Advanced Algorithms – COMS31900

Hashing part three

Cuckoo Hashing

Raphaël Clifford

Slides by Benjamin Sach

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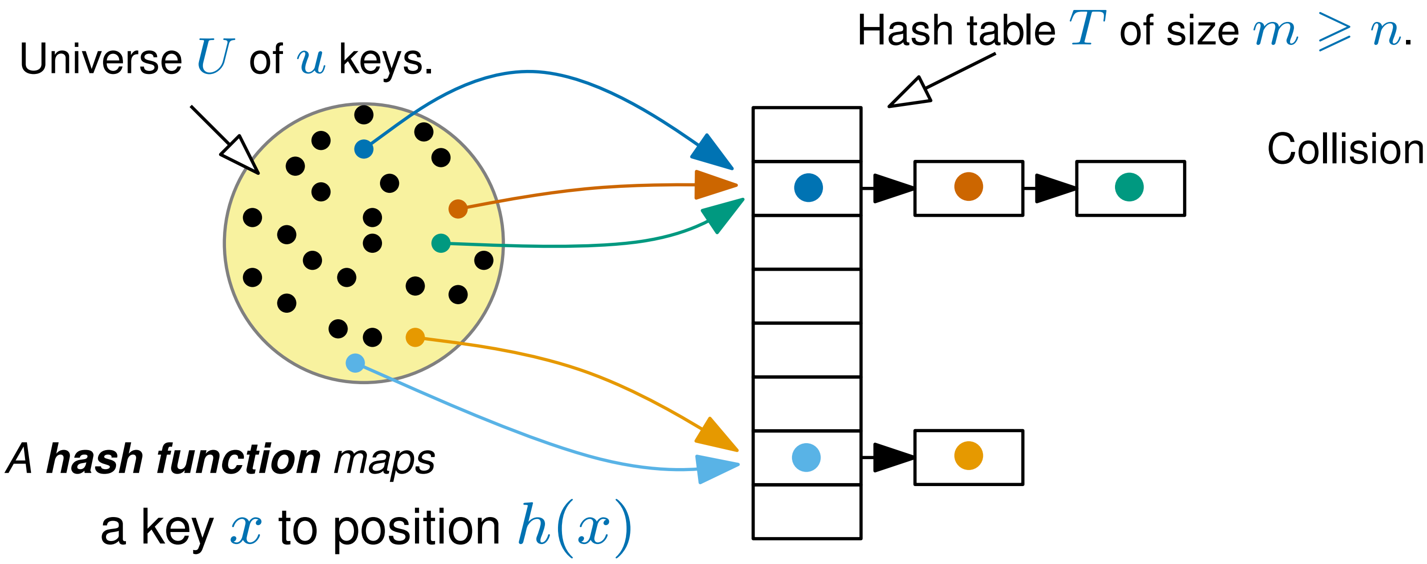
- A **dynamic dictionary** stores $(key, value)$-pairs and supports:

  add$(key, value)$, lookup(key) (which returns *value*) and delete(key)



Universe U of u keys.

Hash table T of size $m \geqslant n$.

Collisions are fixed by **chaining**

*A **hash function** maps*

a key x to position $h(x)$

n *arbitrary operations arrive online, one at a time.*

A set H of hash functions is **weakly universal** if for any two keys $x, y \in U$ (with $x \neq y$),

$$\Pr\left(h(x) = h(y)\right) \leqslant \frac{1}{m}$$

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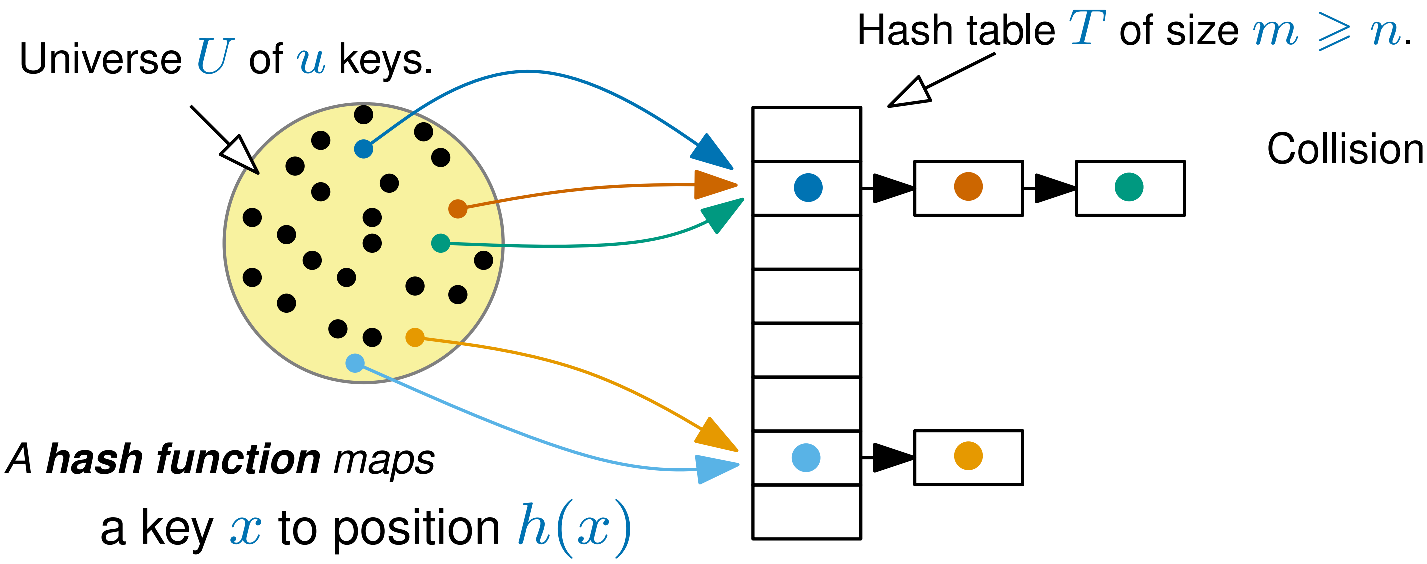
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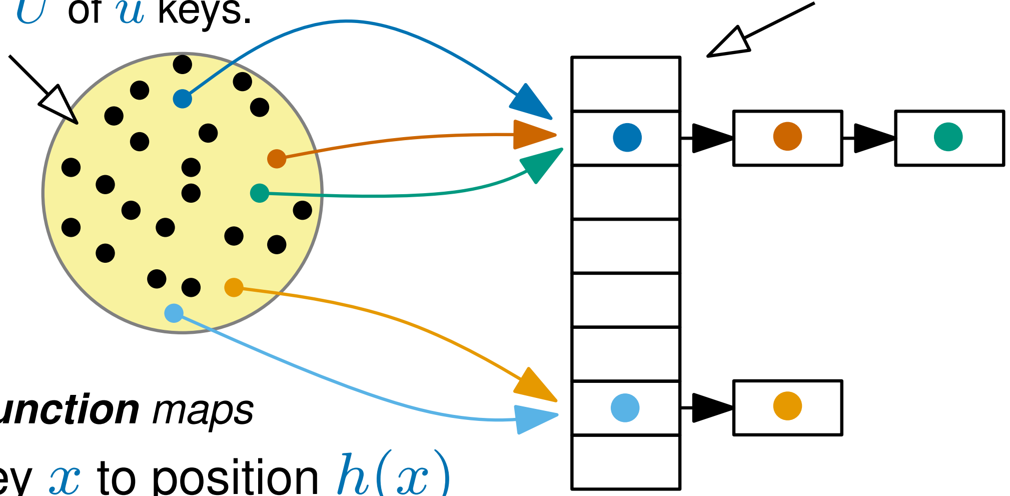

