Advanced Algorithms – COMS31900

Hashing part one

Chaining, true randomness and universal hashing

Raphaël Clifford

Slides by Benjamin Sach and Markus Jalsenius

Dictionaries

In a **dictionary** data structure we store $(\textit{key}, \textit{value})$-pairs

such that for any *key* there is at most one pair $(\textit{key}, \textit{value})$ in the dictionary.

Often we want to perform the following three operations:

add(x, v)	Add the the pair (x, v).
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so *maybe* there is room for improvement?

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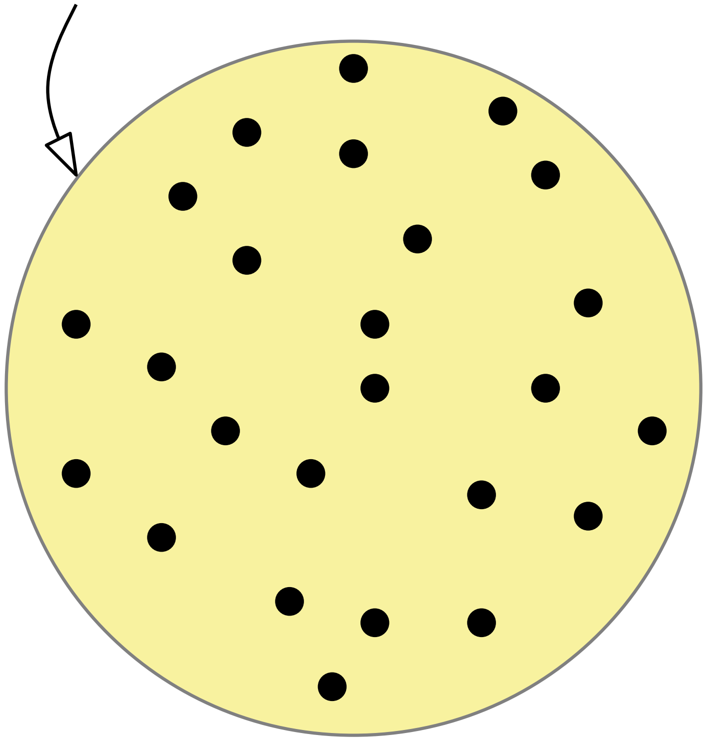
We want to store n elements from the universe, U in a dictionary.

Typically $u = |U|$ *is much, much larger than* n.

Universe U containing u keys.



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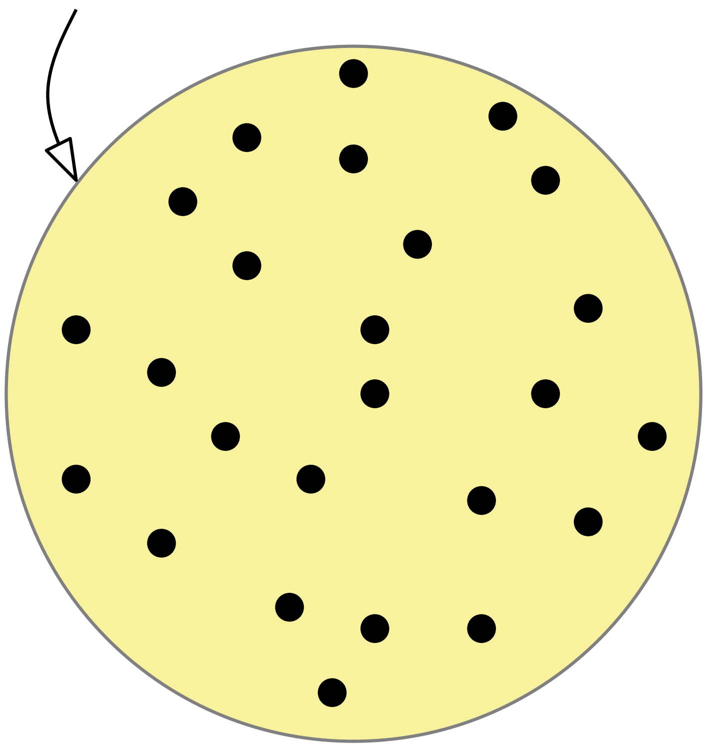

