Advanced Algorithms – COMS31900

Hashing part one
Chaining, true randomness and universal hashing

Raphaël Clifford

Slides by Benjamin Sach and Markus Jalsenius
Dictionaries

In a **dictionary** data structure we store \((key, value)\)-pairs such that for any \(key\) there is at most one pair \((key, value)\) in the dictionary.

Often we want to perform the following three operations:

- **add\((x, v)\)**  
  Add the the pair \((x, v)\).

- **lookup\((x)\)**  
  Return \(v\) if \((x, v)\) is in dictionary, or **NULL** otherwise.

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  Remove pair \((x, v)\) (assuming \((x, v)\) is in dictionary).
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- Binary search trees
- \((2,3,4)\)-trees
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but none of them take \(O(1)\) worst case time for all operations... so *maybe* there is room for improvement?
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We cannot avoid collisions entirely since \( u \gg m \);

*some keys from the universe are bound to be mapped to the same position.*

(remember \( u \) is the size of the universe and \( m \) is the size of the table)

By building a hash table with chaining, we get the following time complexities:

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So how long are these chains?
**Theorem**

Consider any *n* fixed inputs to the hash table (*which has size* *m*), i.e. any sequence of *n* add/lookup/delete operations.

Pick *h* uniformly at random from the set of all functions $U \rightarrow [m]$.

The expected run-time per operation is $O(1 + \frac{n}{m})$, or simply $O(1)$ if $m \geq n$. 

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Finally, we have that
\[
\mathbb{E}(N_x) = \mathbb{E}\left( \sum_{y \in T} I_{x,y} \right) = \sum_{y \in T} \mathbb{E}(I_{x,y}) = n \cdot \frac{1}{m} = \frac{n}{m}
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*linearity of expectation.*
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This has become rather cyclic... let’s try something else!
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Instead, we define a set, or *family of hash functions*: $H = \{h_1, h_2, \ldots \}$.

As part of initialising the hash table,
we choose the hash function $h$ from $H$ randomly.

How should we specify the hash functions in $H$ and how do we pick one at random?
Weakly universal hashing

- A set $H$ of hash functions is **weakly universal** if for any two distinct keys $x, y \in U$,

$$\Pr (h(x) = h(y)) \leq \frac{1}{m}$$

where $h$ is chosen uniformly at random from $H$. 
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**Observe**

The randomness here comes from the fact that $h$ is picked randomly.
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**Theorem**

Consider any $n$ fixed inputs to the hash table (which has size $m$), i.e. any sequence of $n$ add/lookup/delete operations.

Pick $h$ uniformly at random from a weakly universal set $H$ of hash functions.

The expected run-time per operation is $O(1)$ if $m \geq n$. 
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The expected run-time per operation is $O(1)$ if $m \geq n$.

**Proof**
The proof we used for true randomness works here too (which is nice).
Constructing a weakly universal family of hash functions

- Suppose $U = [u]$, i.e. the keys in the universe are integers 0 to $u - 1$.
- Let $p$ be any prime bigger than $u$.
- For $a, b \in [p]$, let

$$h_{a,b}(x) = ((ax + b) \mod p) \mod m,$$

$$H_{p,m} = \{h_{a,b} \mid a \in \{1, \ldots, p - 1\}, b \in \{0, \ldots, p - 1\}\}.$$
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$H_{p,m}$ is a weakly universal set of hash functions.
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\]

**Theorem**

\( H_{p,m} \) is a weakly universal set of hash functions.

**Proof**


**Observe**

- \( ax + b \) is a linear transformation which “spreads the keys” over \( p \) values when taken modulo \( p \). This does not cause any collisions.
- Only when taken modulo \( m \) do we get collisions.
True randomness vs. weakly universal hashing

For both,

**true randomness**

(h is picked uniformly from the set of all possible hash functions)

and **weakly universal hashing**

(h is picked uniformly from a weakly universal set of hash functions)

we have seen that when $m \geq n$,

the expected lookup time in the hash table is $O(1)$. 