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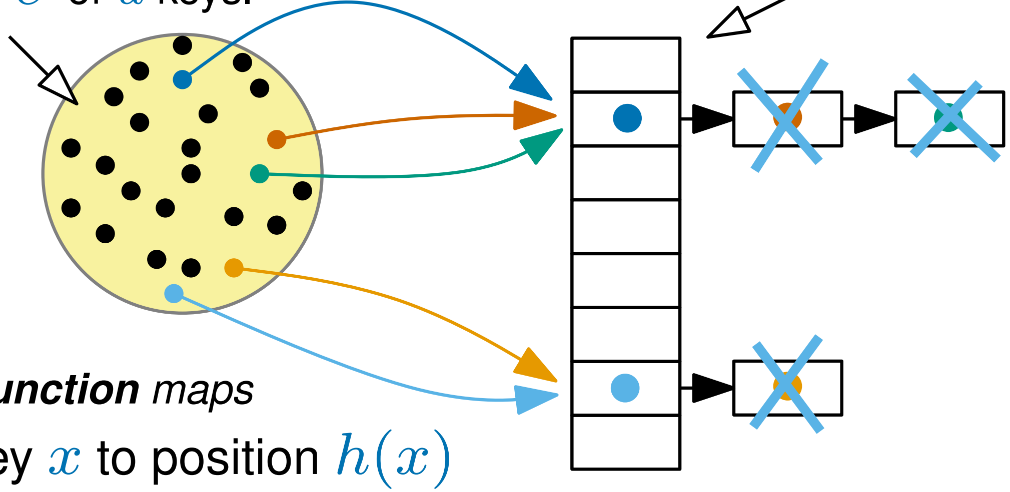

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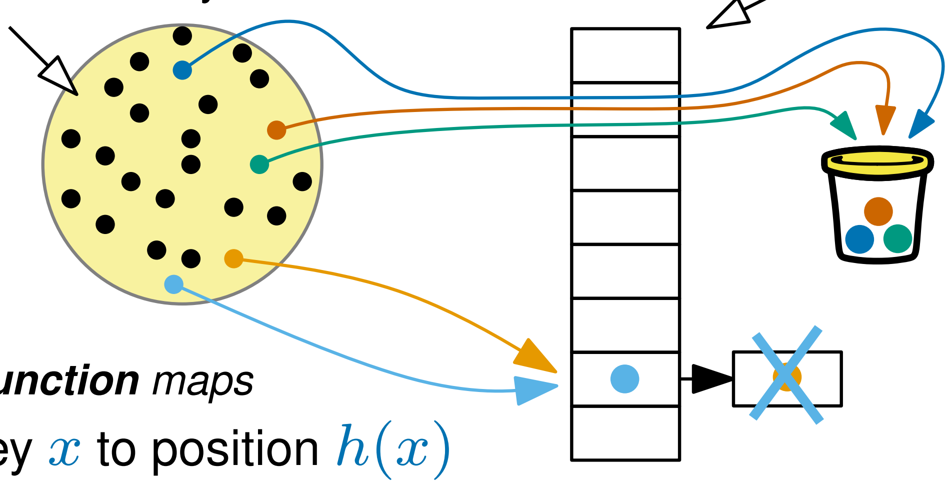

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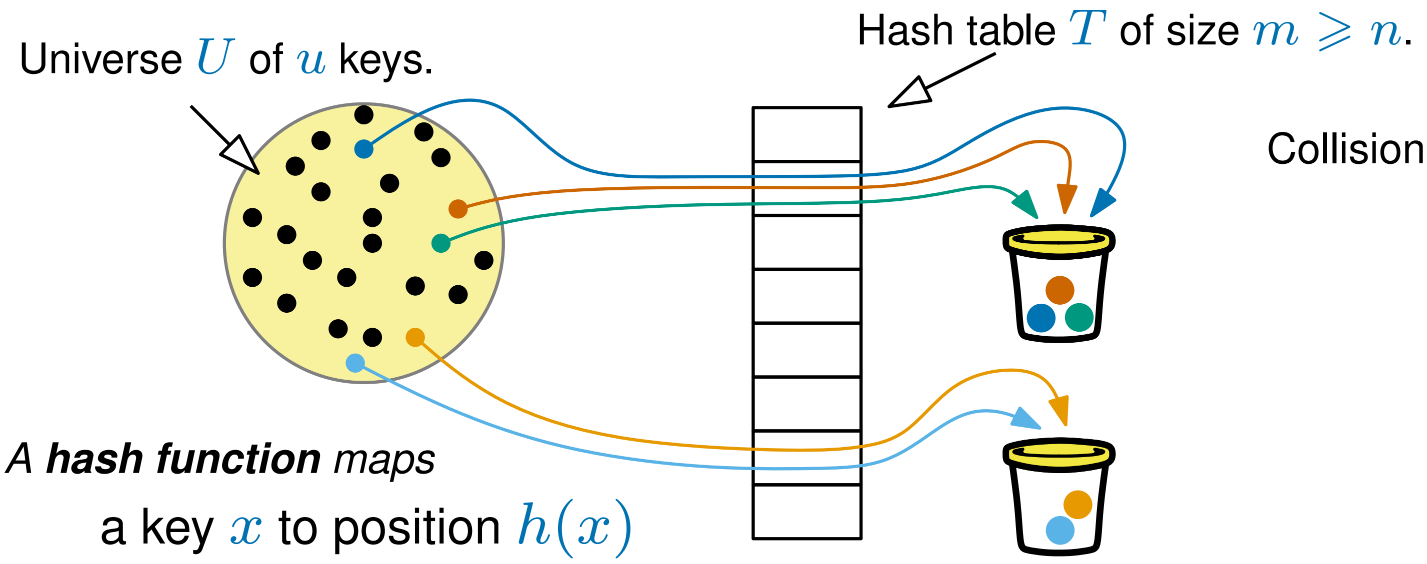

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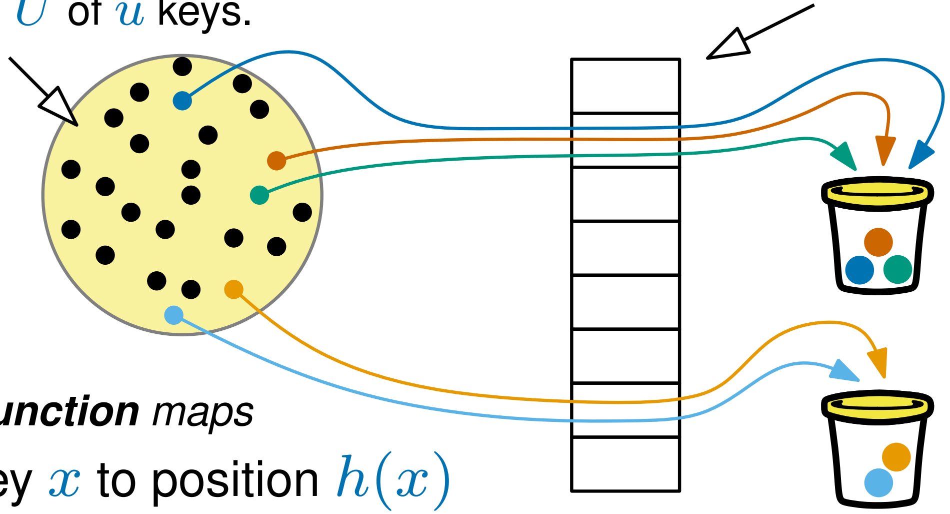

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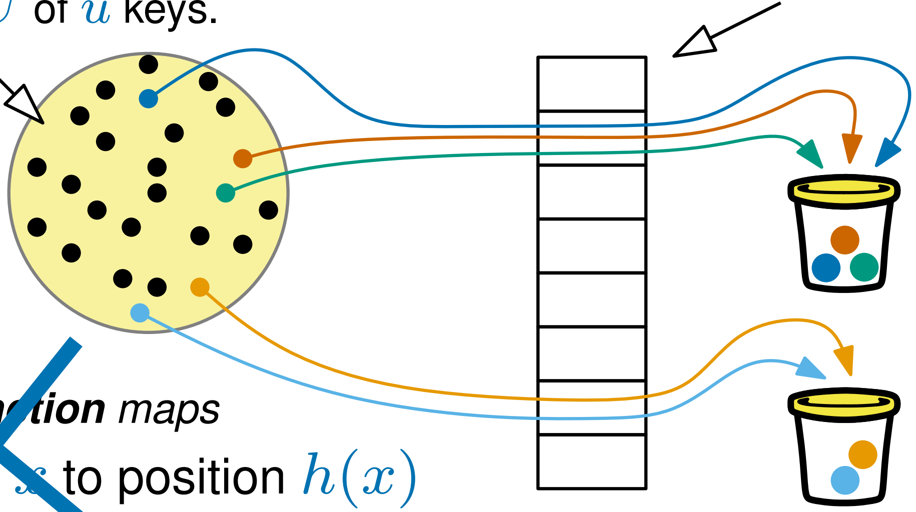