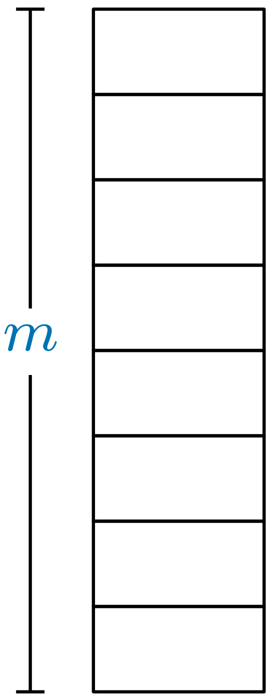

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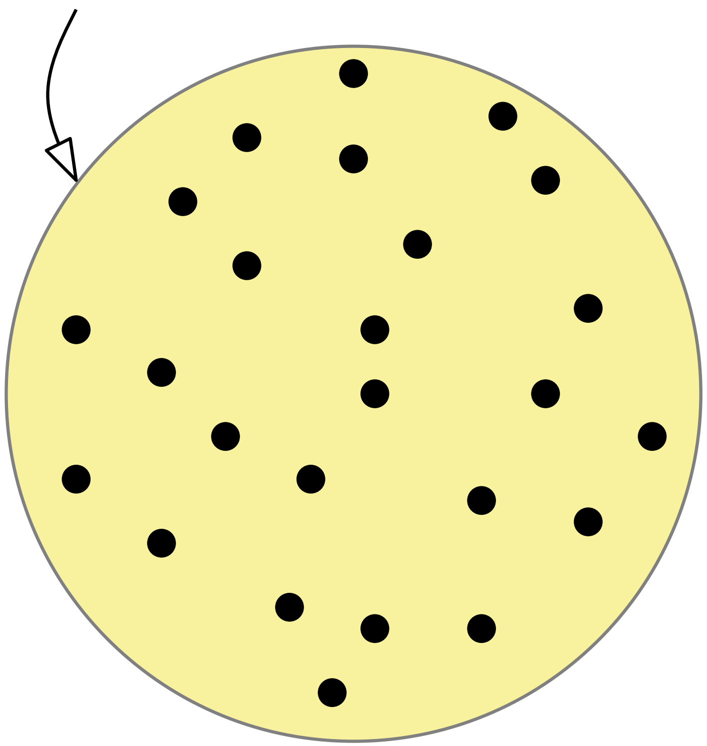

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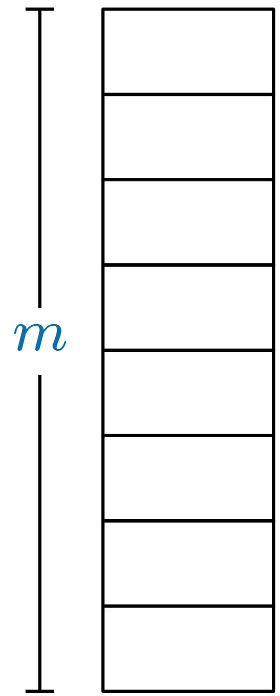

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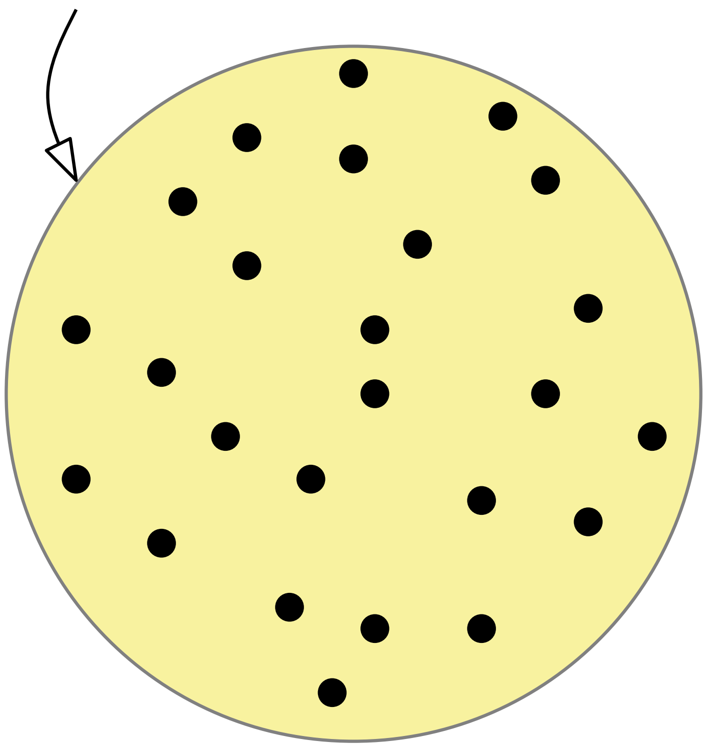