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*Since constructing a weakly universal set of hash functions seems much easier than obtaining true randomness, this is all good news!*
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What about the length of the *longest* chain? *(the longest linked list)*

If it is very long, some lookups could take a very long time...
Longest chain – true randomness

**Lemma**

If $h$ is selected uniformly at random from all functions $U \rightarrow [m]$ then, over $m$ fixed inputs,

$$\Pr(\text{any chain has length } \geq 3 \log m) \leq \frac{1}{m}.$$
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**Observe**

In this lemma we insert \( m \) keys, i.e. \( n = m \).
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**Observe**

In this lemma we insert $m$ keys, i.e. $n = m$.

**Proof**

The problem is equivalent to showing that if we randomly throw $m$ balls into $m$ bins, the probability of having a bin with at least $3 \log m$ balls is at most $\frac{1}{m}$.

![Diagram of balls and bins](https://via.placeholder.com/150)
Let $X_1$ be the number of balls in the first bin.
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So, the union bound gives us

$$\Pr(X_1 \geq k) \leq \binom{m}{k} \cdot \frac{1}{m^k} \leq \frac{1}{k!}.$$
Let $X_1$ be the number of balls in the first bin.

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**Proof continued...**

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---

**Theorem**

Let $V_1, \ldots, V_q$ be $q$ events. Then

$$\Pr\left( \bigcup_{i=1}^{q} V_i \right) \leq \sum_{i=1}^{q} \Pr(V_i).$$
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$$\Pr(X_1 \geq k) \leq \binom{m}{k} \cdot \frac{1}{m^k} \leq \frac{1}{k!}.$$

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m \cdot (m-1) \cdot (m-2) \cdots (m-k+1) \cdot (m-k)!}{k!(m-k)!} \leq \frac{m \cdot (m) \cdot (m) \cdots (m)}{k!}$$
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By using the union bound again, we have that

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$$\Pr(X_1 \geq k) \leq \left(\binom{m}{k}\right) \cdot \frac{1}{m^k} \leq \frac{1}{k!}.$$ 

By using the union bound *again*, we have that

$$\Pr(\text{at least one bin receives at least } k \text{ balls}) \leq m \cdot \Pr(X_1 \geq k) \leq \frac{m}{k!}.$$ 

Now we set $k = 3 \log m$ and observe that $\frac{m}{k!} \leq \frac{1}{m}$ for $m \geq 2$, and we are done.
Let $X_1$ be the number of balls in the first bin.

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Why is $\frac{m}{k!} \leq \frac{1}{m}$? (when $k = 3 \log m$)

By using the union bound again, we have that

So, the probability that all of these $k$ balls go into the first bin is $\frac{1}{m^k}$.

Now we set $k = 3 \log m$ and observe that $\frac{k!}{m} \leq \frac{1}{m}$ for $m \geq 2$, and we are done.
Longest chain – true randomness

**Lemma**
If $h$ is selected uniformly at random from all functions $U \rightarrow [m]$ then, over $m$ fixed inputs,

$$\Pr(\text{any chain has length } \geq 3 \log m) \leq \frac{1}{m}.$$ 

**Observe**
In this lemma we insert $m$ keys, i.e. $n = m$.

**Proof**
The problem is equivalent to showing that if we randomly throw $m$ balls into $m$ bins, the probability of having a bin with at least $3 \log m$ balls is at most $\frac{1}{m}$. 

![Diagram showing random distribution of balls into bins](image)
Longest chain – weakly universal hashing

The conclusion from previous slides is that with true randomness, the longest chain is very short (at most $3 \log m$) with high probability.
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**Lemma**

If $h$ is picked uniformly at random from a weakly universal set of hash functions then, over $m$ fixed inputs,

$$\Pr \left( \text{any chain has length} \geq 1 + \sqrt{2m} \right) \leq \frac{1}{2}.$$
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$$\Pr \left( \text{any chain has length} \geq 1 + \sqrt{2m} \right) \leq \frac{1}{2}.$$

**Observe**

This rubbish upper bound of $\frac{1}{2}$ does not necessarily rule out the possibility that the tightest upper bound is indeed very small. However, the upper bound of $\frac{1}{2}$ is in fact tight!
Longest chain – weakly universal hashing

**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$. 
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- For any two keys \( x, y \), let indicator r.v. \( I_{x,y} \) be 1 iff \( h(x) = h(y) \).
- Let r.v. \( C \) be the total number of collisions: \( C = \sum_{x,y \in T, x < y} I_{x,y} \).
Proof

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x, y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),

\[
\mathbb{E}(C) = \mathbb{E}\left( \sum_{x, y \in T, x < y} I_{x,y} \right) = \sum_{x, y \in T, x < y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.
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  \]
- by Markov’s inequality, \( \Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2} \).
For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.