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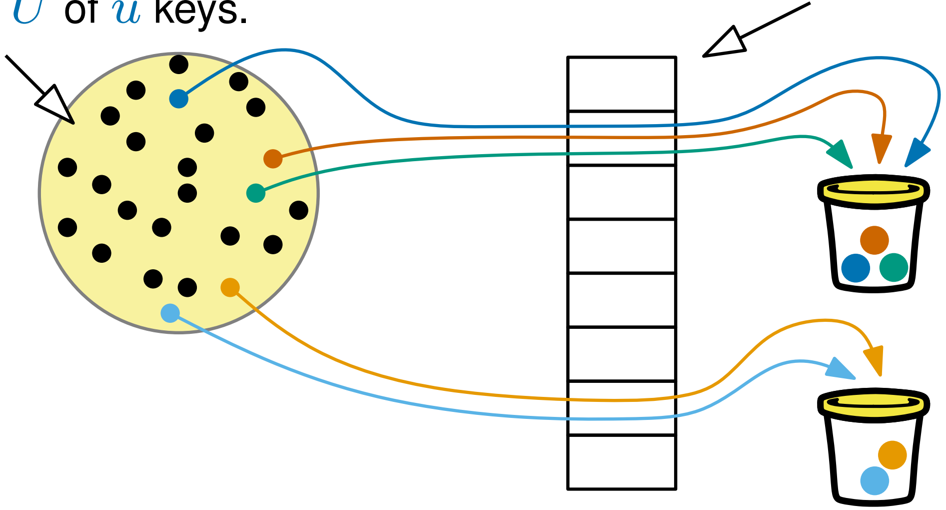

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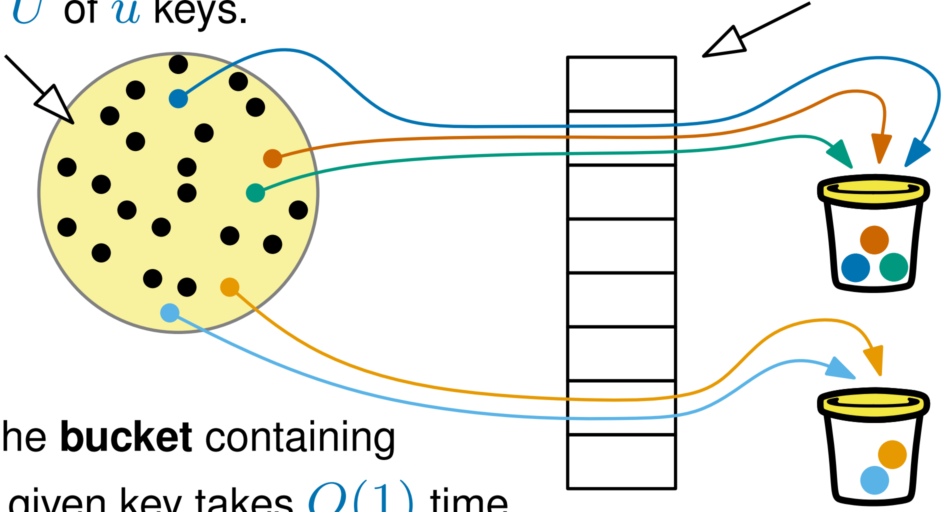

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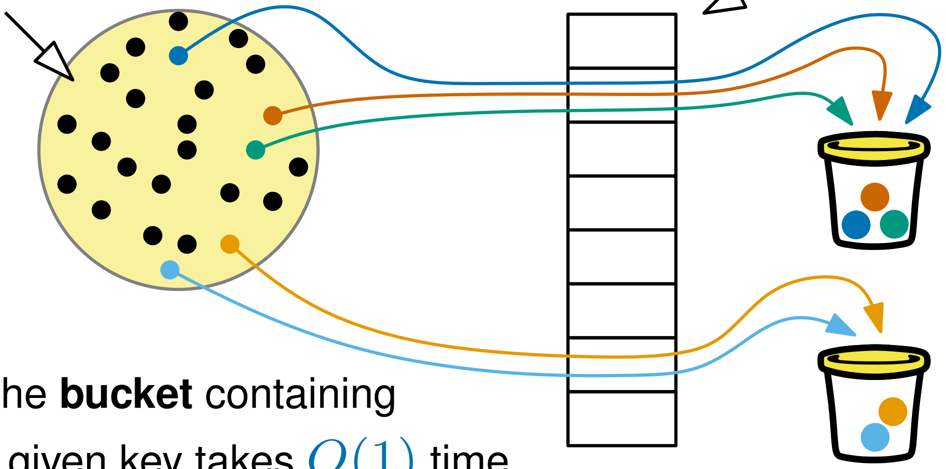

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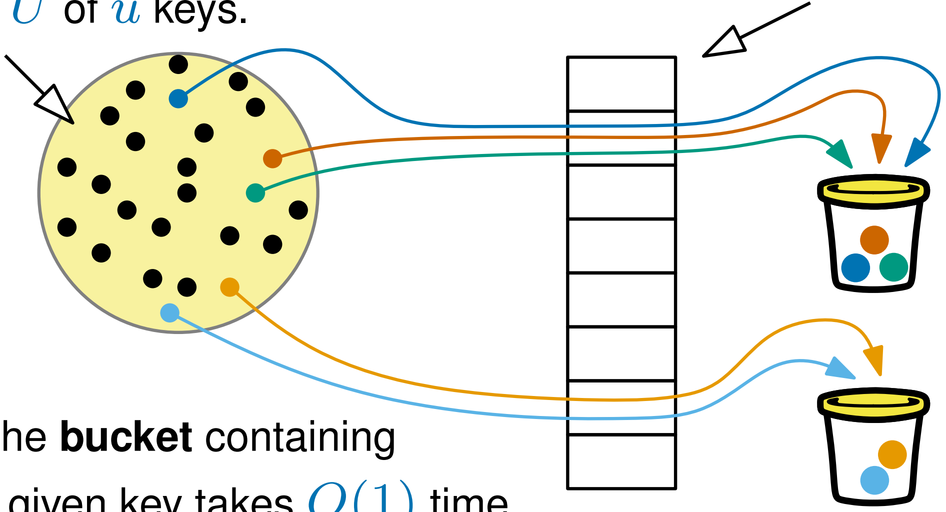

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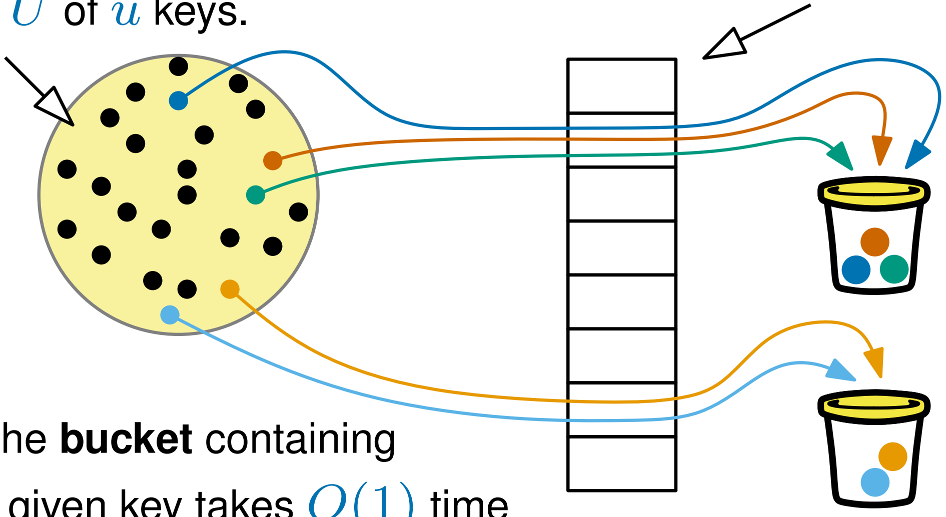