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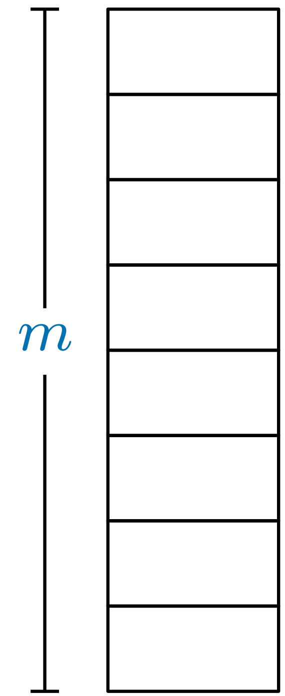

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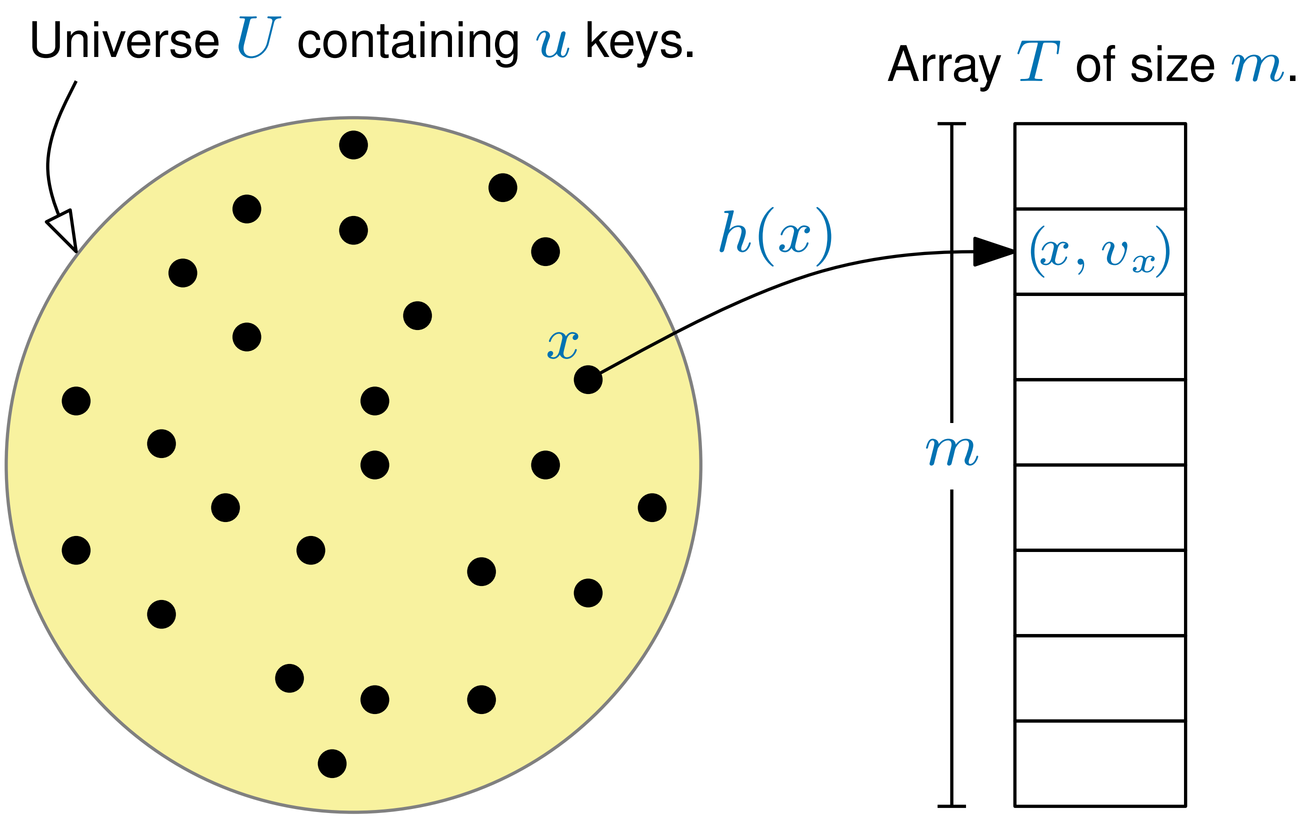

