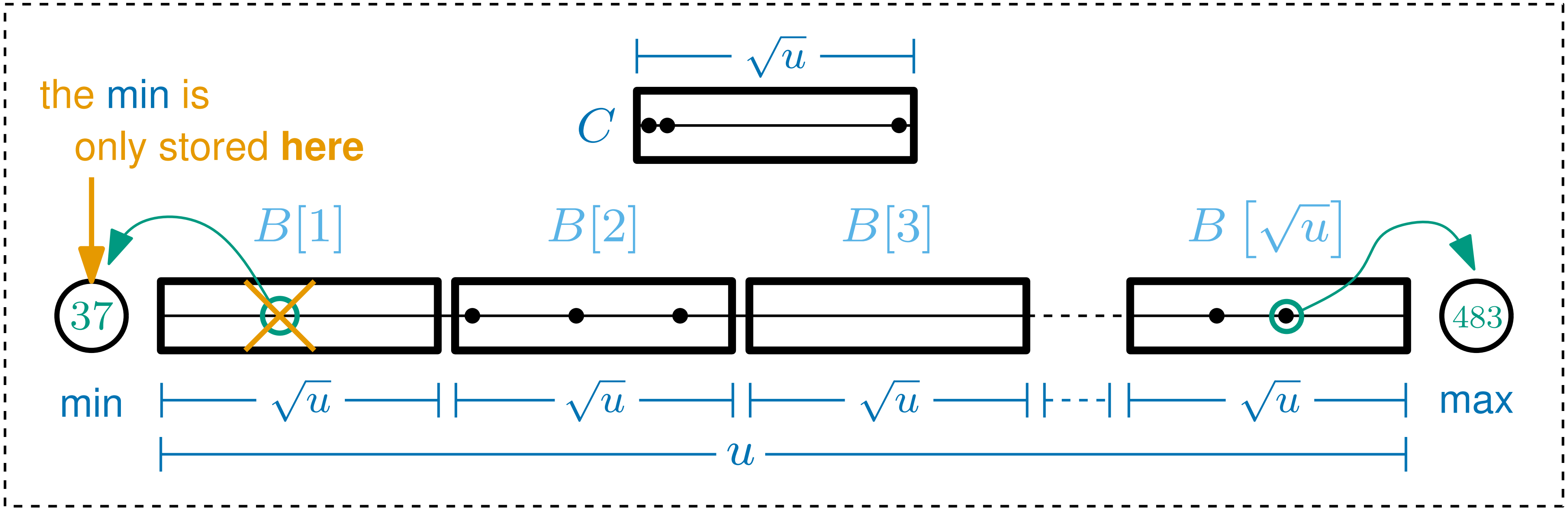

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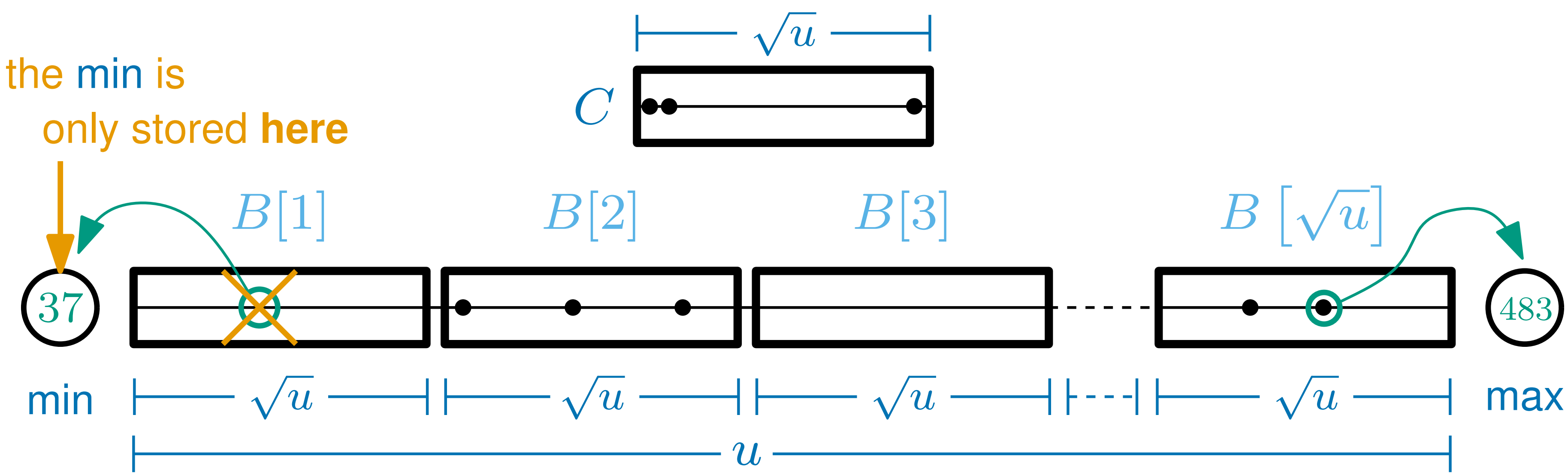
this allows us to avoid making multiple recursive calls when adding an element

Another look at add



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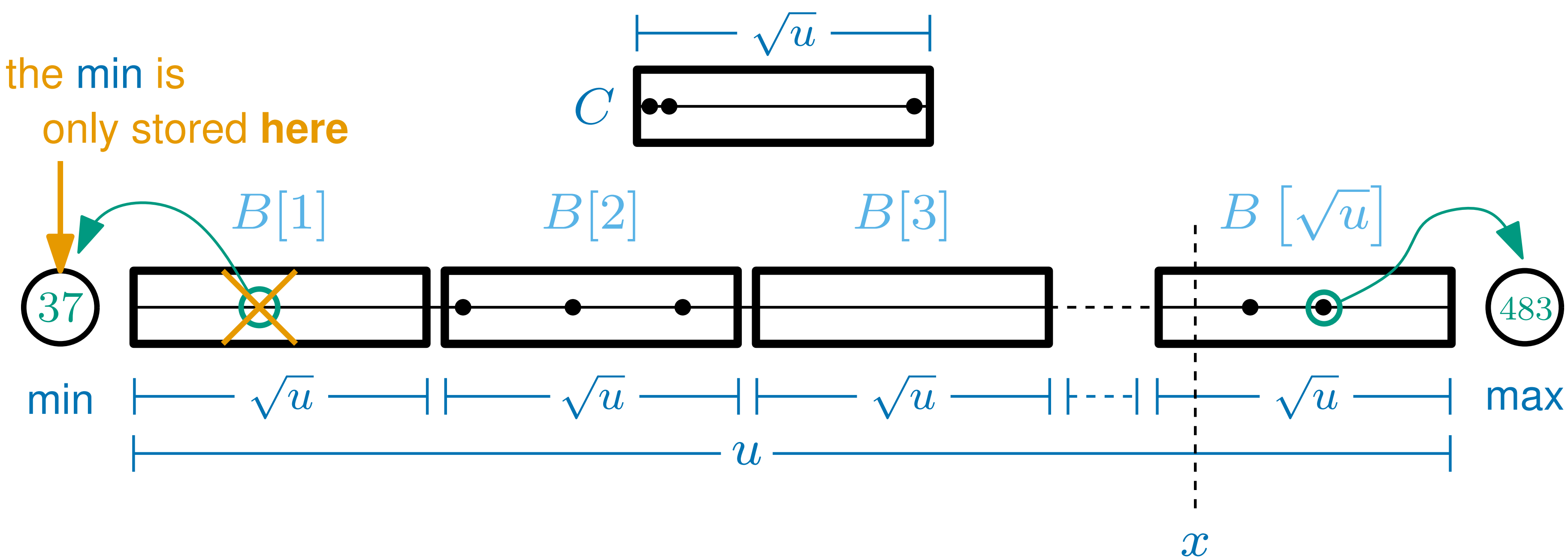

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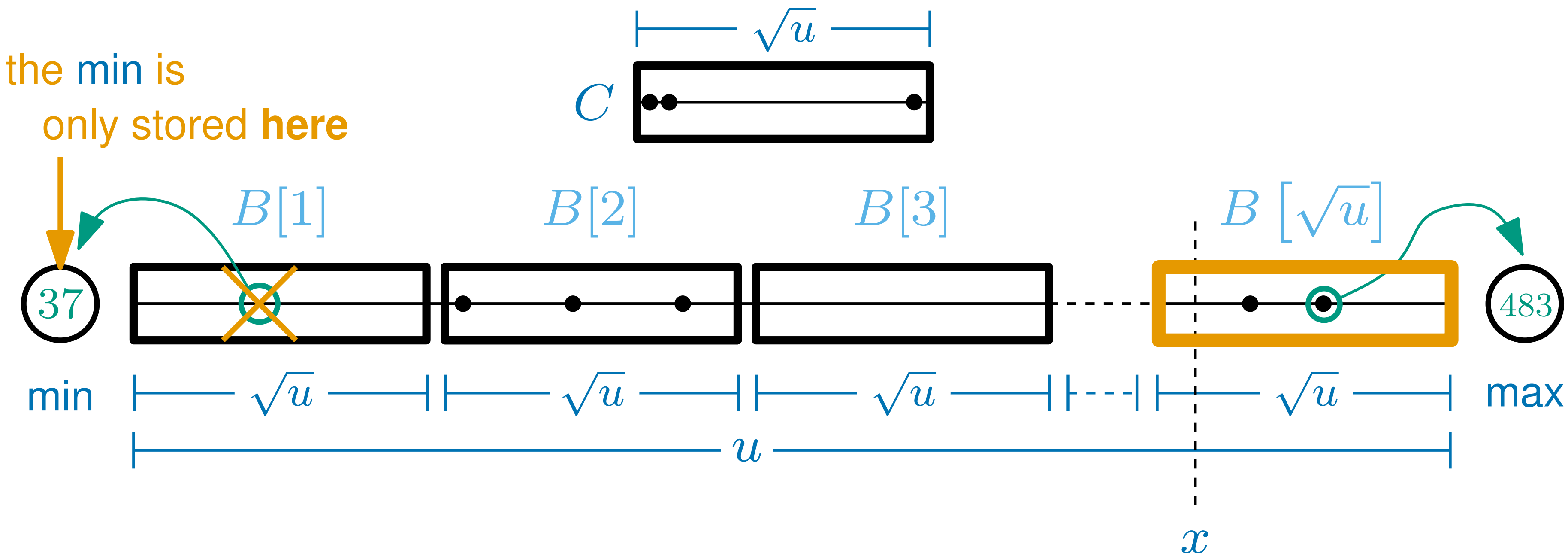