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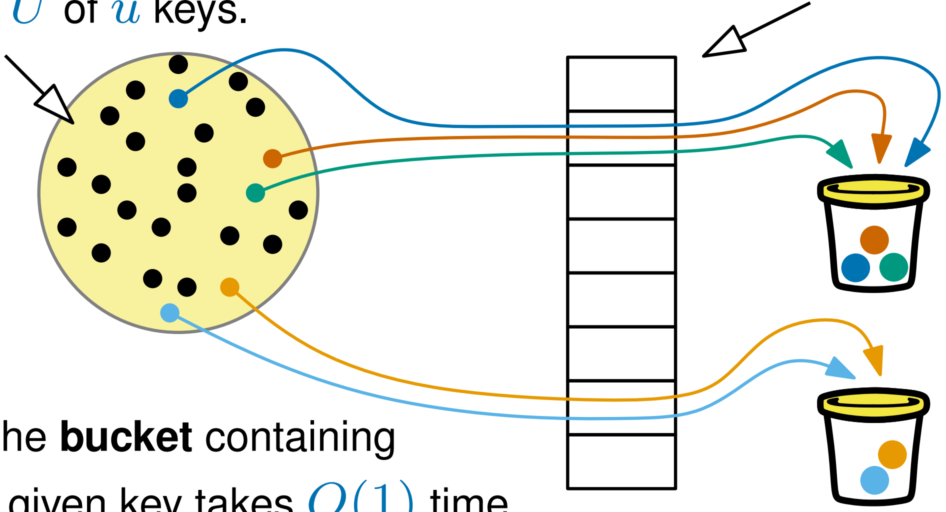

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In the **Cuckoo hashing** scheme:

- Every lookup and every delete takes $O(1)$ *worst-case* time,
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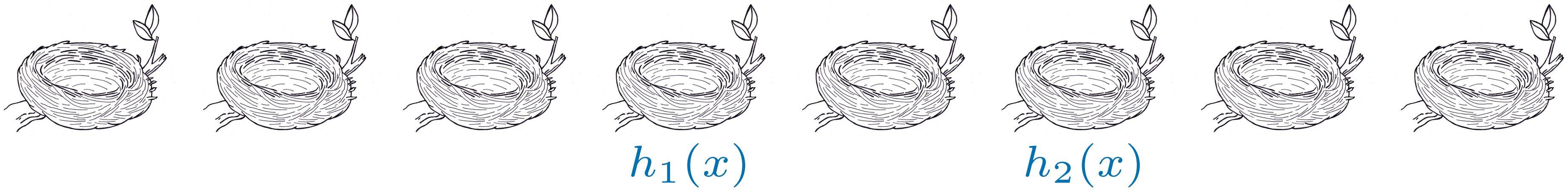

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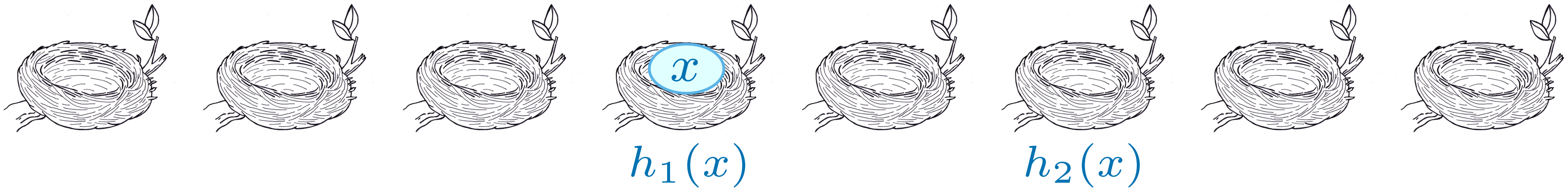

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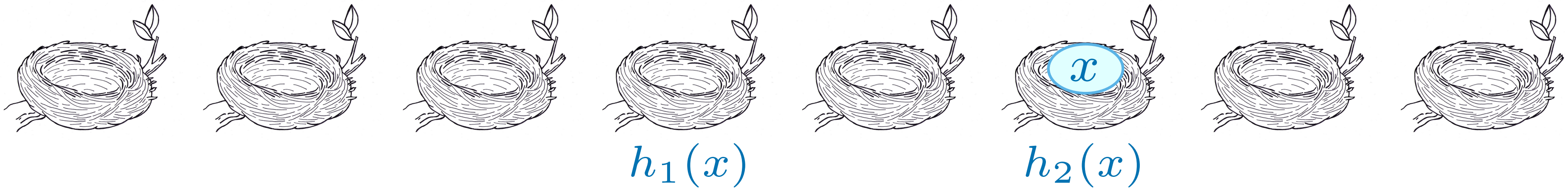

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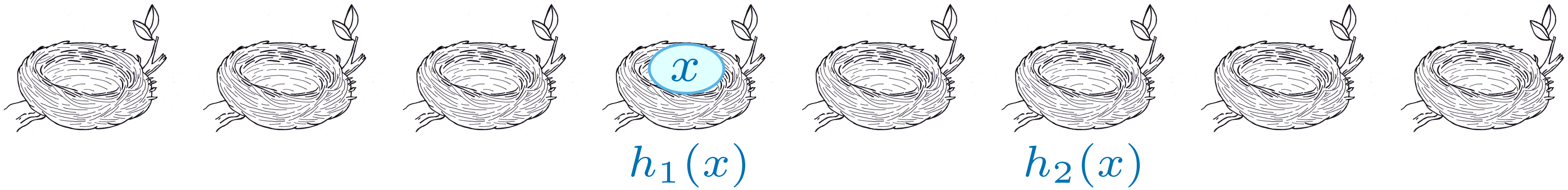

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- Every lookup and every delete takes $O(1)$ *worst-case* time,
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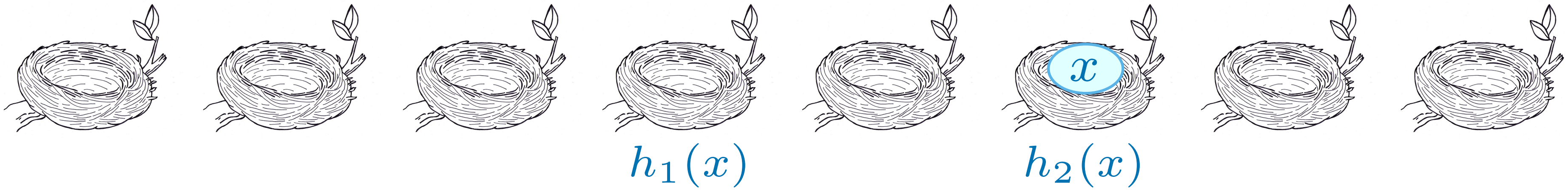

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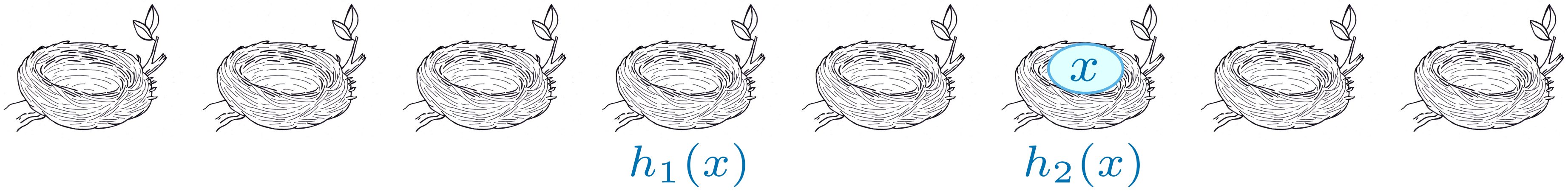