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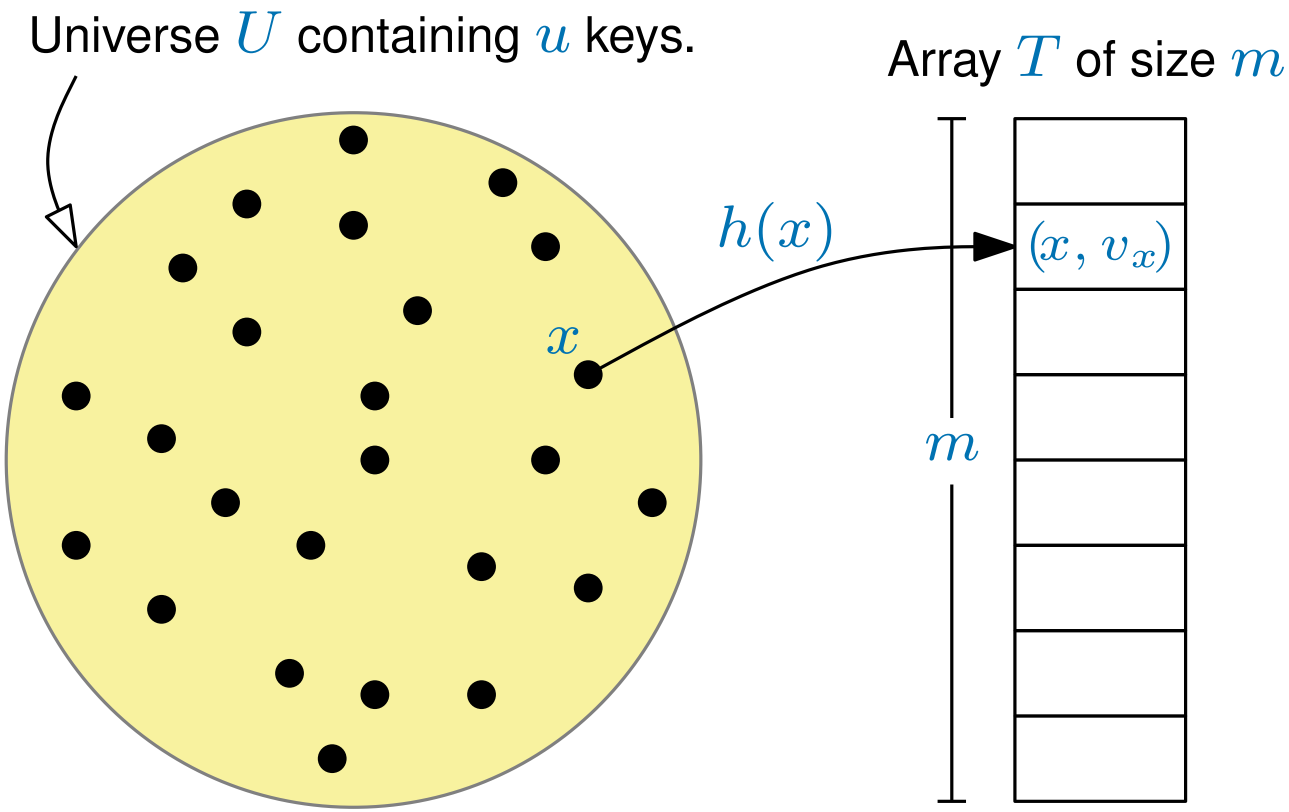