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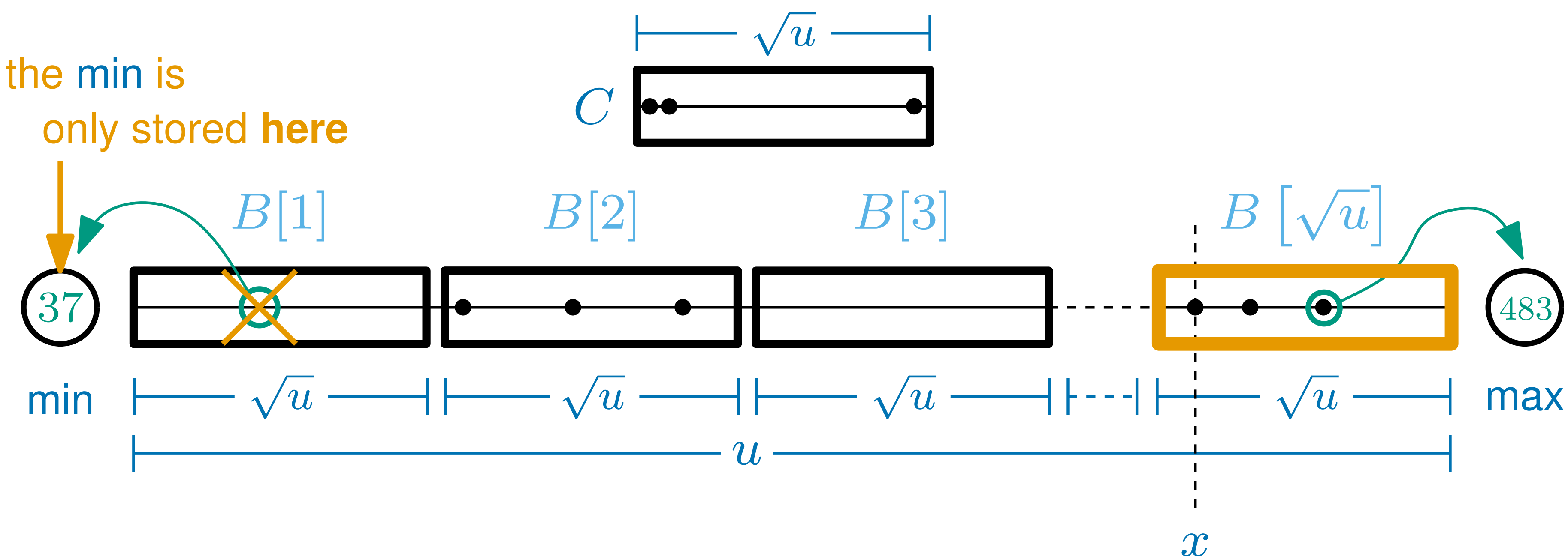

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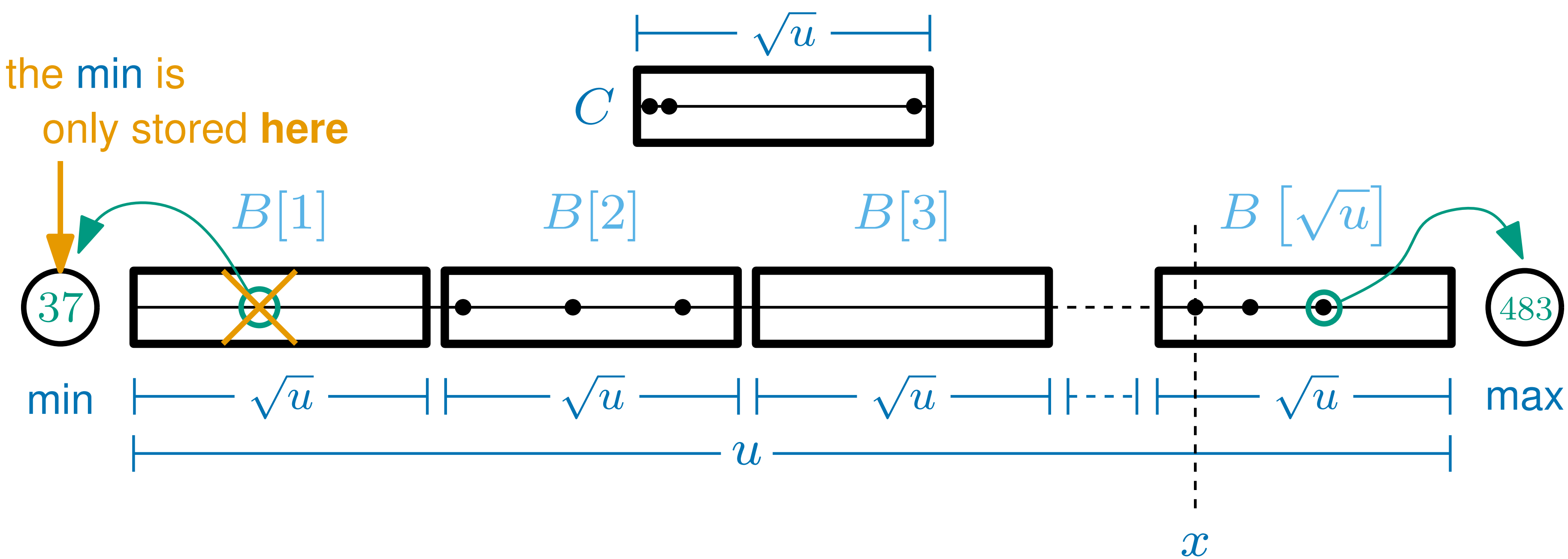

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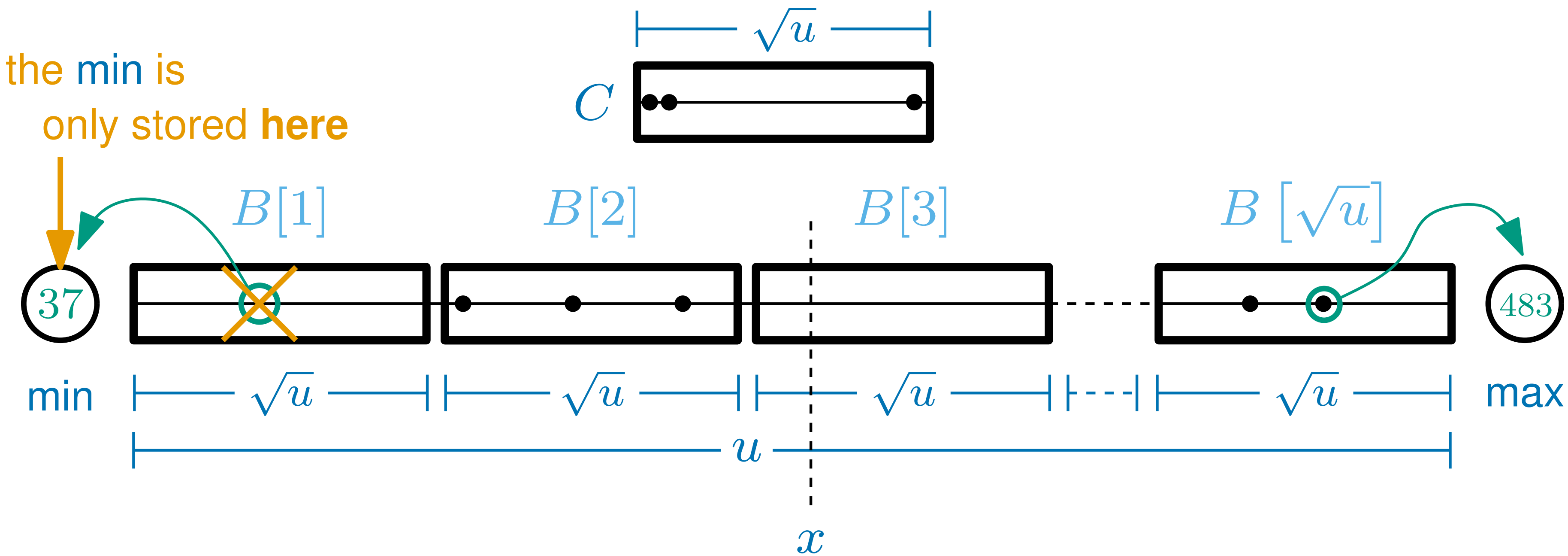

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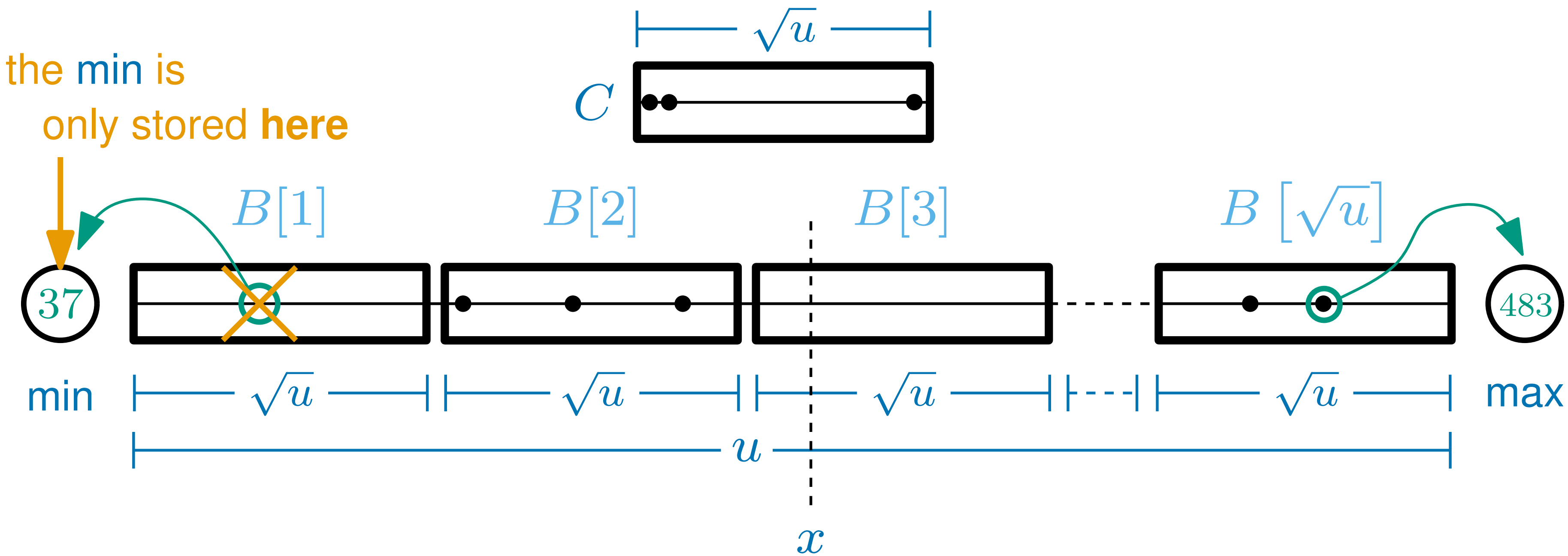