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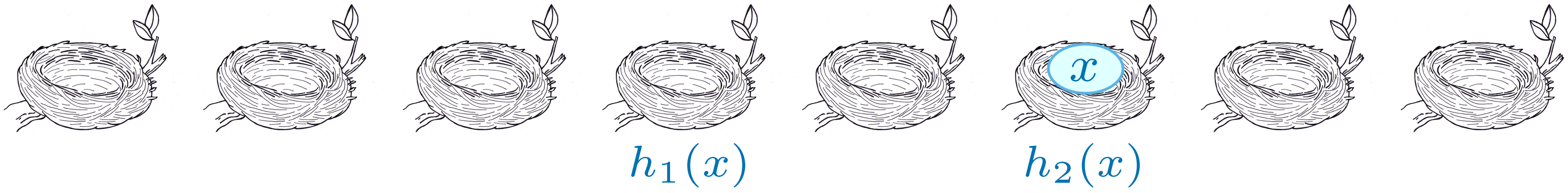

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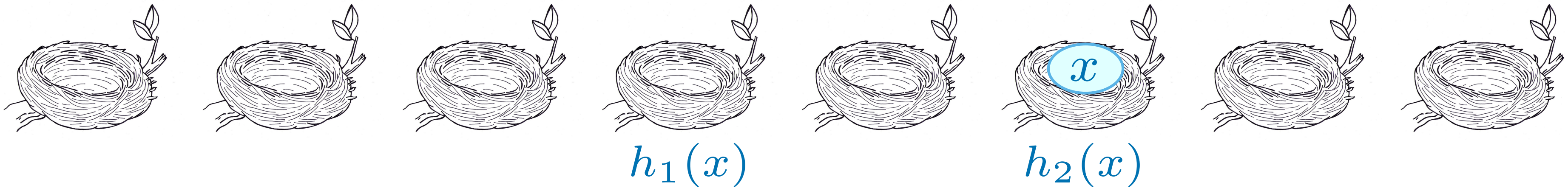
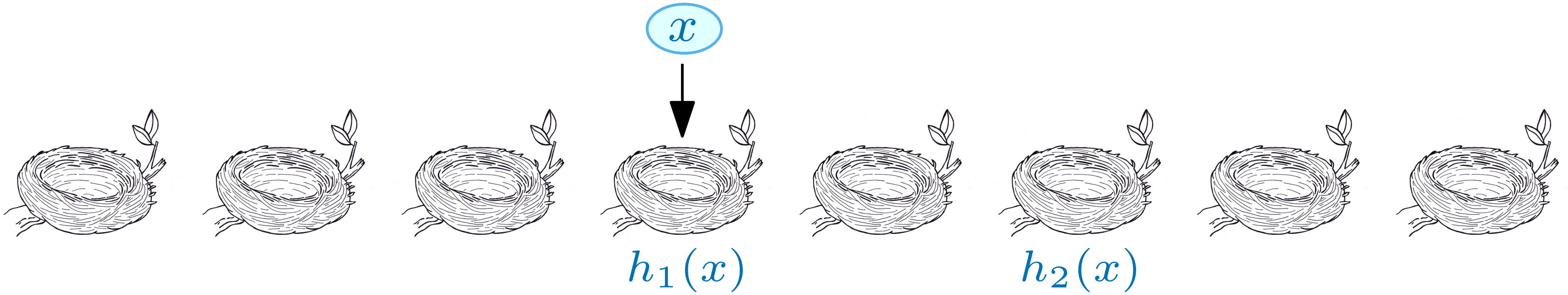


Therefore, as claimed, lookup *takes* $O(1)$ *time… but how do we do inserts?*

Inserts in Cuckoo hashing



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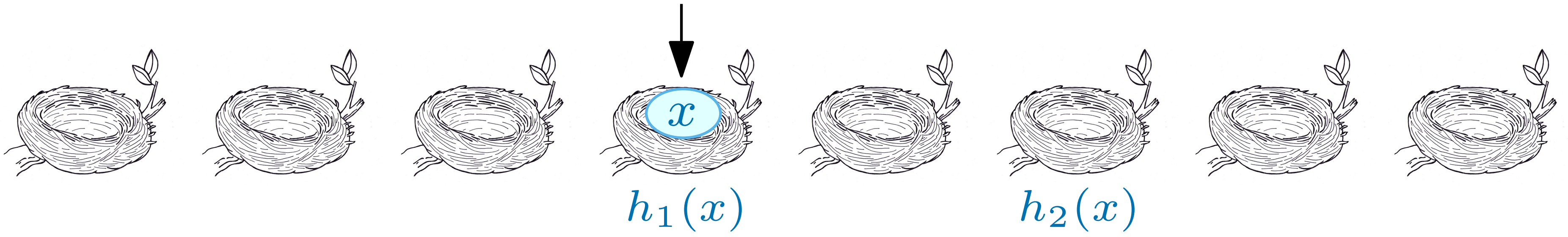

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if that position is empty, stop (and congratulate yourself on a job well done)

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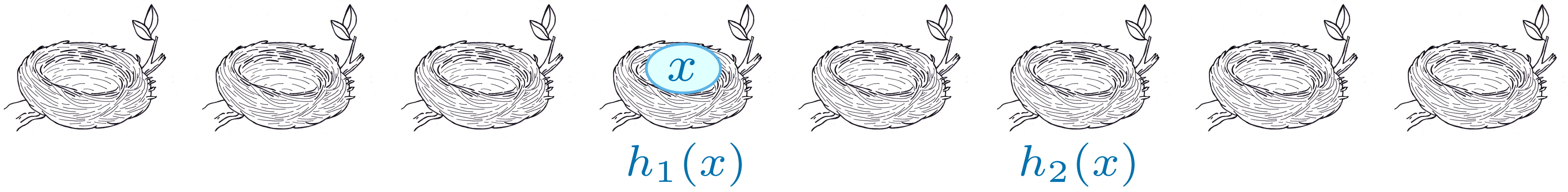

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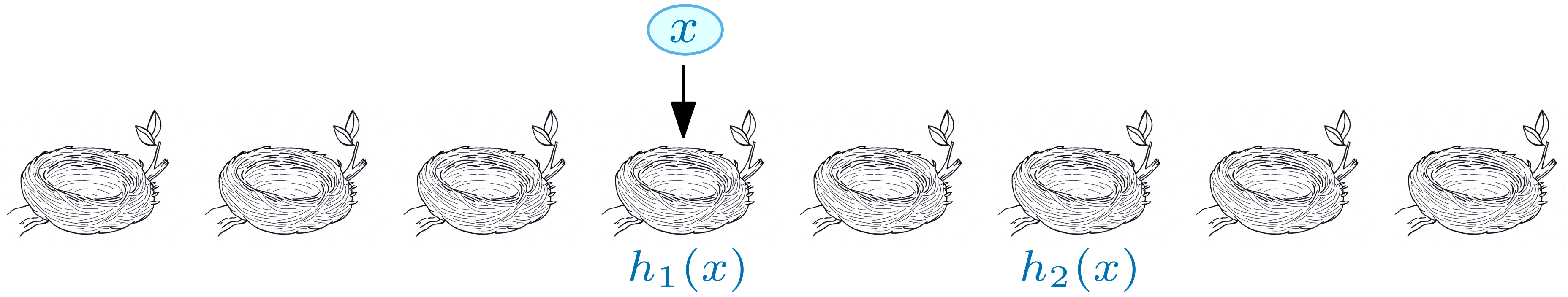