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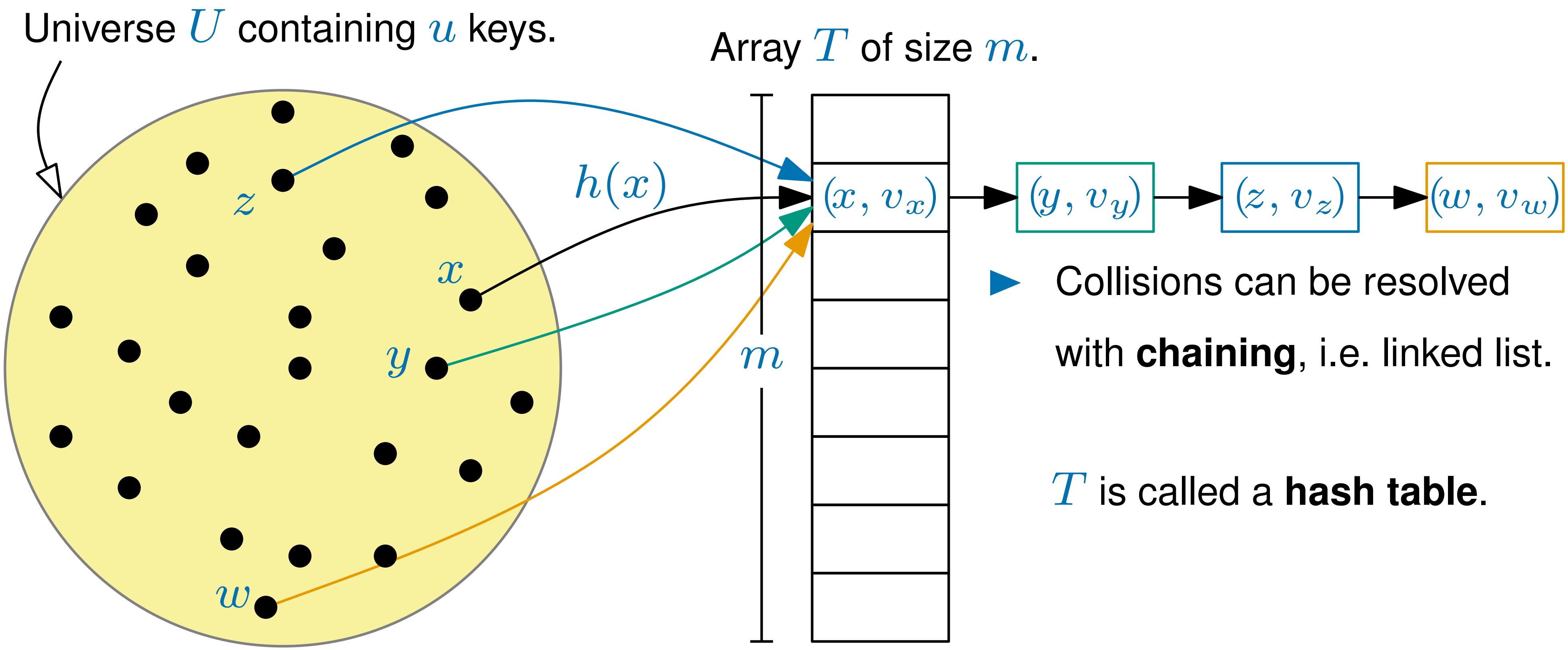

- Collisions can be resolved with **chaining**, i.e. linked list.

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Time complexity

We cannot avoid collisions entirely since $u \gg m$;

some keys from the universe are bound to be mapped to the same position.

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By building a hash table with chaining, we get the following time complexities:

Operation	Worst case time	Comment
add(x, v)	$O(1)$	Simply add item to the list link if necessary.
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So how long are these chains?

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Theorem

Consider any n fixed inputs to the hash table *(which has size m)*,

i.e. any sequence of n add/lookup/delete operations.

Pick h uniformly at random from the set of *all* functions $U \to [m]$.

The expected run-time per operation is $O(1 + \frac{n}{m})$, or simply $O(1)$ if $m \geqslant n$.

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Finally, we have that $\mathbb{E}(N_x) = \mathbb{E}\left(\sum_{y \in T} I_{x,y}\right) = \sum_{y \in T} \mathbb{E}(I_{x,y}) = n \cdot \frac{1}{m} = \frac{n}{m}$

linearity of expectation.

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This has become rather cyclic... let's try something else!

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Instead, we define a set, or *family of hash functions*: $H = \{h_1, h_2, \ldots\}$.

As part of initialising the hash table,

we choose the hash function h from H randomly.

Specifying the hash function

Problem: how do we specify an *arbitrary* (e.g. a truly random) hash function?

For each key in U we need to specify an arbitrary position in T,

this is a number in $[m]$, so requires $\approx \log_2 m$ bits.

So in total we need $\approx u \log_2 m$ bits, which is a ridiculous amount of space!

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How should we specify the hash functions in H and how do we pick one at random?