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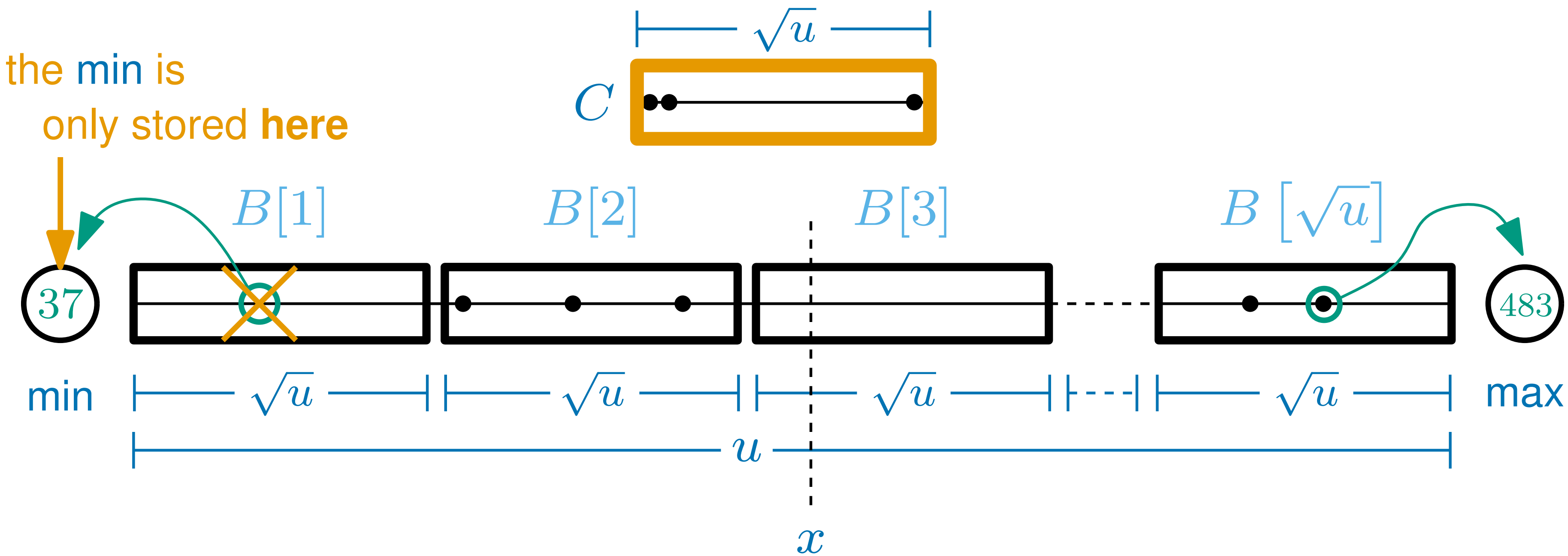

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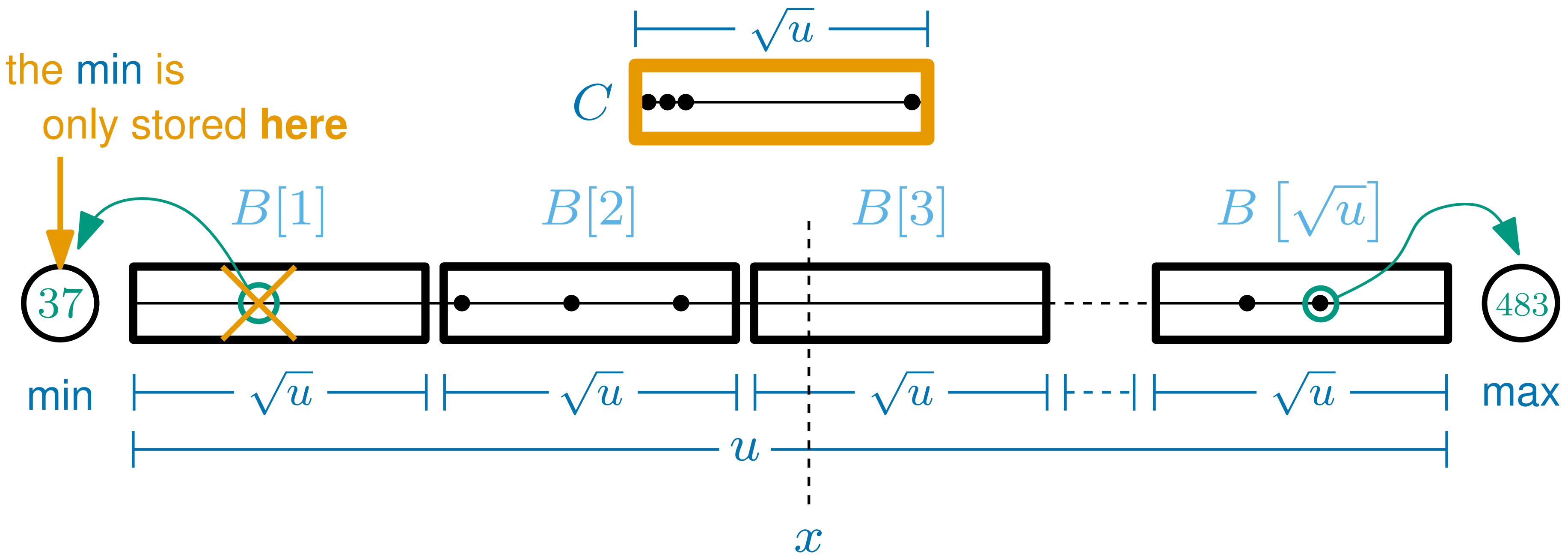

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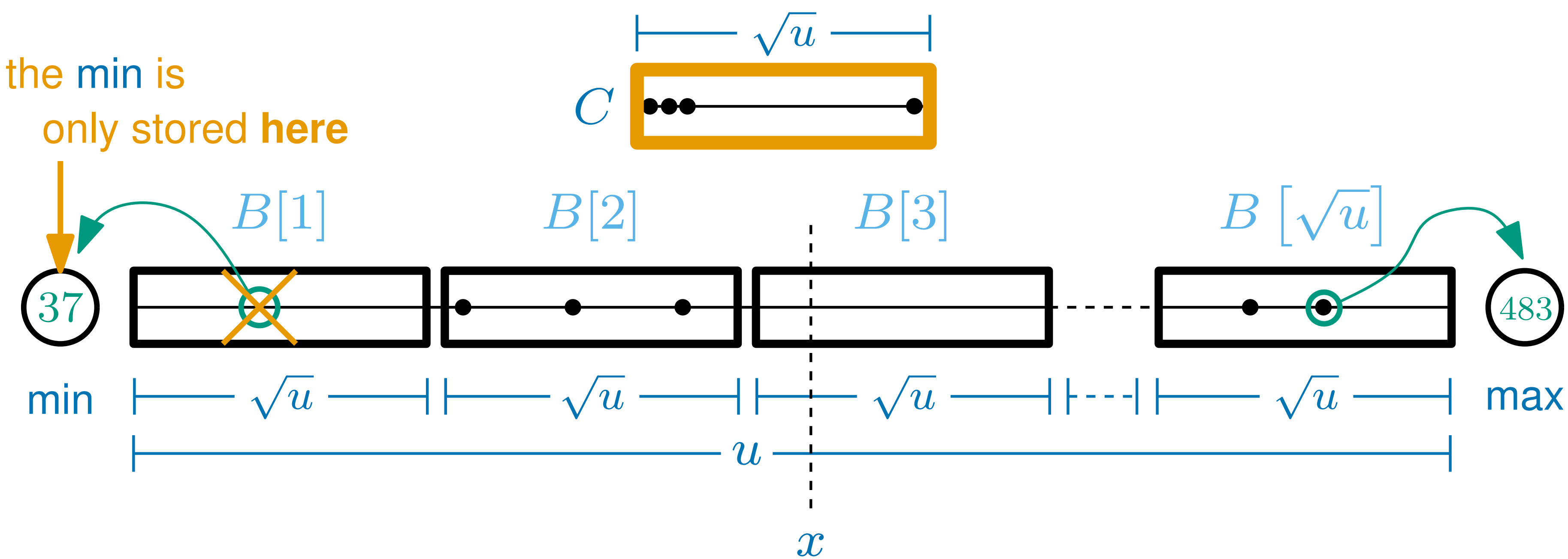

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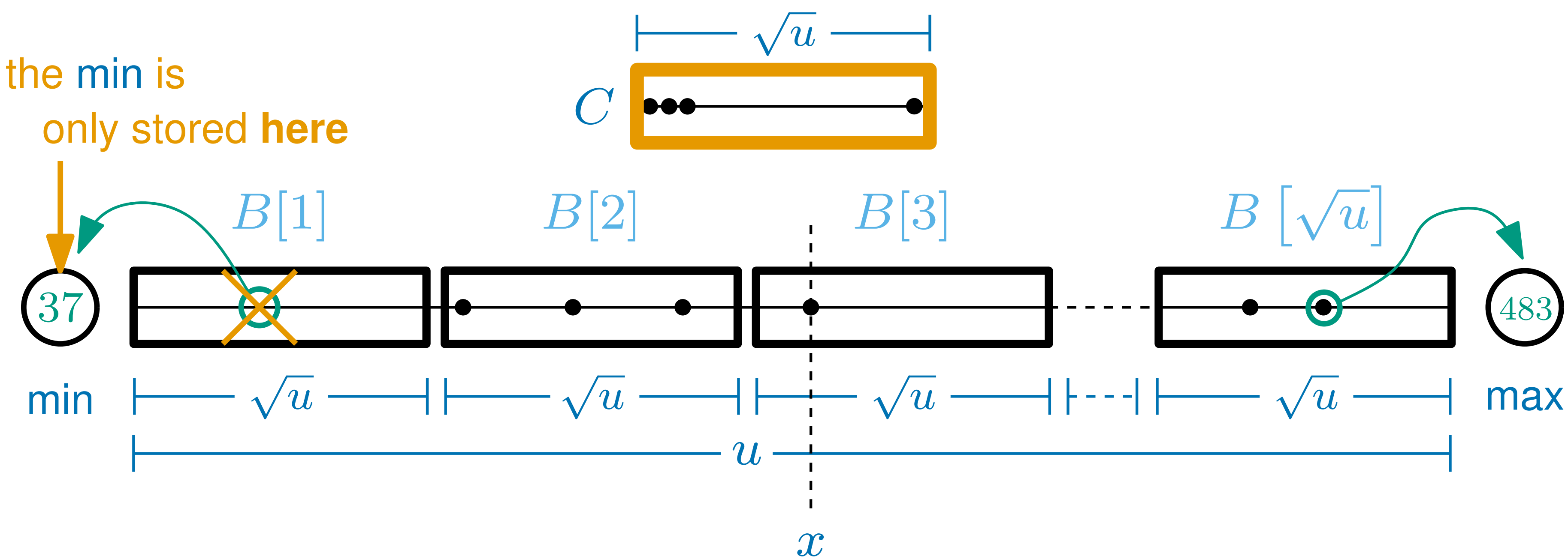
this is not recursive