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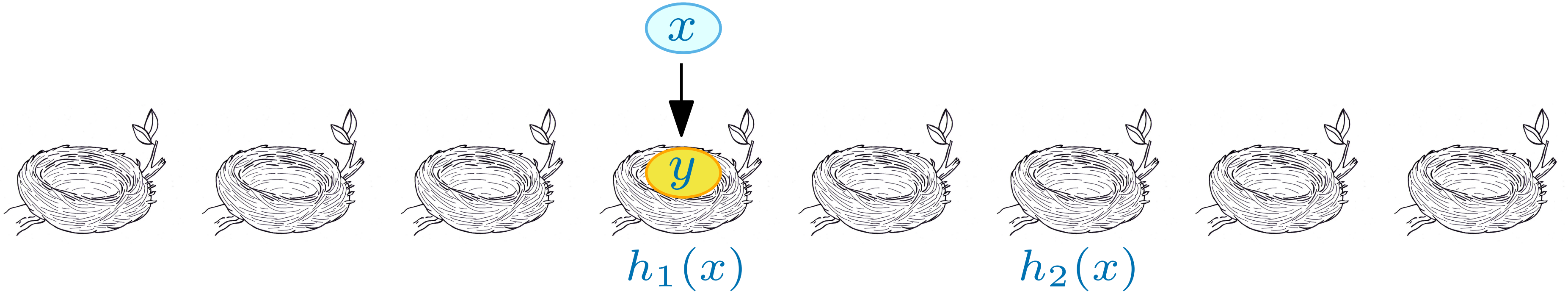

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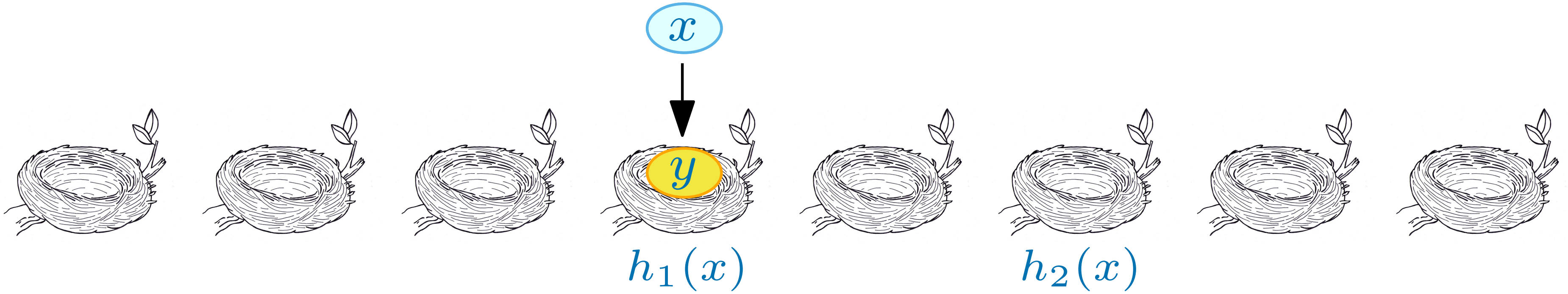

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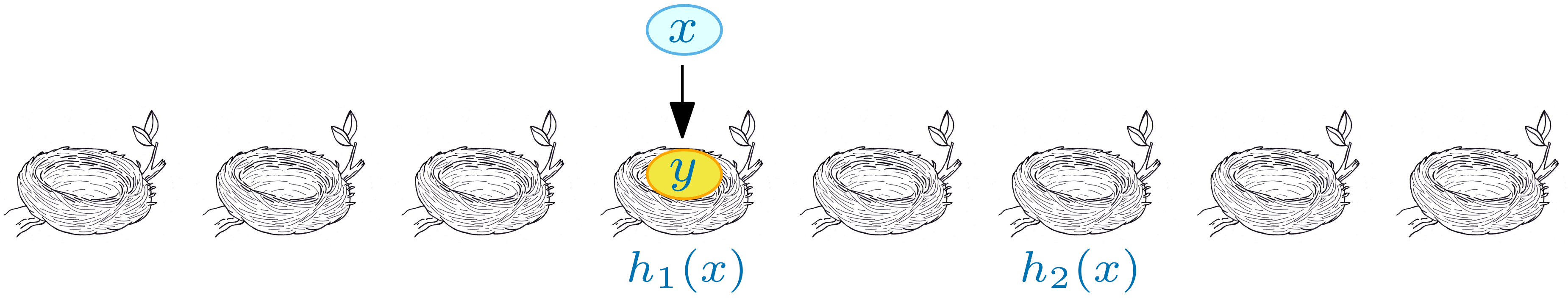

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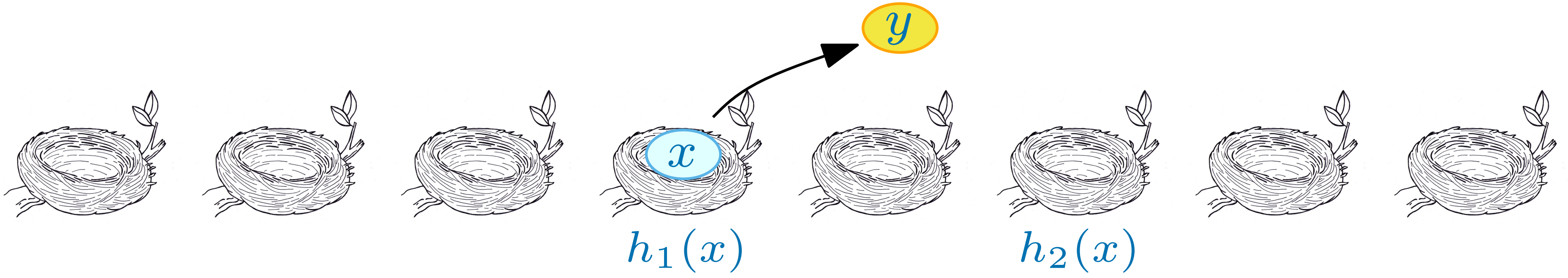

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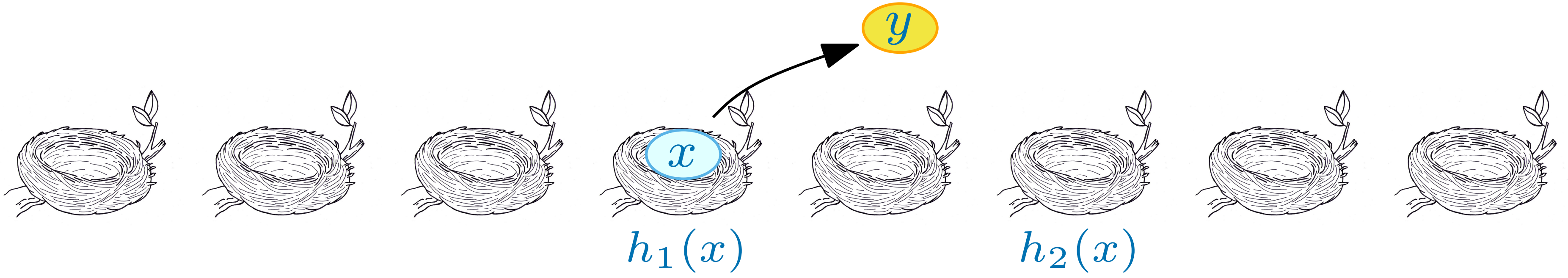

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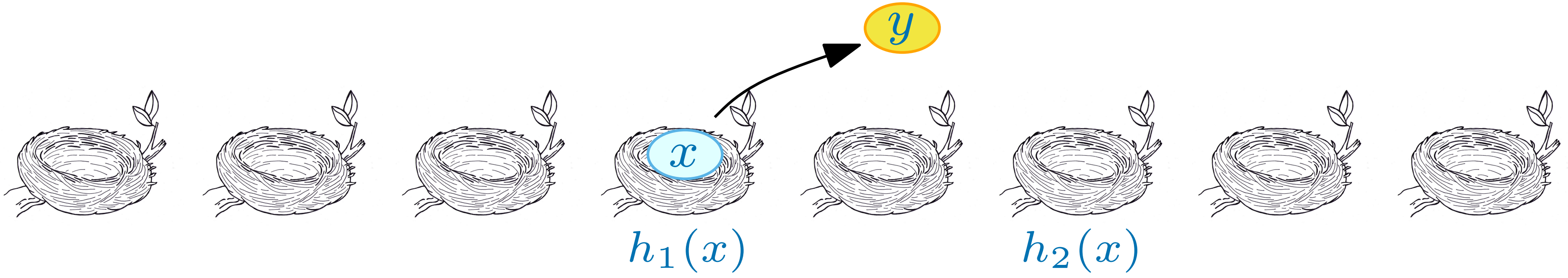