Weakly universal hashing

- A set H of hash functions is **weakly universal** if for any two distinct keys $x, y \in U$,

$$\Pr\left(h(x) = h(y)\right) \leqslant \frac{1}{m}$$

where h is chosen uniformly at random from H.

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Theorem

Consider any n fixed inputs to the hash table *(which has size m)*,

i.e. any sequence of n add/lookup/delete operations.

Pick h uniformly at random from a weakly universal set H of hash functions.

The expected run-time per operation is $O(1)$ if $m \geqslant n$.

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Proof

The proof we used for true randomness works here too (which is nice)

Constructing a weakly universal family of hash functions

- Suppose $U = [u]$, i.e. the keys in the universe are integers 0 to $u-1$.
- Let p be any prime bigger than u.
- For $a, b \in [p]$, let

$$h_{a,b}(x) = ((ax + b) \bmod p) \bmod m,$$

$$H_{p,m} = \{h_{a,b} \mid a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\}.$$

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See CLRS, Theorem 11.5, (page 267 in 3rd edition).

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Observe

- $ax + b$ is a linear transformation which "spreads the keys" over p values when taken modulo p. This does not cause any collisions.
- Only when taken modulo m do we get collisions.

True randomness vs. weakly universal hashing

For both,

true randomness

(h is picked uniformly from the set of all possible hash functions)

and **weakly universal hashing**

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we have seen that when $m \geqslant n$,

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If it is very long, some lookups could take a very long time. . .

Longest chain – true randomness

Lemma

If h is selected uniformly at random from all functions $U \to [m]$ then,
over m fixed inputs,

$$\Pr\left(\text{any chain has length } \geqslant 3\log m\right) \leqslant \frac{1}{m}.$$

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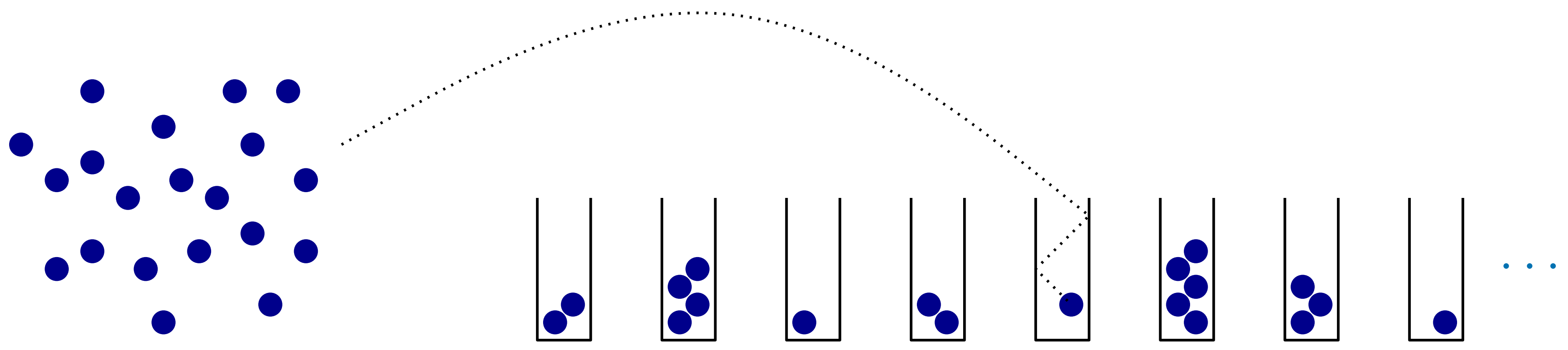
PROOF

The problem is equivalent to showing that if we randomly throw m balls into m bins, the probability of having a bin with at least $3 \log m$ balls is at most $\frac{1}{m}$.



Longest chain – true randomness

PROOF

continued...

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So, the union bound gives us

$$\Pr(X_1 \geqslant k) \leqslant \binom{m}{k} \cdot \frac{1}{m^k} \leqslant \frac{1}{k!}.$$

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THEOREM

Let $V_1, \ldots, V_q$ be q events. Then

$$\Pr\left(\bigcup_{i=1}^{q} V_i\right) \leqslant \sum_{i=1}^{q} \Pr(V_i).$$

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$\Pr \ldots \leqslant \dfrac{m}{k!}$.