Another look at add



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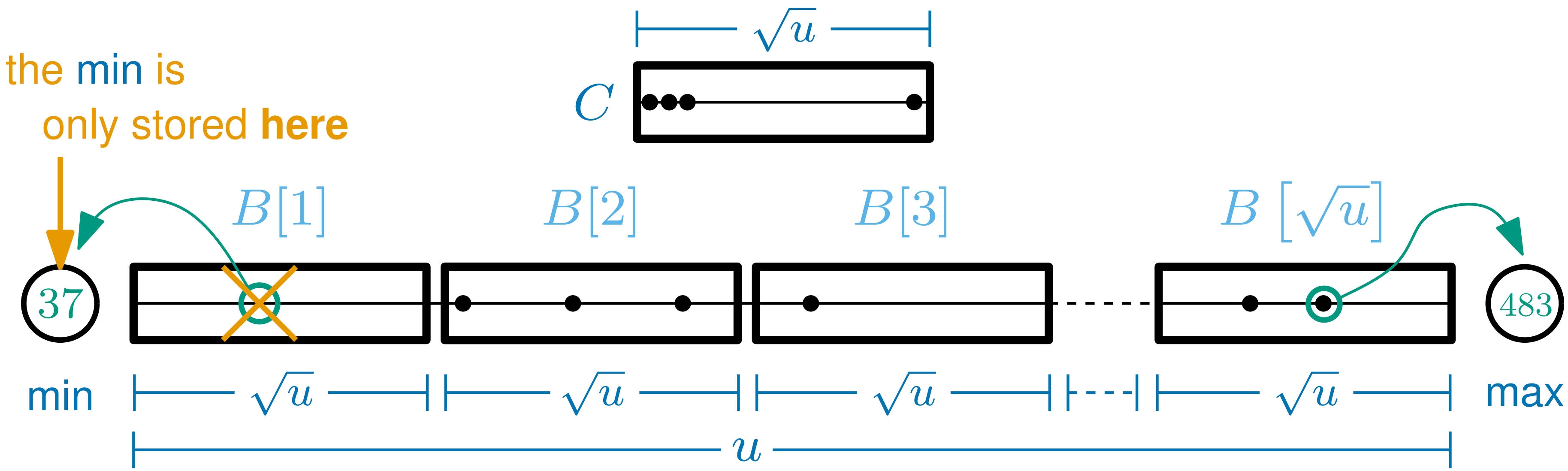

Now we always make exactly one recursive call

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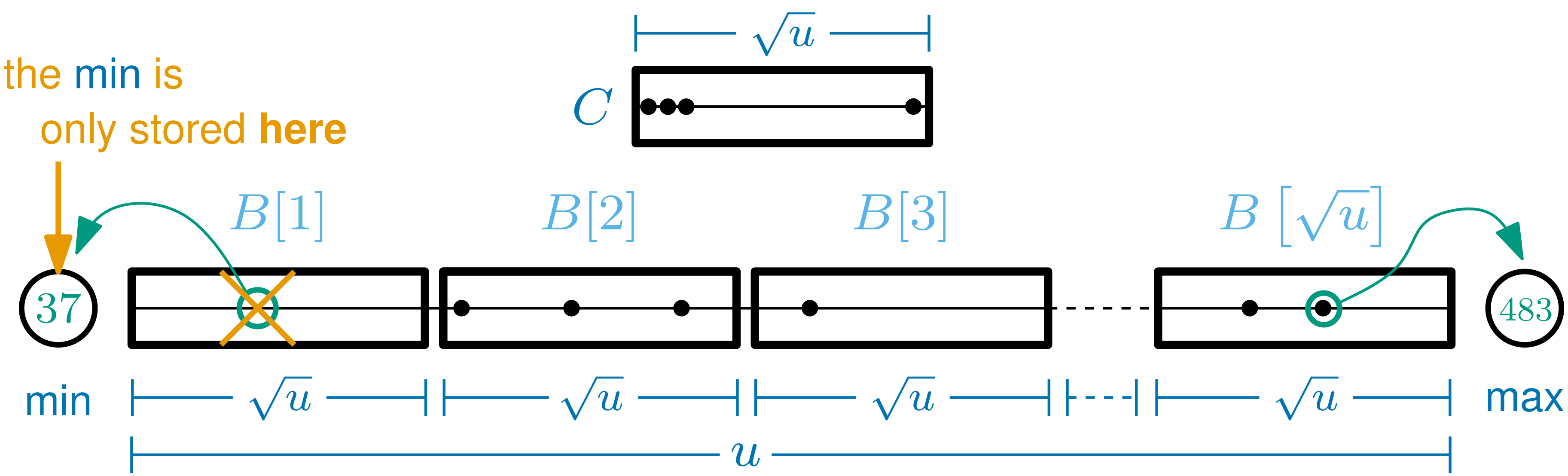
Another look at add

the min is
only stored **here**



Now we always make exactly one recursive call
but what happens when the min/max change?

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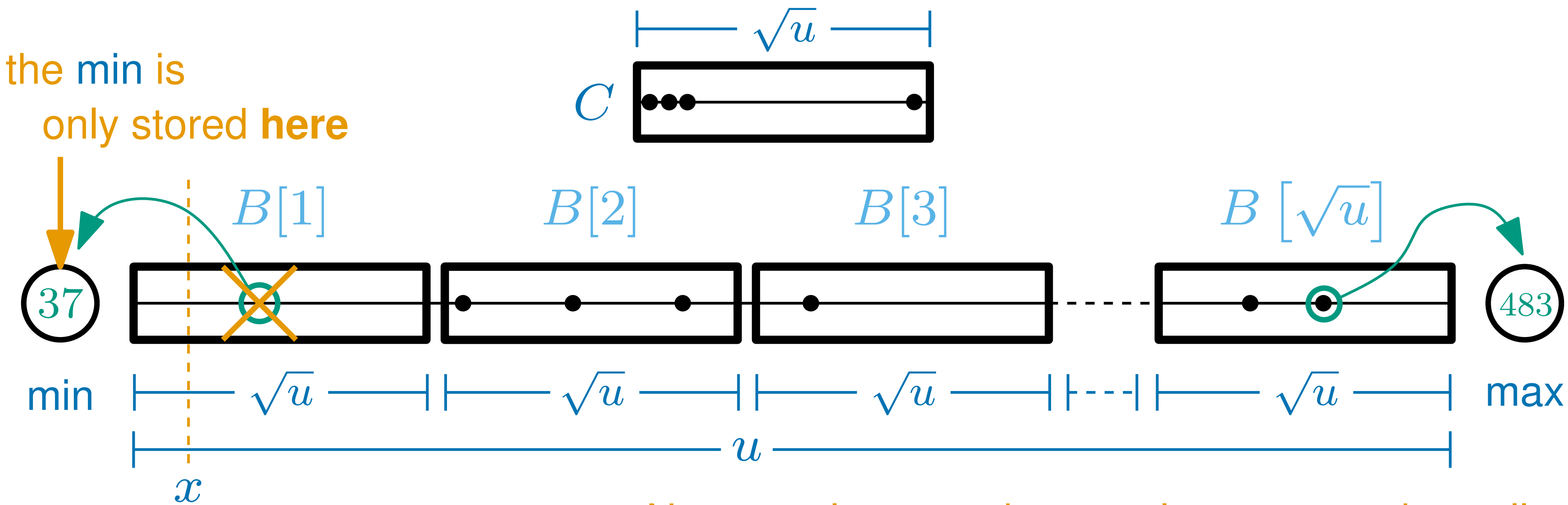

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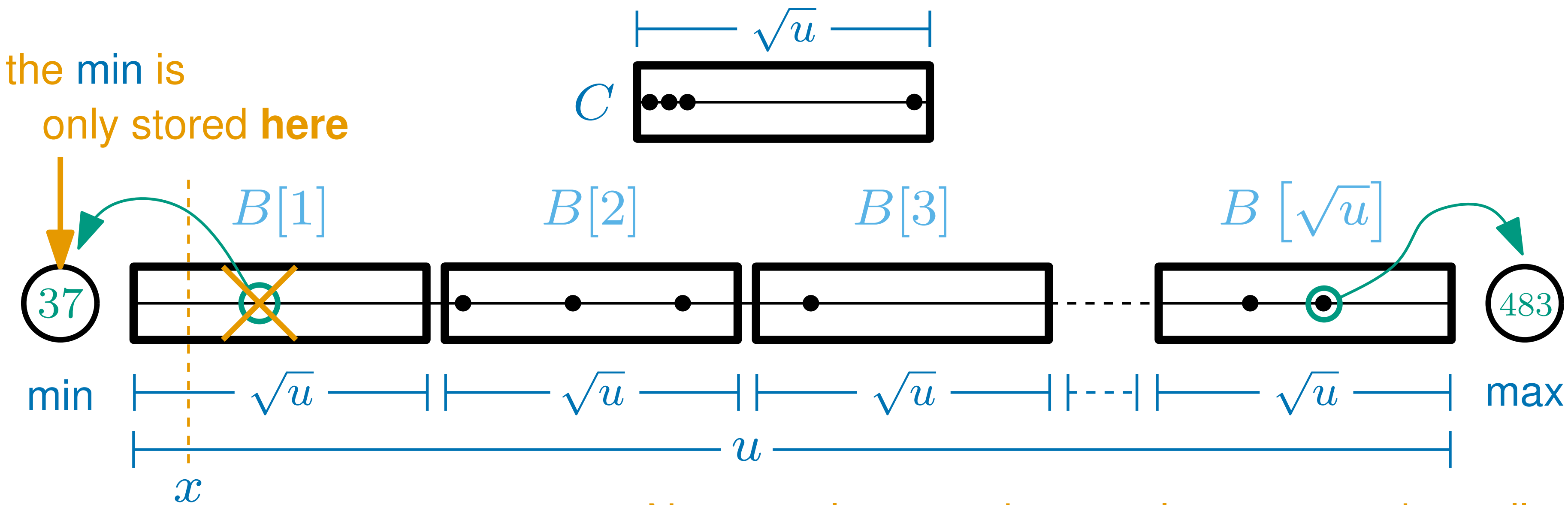