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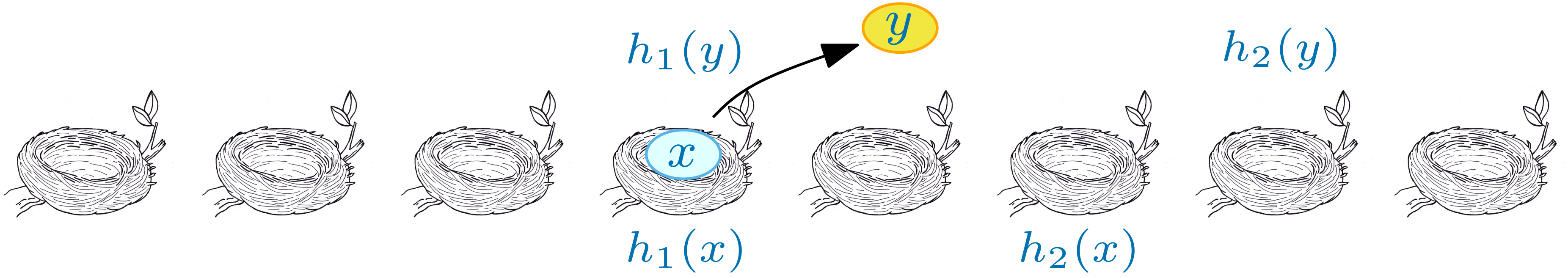

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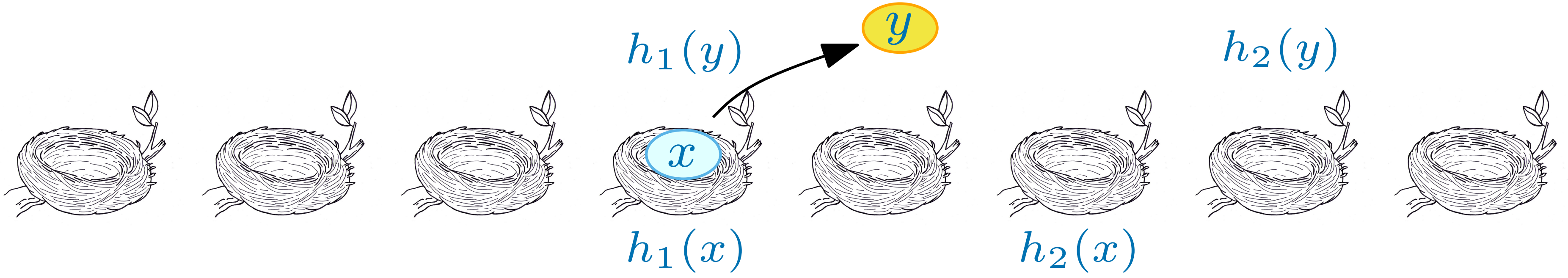

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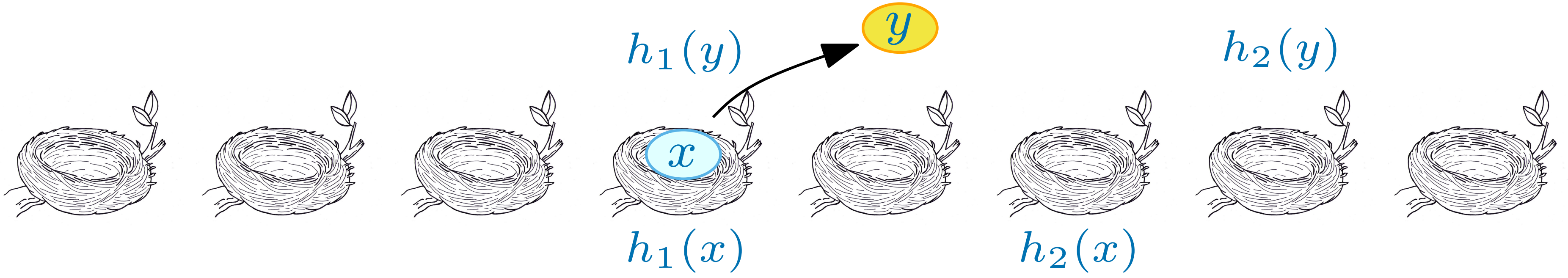

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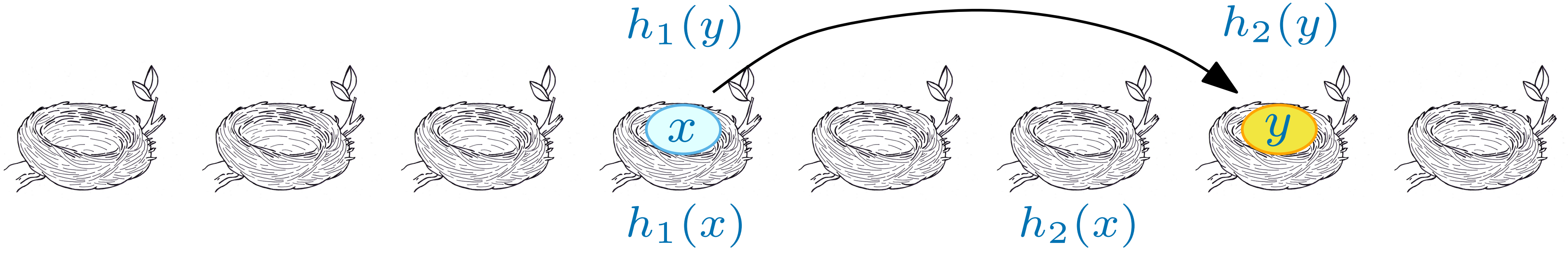

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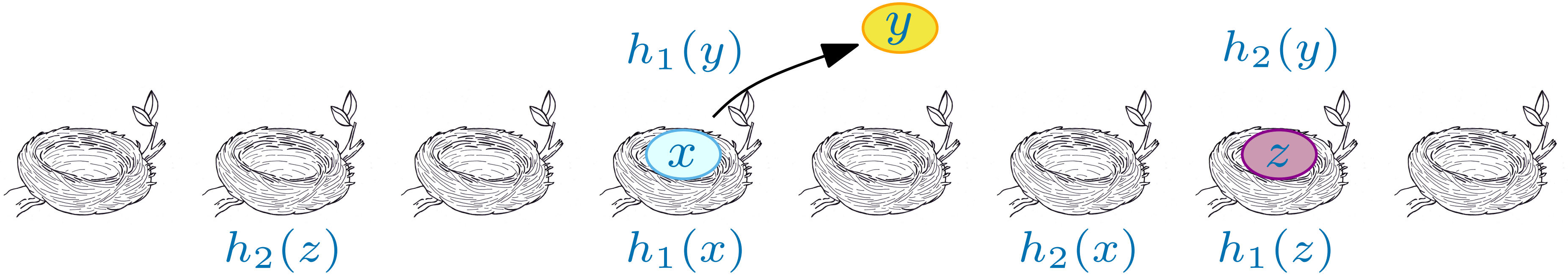

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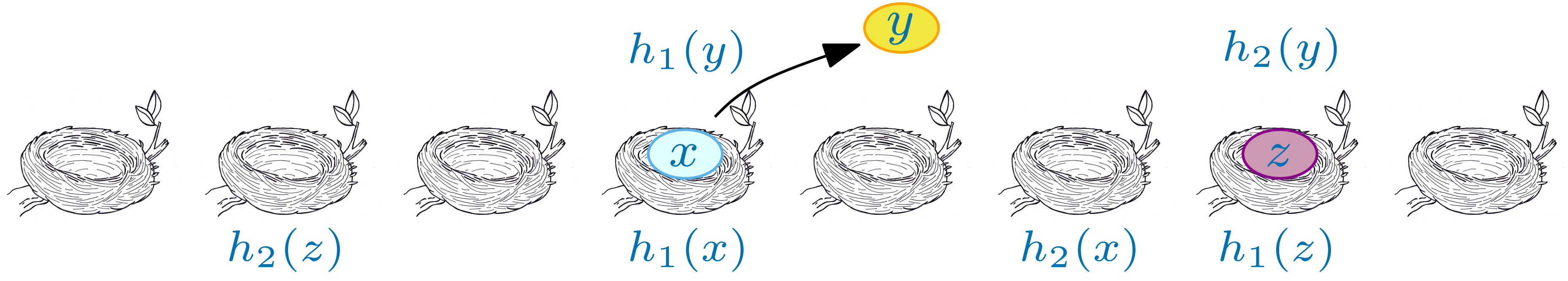