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$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.$$

by Markov’s inequality, $Pr(C \geq m) \leq \frac{E(C)}{m} \leq \frac{1}{2}$.

Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$. 
**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$.
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- By Markov’s inequality, $\Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2}$.
- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$.

This is because a chain of length $L$ causes $\binom{L}{2}$ collisions!
Longest chain – weakly universal hashing

**Proof**

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- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$.
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- by Markov’s inequality, \( \Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2} \).
- Let r.v. \( L \) be the length of the longest chain. Then \( C \geq \binom{L}{2} \).
- Now, \( \Pr\left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr\left( \binom{L}{2} \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2} \).
Proof

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),
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- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$.
- Now, $\Pr \left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr \left( \frac{L}{2} \geq m \right) \leq \Pr \left( C \geq m \right) \leq \frac{1}{2}$.

This is because $\binom{L}{2} = \frac{L!}{2!(L-2)!} = \frac{L \cdot (L-1)}{2} \geq \frac{(L-1)^2}{2}$.
Longest chain – weakly universal hashing

**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),
  $$\mathbb{E}(C') = \mathbb{E}\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.$$  
- by Markov’s inequality, $\Pr(C \geq m) \leq \frac{\mathbb{E}(C')}{m} \leq \frac{1}{2}$.
- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$.
- Now, $\Pr\left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr\left( \binom{L}{2} \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2}$. 
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- Using linearity of expectation and $E(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),
  \[
  E(C) = \sum_{x,y \in T, x < y} E(I_{x,y}) = \sum_{x,y \in T, x < y} \frac{1}{m} = \left( \frac{m^2}{2} \right) \cdot \frac{1}{m} \leq \frac{m}{2}.
  \]
- by Markov’s inequality, $\Pr(C \geq m) \leq \frac{E(C)}{m} \leq \frac{1}{2}$.
- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$.
- Now, $\Pr\left(\frac{(L-1)^2}{2} \geq m\right) \leq \Pr\left(\binom{L}{2} \geq m\right) \leq \Pr(C \geq m) \leq \frac{1}{2}$.
**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be $1$ iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $E(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),

\[
E(C) = E(\sum_{x,y \in T, x < y} I_{x,y}) = \sum_{x,y \in T, x < y} E(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.
\]

- By Markov’s inequality, $\Pr(C \geq m) \leq \frac{E(C)}{m} \leq \frac{1}{2}$.
- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$.
- Now, $\Pr\left(\frac{(L-1)^2}{2} \geq m\right) \leq \Pr\left(\binom{L}{2} \geq m\right) \leq \Pr\left(C \geq m\right) \leq \frac{1}{2}$.

By rearranging, we have that $\Pr\left(L \geq 1 + \sqrt{2m}\right) \leq \frac{1}{2}$, and we are done.
Conclusions

For both, 

true randomness \((h)\) is picked uniformly from the set of all possible hash functions\)

and weakly universal hashing \((h)\) is picked uniformly from a weakly universal set of hash functions\)

we have seen that when \(m \geq n\),

the expected lookup time in a hash table with chaining is \(O(1)\).

**Lemma**

If \(h\) is selected uniformly at random from all functions \(U \rightarrow [m]\) then,

\[
\Pr(\text{any chain has length } \geq 3 \log m) \leq \frac{1}{m}.
\]

**Lemma**

If \(h\) is picked uniformly at random from a weakly universal set of hash functions, 

\[
\Pr\left(\text{any chain has length } \geq 1 + \sqrt{2m}\right) \leq \frac{1}{2}.
\]

(both Lemmas hold for \(m\) any fixed inputs)