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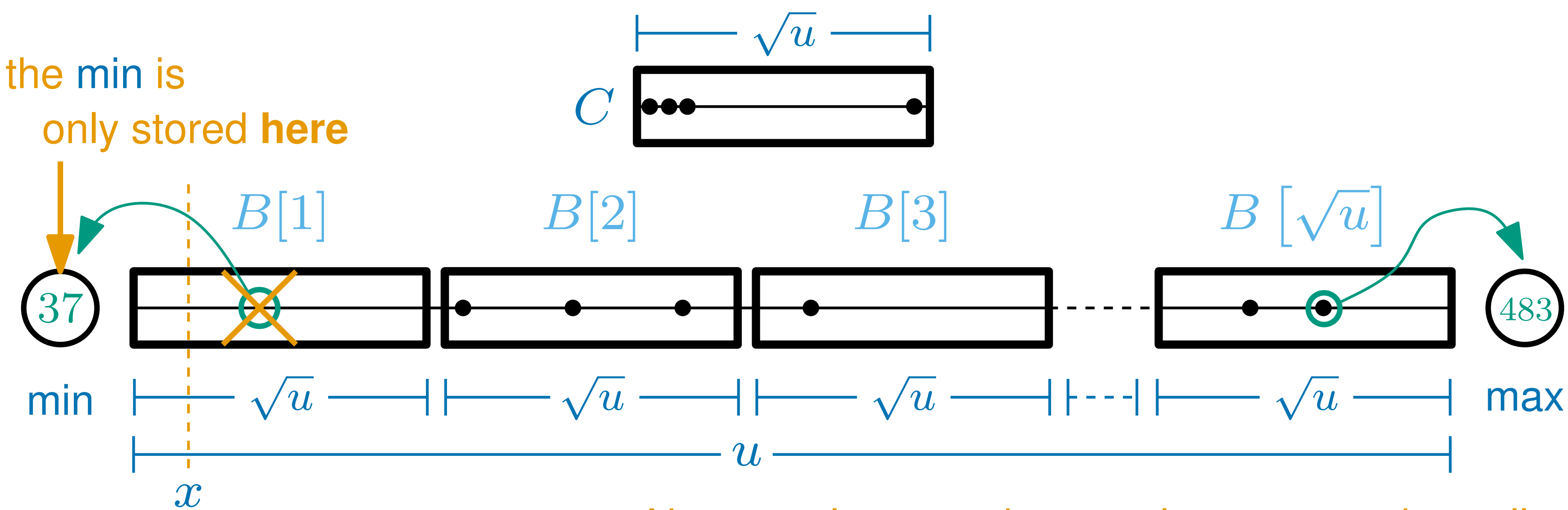

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To perform add(x):

Step 0 If $x <$ min then swap x and min

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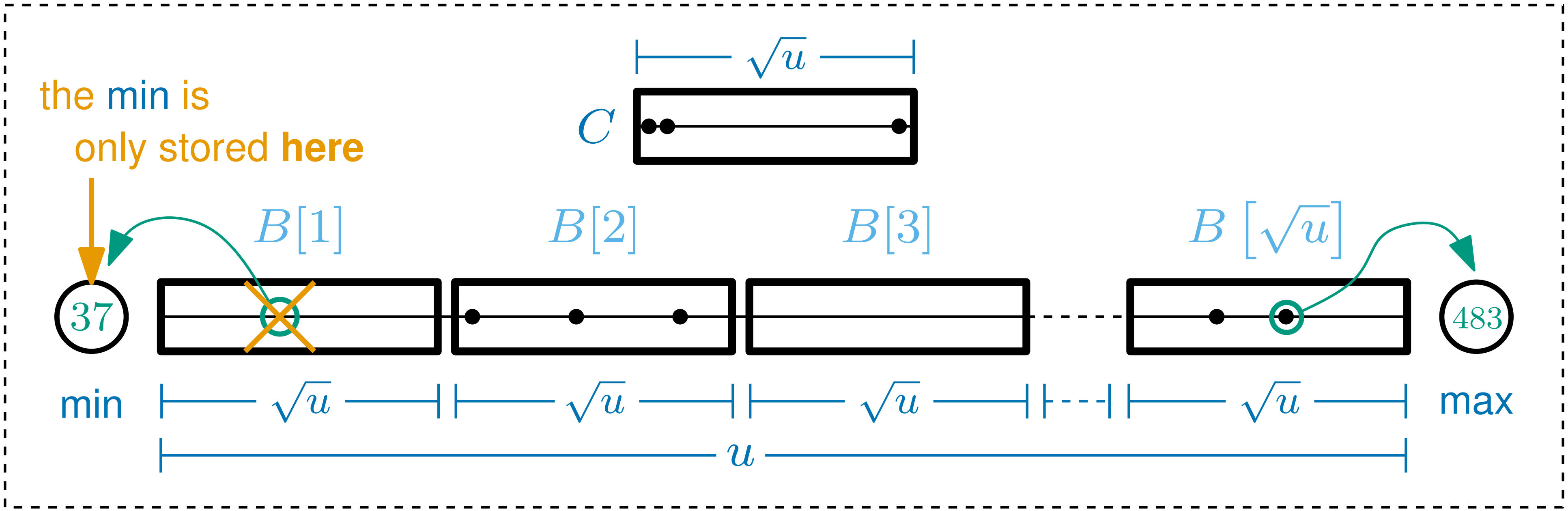
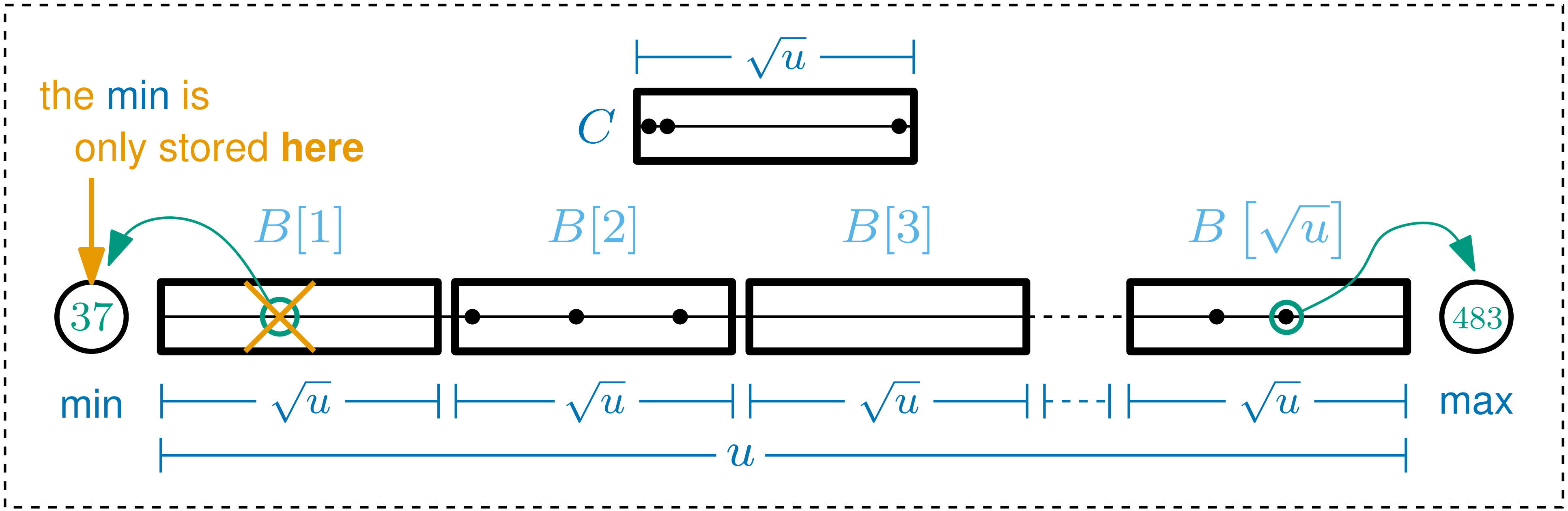
Step 3 If $B[i]$ is not empty, add x to $B[i]$

Step 4 Update the max

Time Complexity

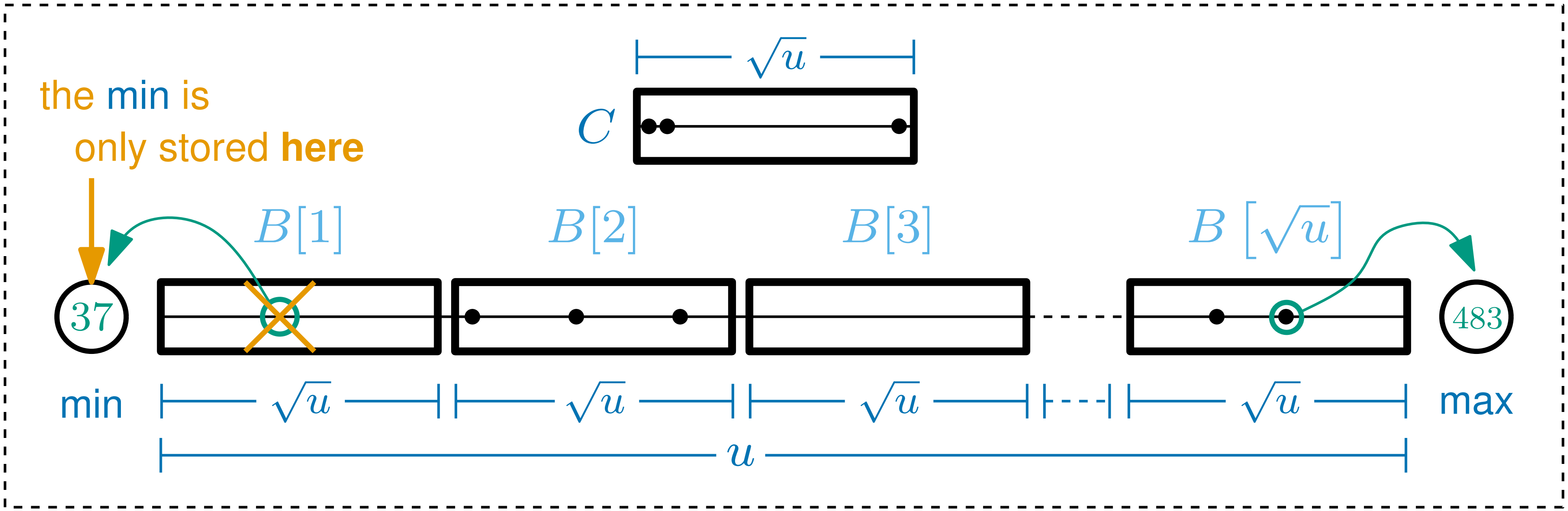


Time Complexity



We have seen that the operations add and predecessor can be defined so that they make only one recursive call

Time Complexity



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The operations lookup, delete and successor can all also be defined in a similar, recursive manner so that they make only one recursive call