Why is $\frac{m}{k!} \leqslant \frac{1}{m}$? (when $k = 3\log m$)

$$k! = \overbrace{k \times (k-1) \times (k-2) \ldots \times 2 \times 1}^{k \text{ terms}}$$

$$k! > 2 \times 2 \times 2 \ldots \times 2 \times 1 = 2^{k-1}$$

Let $k = 3\log m \ldots$

$$k! > 2^{(3\log m - 1)} \geqslant 2^{2\log m} = (2^{\log m})^2 = m^2$$

so $\dfrac{m}{k!} \leqslant \dfrac{m}{m^2} = \dfrac{1}{m}$

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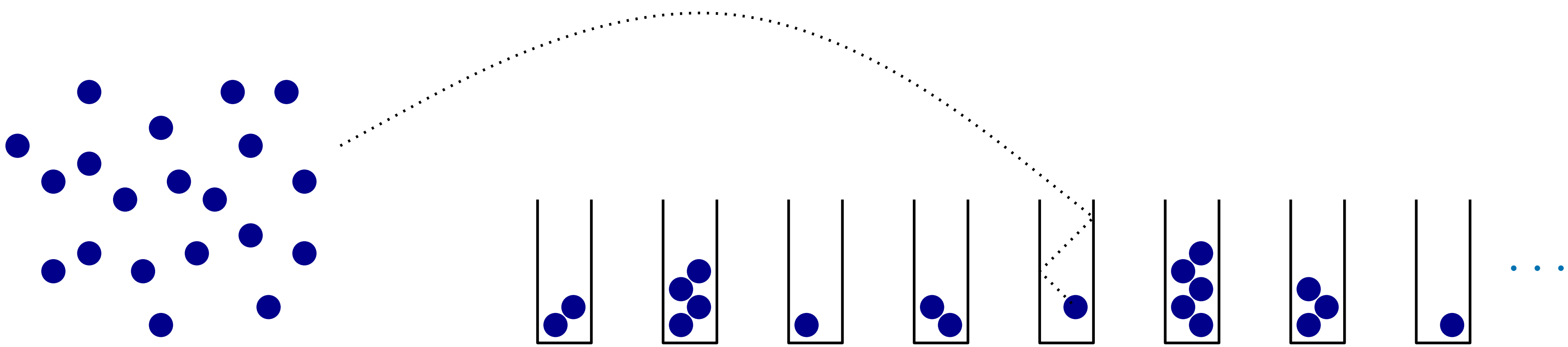

Longest chain – weakly universal hashing

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This rubbish upper bound of $\frac{1}{2}$ does not necessarily rule out the possibility that the *tightest* upper bound is indeed very small. However, the upper bound of $\frac{1}{2}$ is in fact tight!

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- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),

$$\mathbb{E}(C) = \mathbb{E}\Big(\sum_{x,y \in T,\, x<y} I_{x,y}\Big) = \sum_{x,y \in T,\, x<y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leqslant \frac{m}{2}.$$

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This is because a chain of length L causes $\binom{L}{2}$ collisions!

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- Let r.v. L be the length of the longest chain. Then $C \geqslant \binom{L}{2}$.
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Longest chain – weakly universal hashing

PROOF

- For any two keys x, y, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. C be the total number of collisions: $C = \sum_{x,y \in T,\, x<y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),

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this is because $\binom{L}{2} = \frac{L!}{2!(L-2)!} = \frac{L \cdot (L-1)}{2} \geqslant \frac{(L-1)^2}{2}$

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Longest chain – weakly universal hashing

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By rearranging, we have that $\Pr\left(L \geqslant 1 + \sqrt{2m}\right) \leqslant \frac{1}{2}$, and we are done.

Conclusions

For both,

true randomness (h is picked uniformly from the set of all possible hash functions)

and **weakly universal hashing**

(h is picked uniformly from a weakly universal set of hash functions)

we have seen that when $m \geqslant n$,

the expected lookup time in a hash table with chaining is $O(1)$.

Lemma

If h is selected uniformly at random from all functions $U \to [m]$ then,

$$\Pr\left(\text{any chain has length } \geqslant 3\log m\right) \leqslant \frac{1}{m}.$$

Lemma

If h is picked uniformly at random from a weakly universal set of hash functions,

$$\Pr\left(\text{any chain has length } \geqslant 1+\sqrt{2m}\right) \leqslant \frac{1}{2}.$$

(both Lemmas hold for m any fixed inputs)