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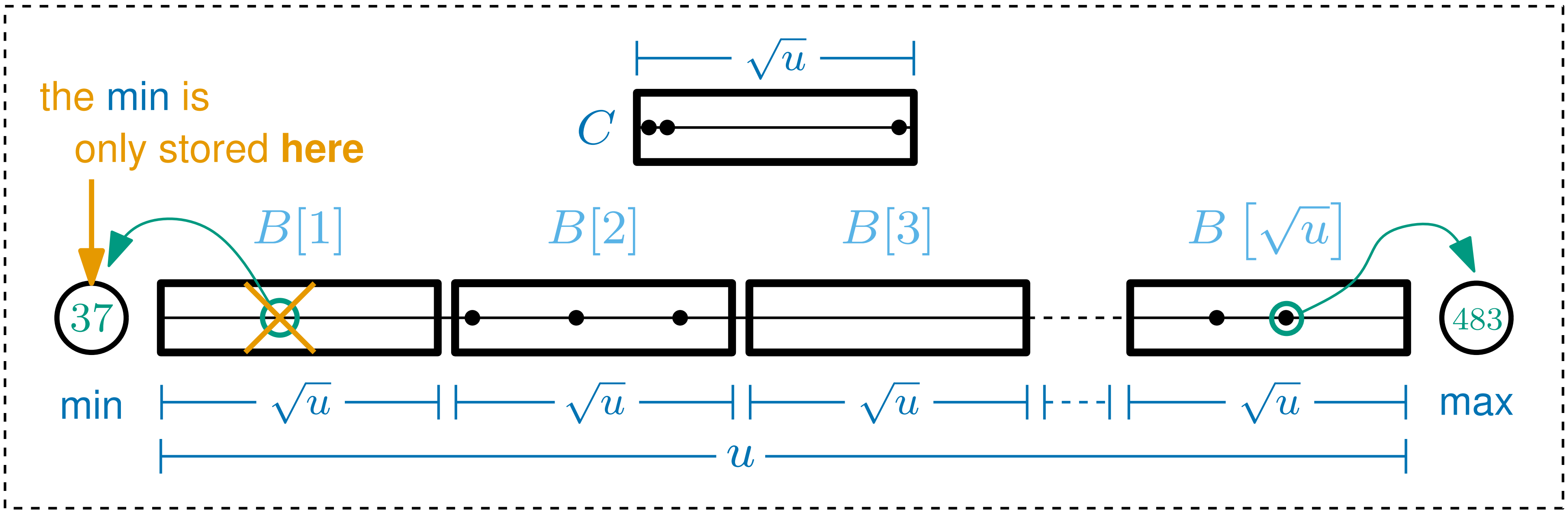

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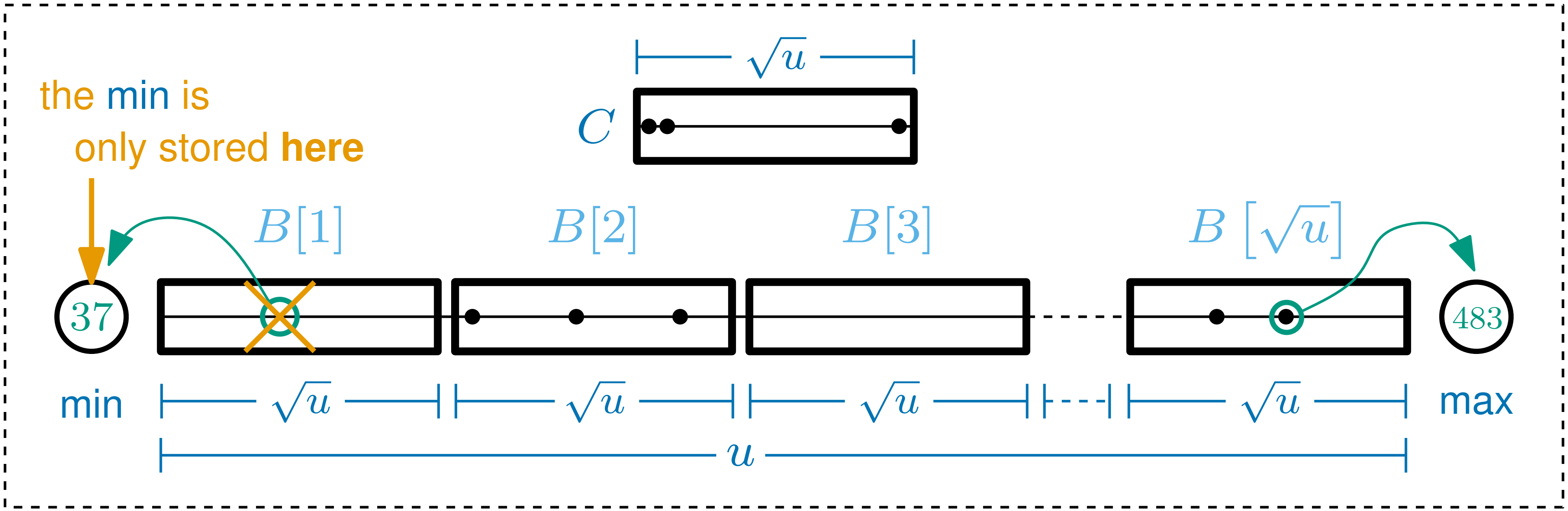
so that they make only one recursive call

How long do the operations take?

Time Complexity



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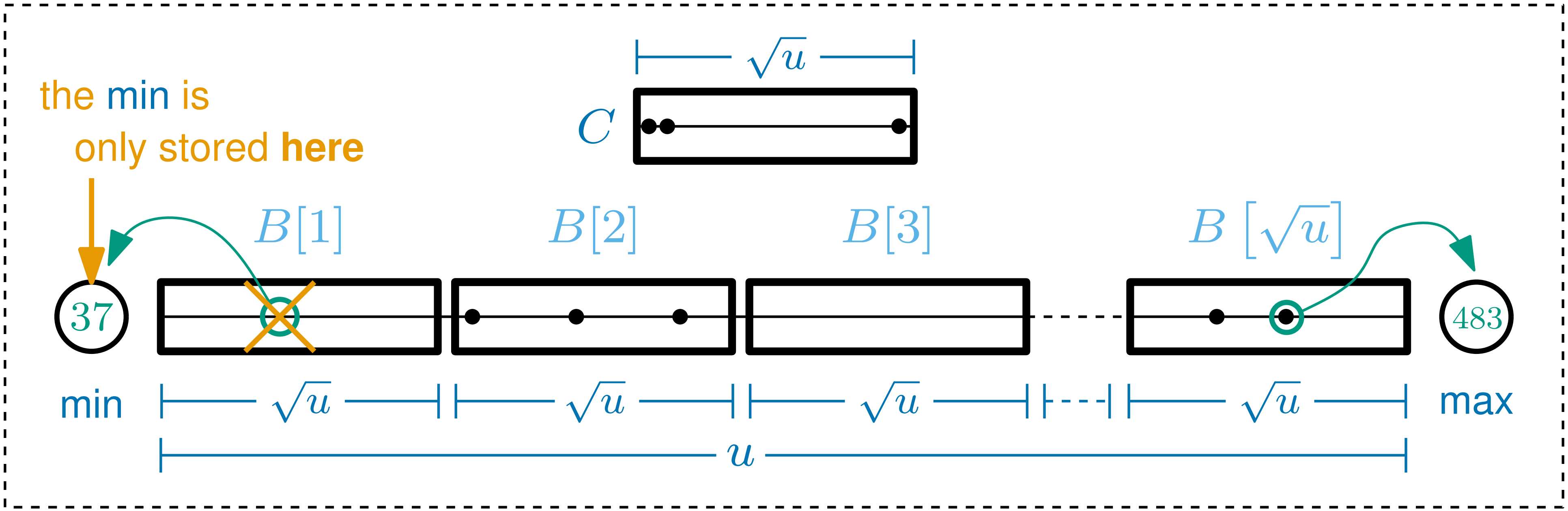
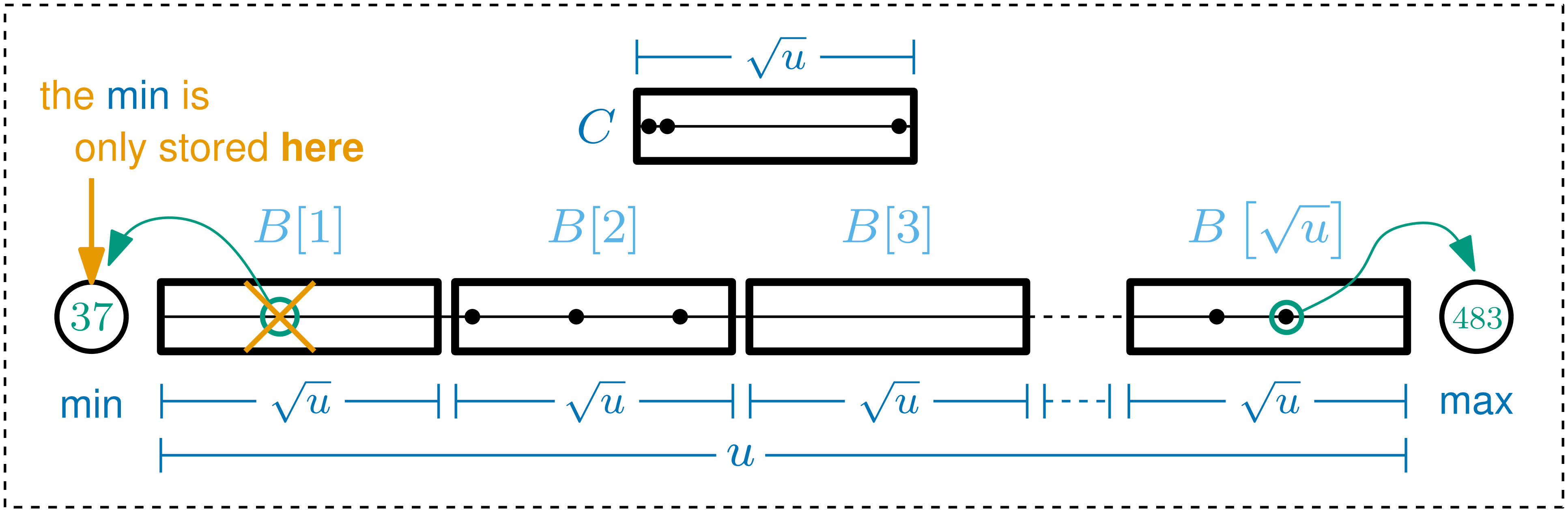


Let $T(u)$ be the time complexity of the add operation

(where u is the universe size)

Time Complexity



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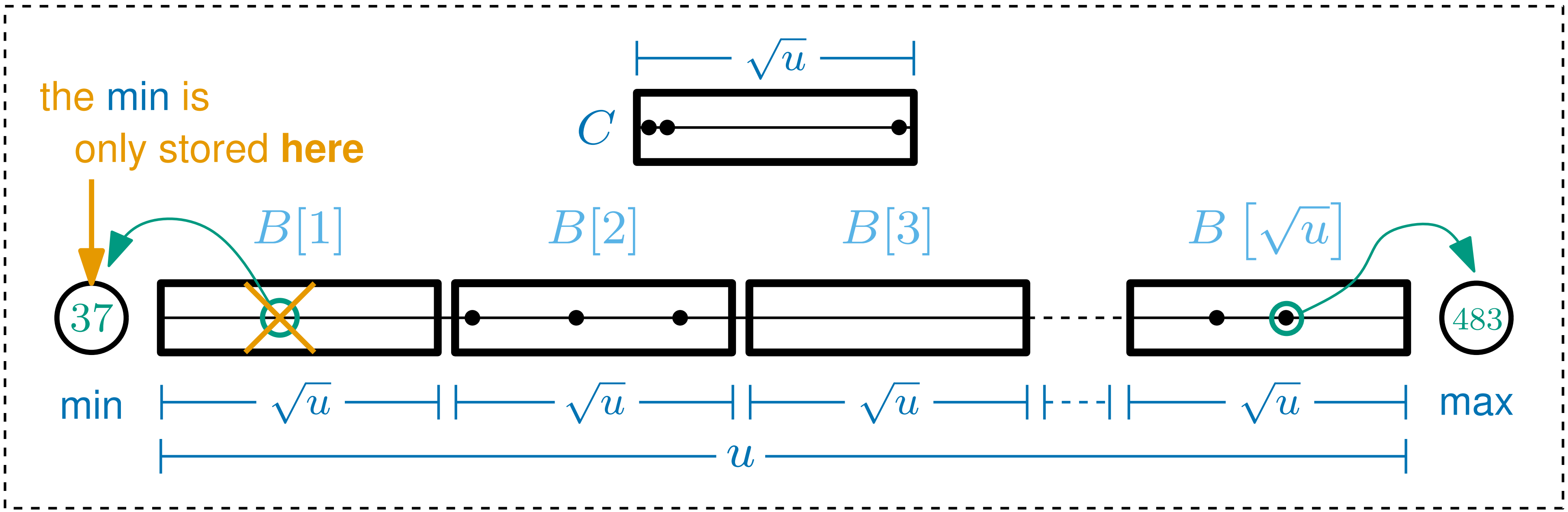

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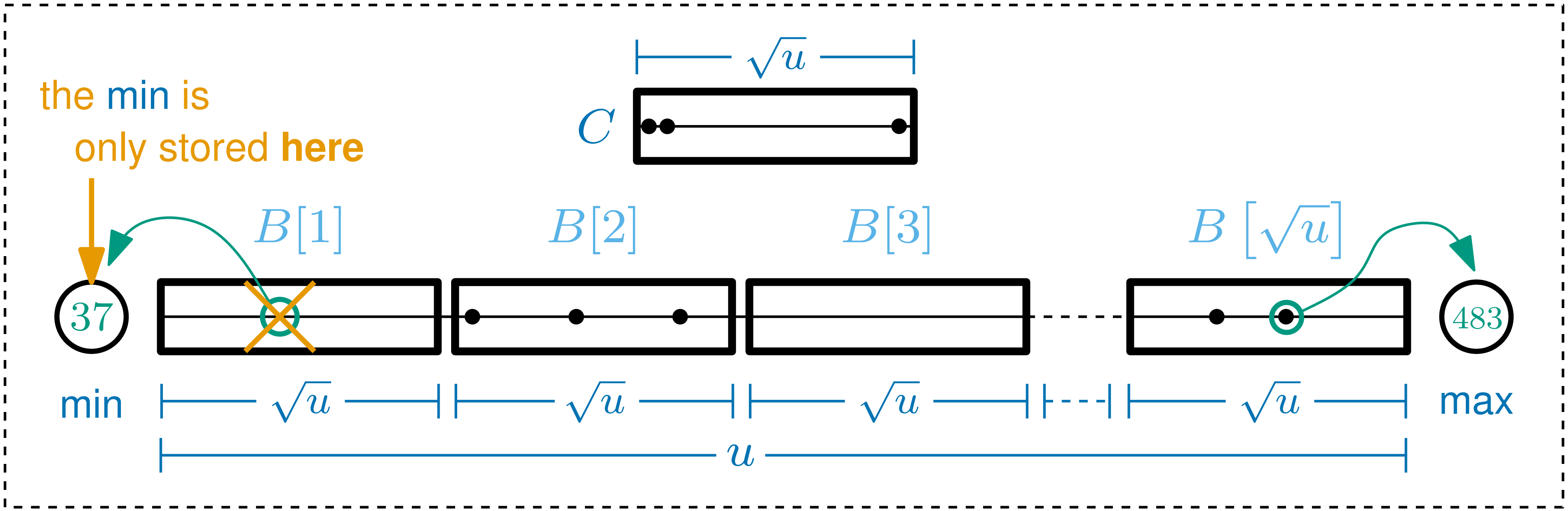

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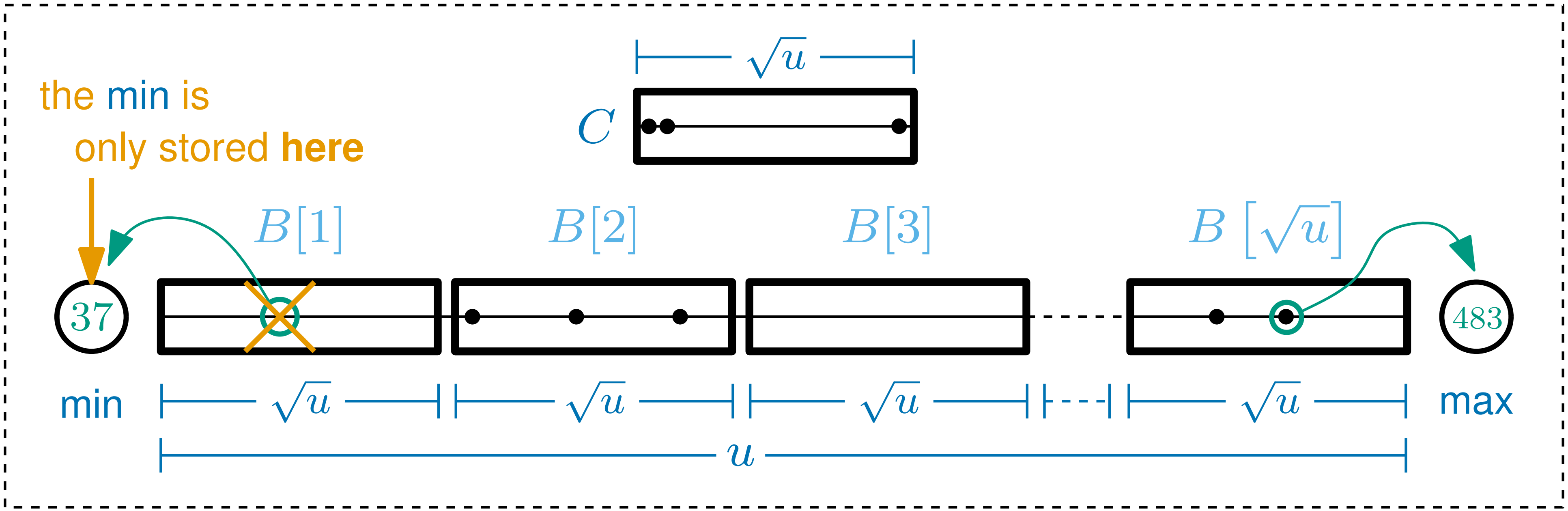

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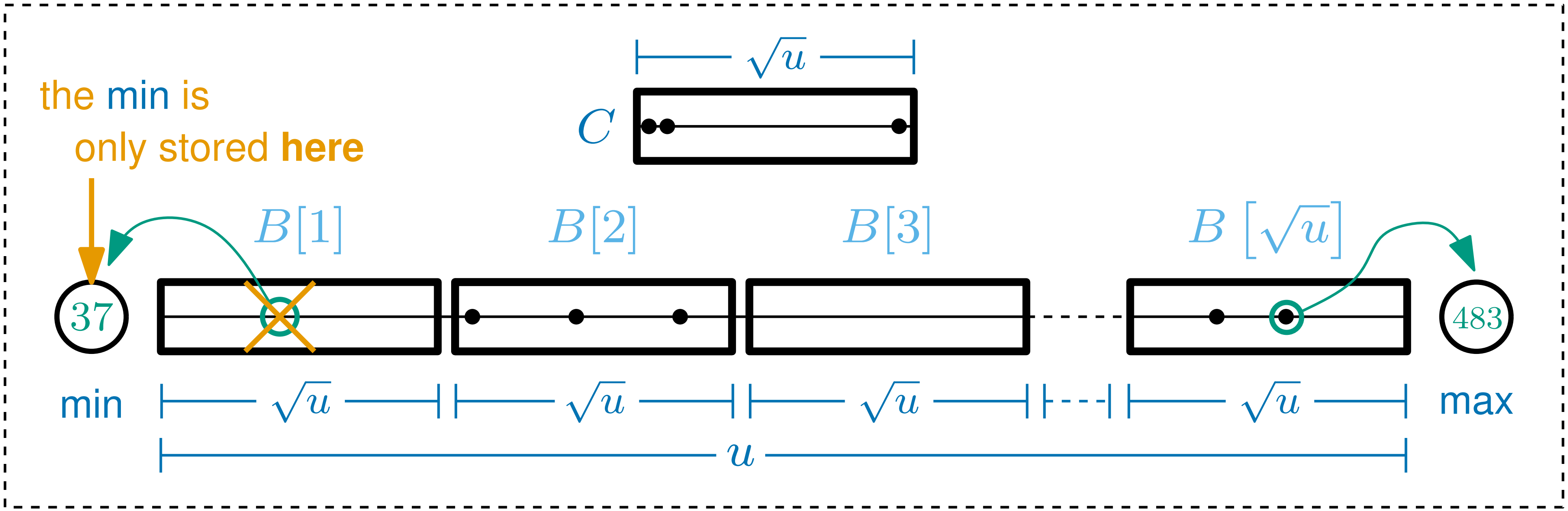
this holds for all the operations

Space Complexity



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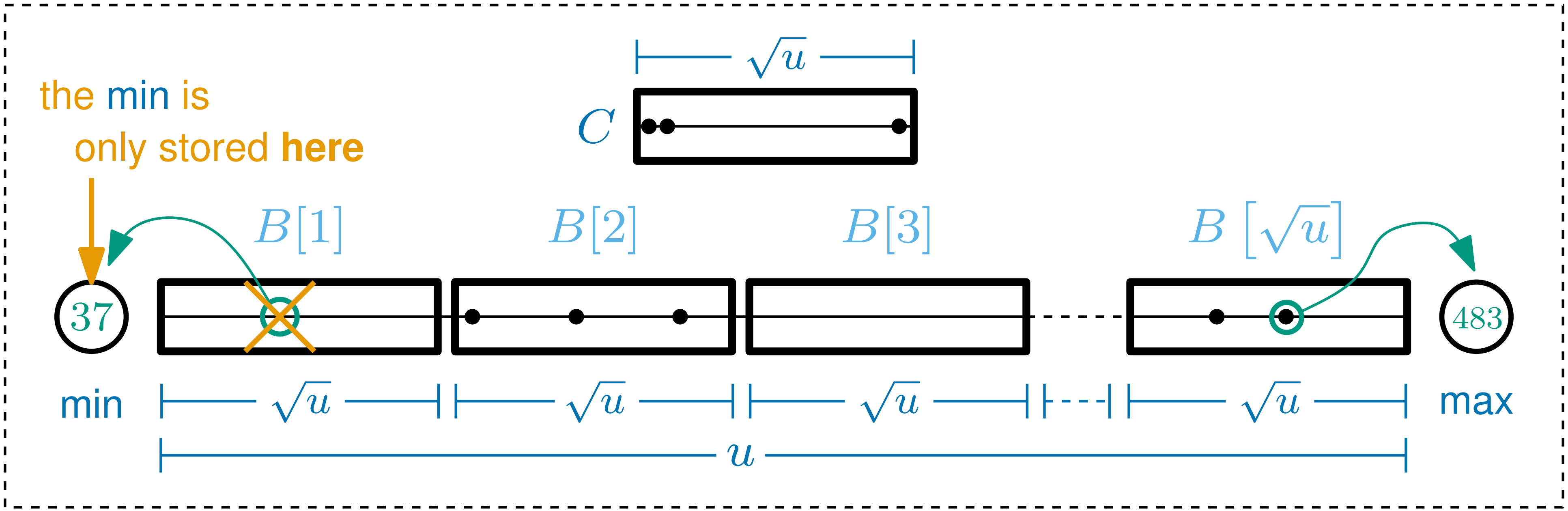
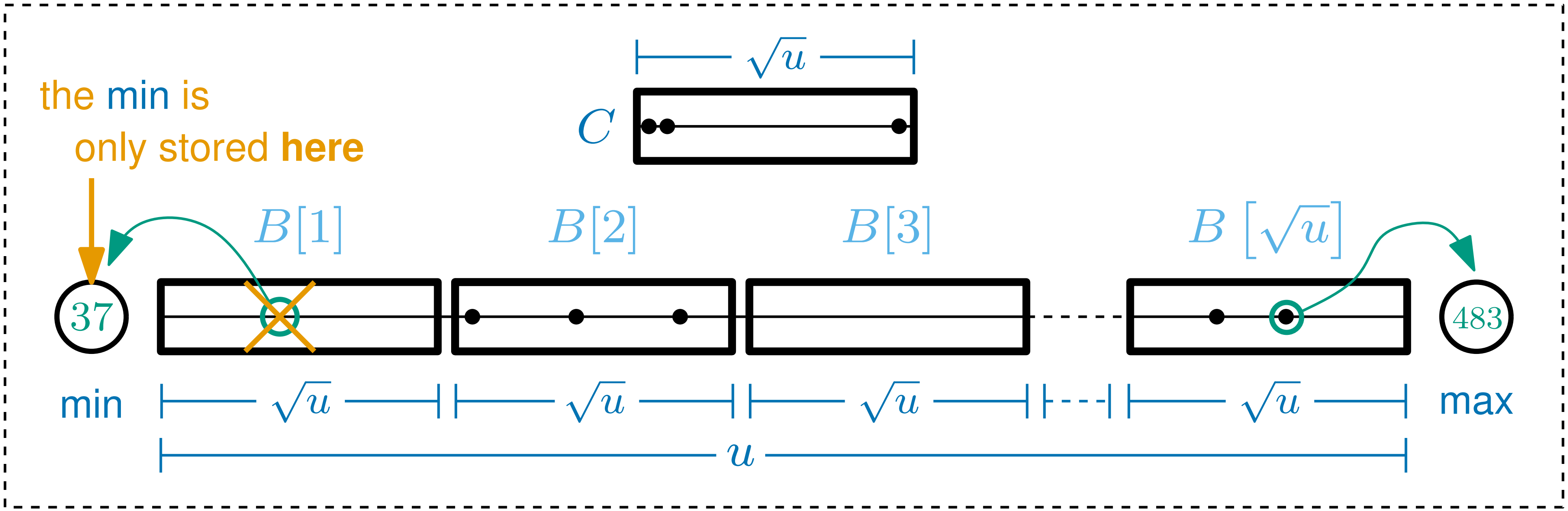


Let $Z(u)$ be the space used by a vEB tree over a universe of size u

Space Complexity



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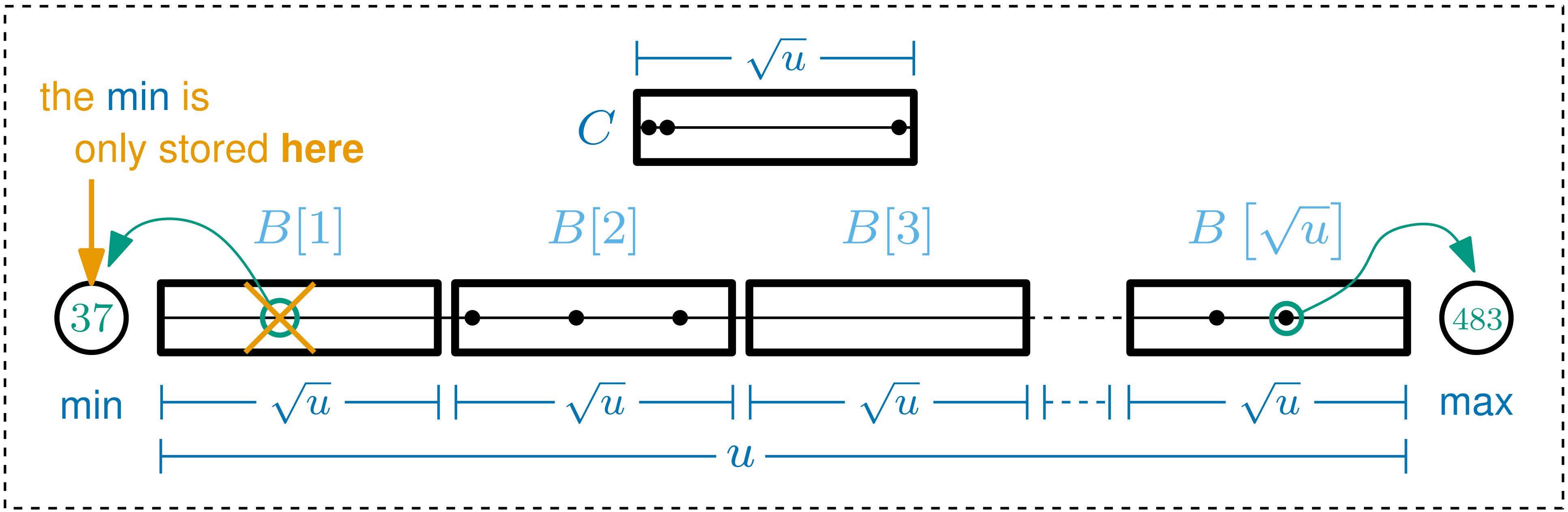

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If you solve this you get that... $Z(u) = O(u)$

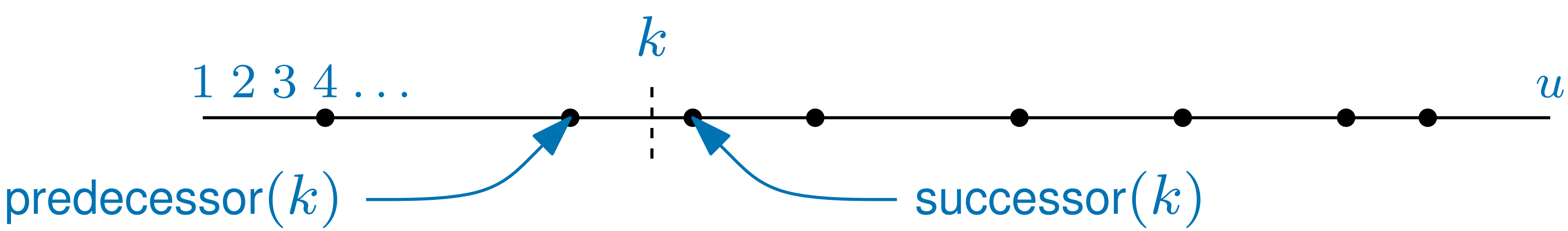
van Emde Boas Trees

The **van Emde Boas (vEB) tree**

stores a set S of integer keys from a universe $U = \{1, 2, 3, 4 \ldots u\}$ (i.e. $u = |U|$).

Five operations are supported:

add(x)	Insert the integer x into S (where $x \in U$)
lookup(x)	Return yes if x is in S, or no otherwise.
delete(x)	Remove x from S
predecessor(k)	Return the largest integer x in S such that $x \leqslant k$
successor(k)	Return the smallest integer x in S such that $x \geqslant k$



All operations take $O(\log \log u)$ worst case time

and the space used is $O(u)$

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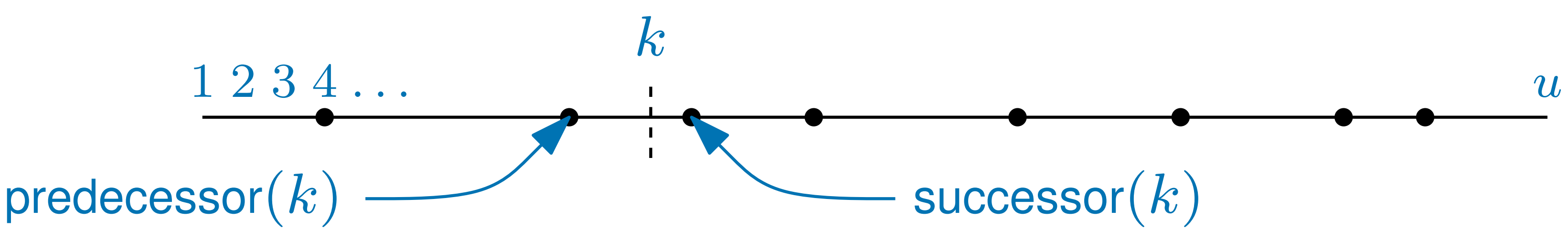

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The space can be improved to $O(n)$ using hashing (see y-fast trees)