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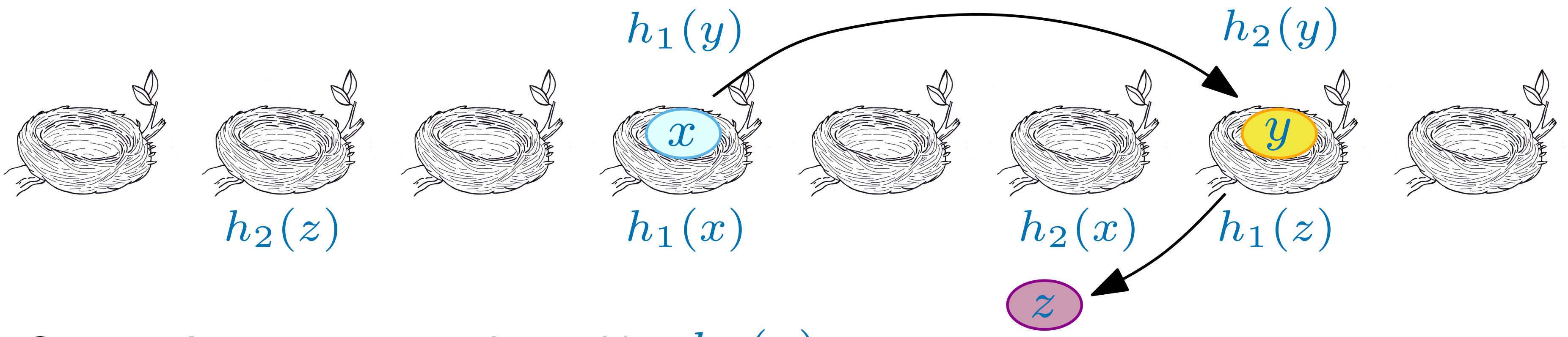

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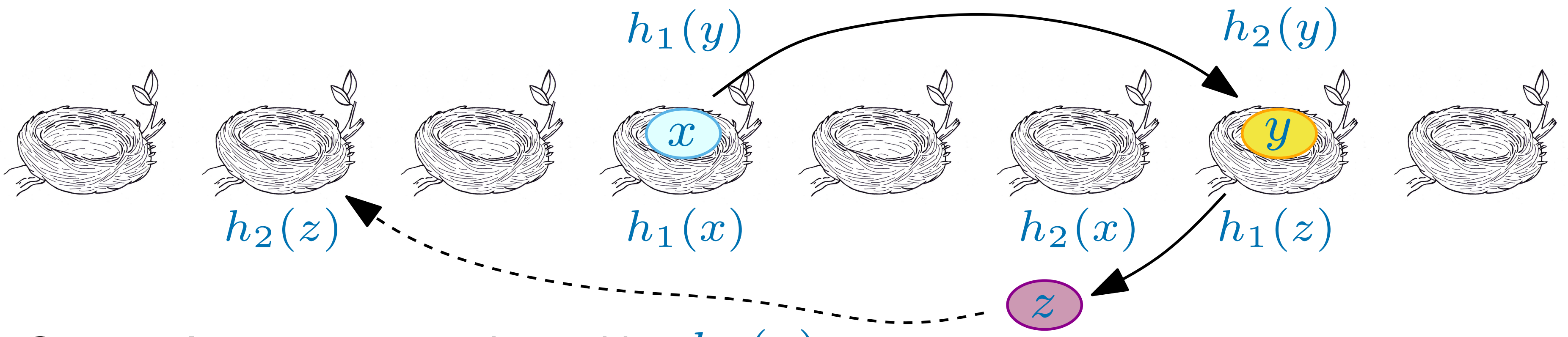

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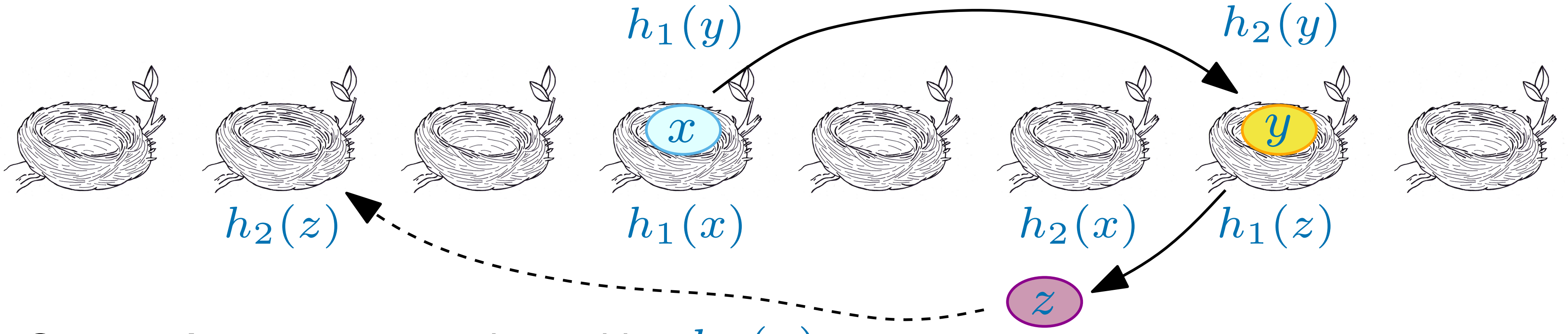

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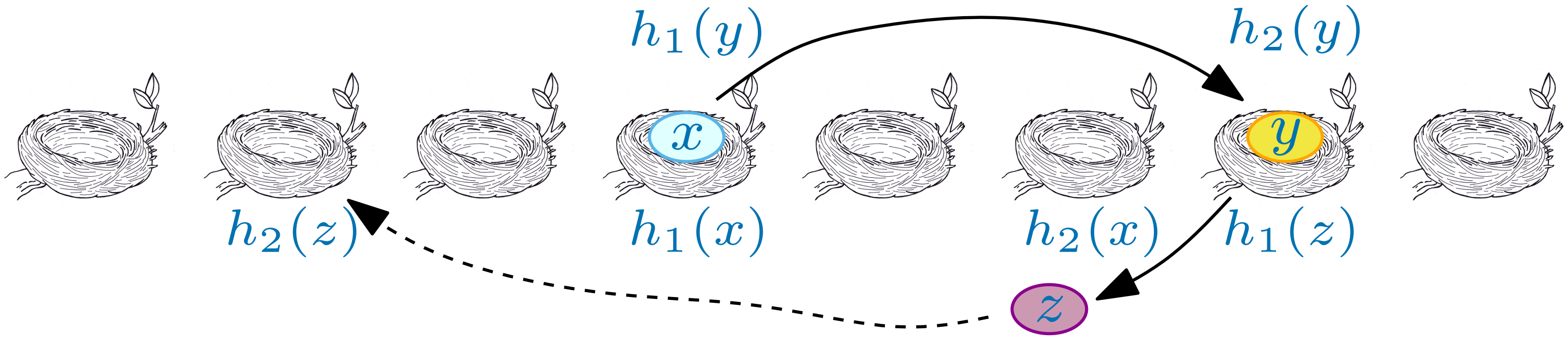
and so on...

Pseudocode

```
add($x$):
  pos ← $h_1(x)$
  Repeat at most $n$ times:
    If $T[\text{pos}]$ is empty then $T[\text{pos}] \leftarrow x$.
    Otherwise,
      $y \leftarrow T[\text{pos}]$,
      $T[\text{pos}] \leftarrow x$,
      pos ← the other possible location for $y$.
        (i.e. if $y$ was evicted from $h_1(y)$ then pos ← $h_2(y)$, otherwise pos ← $h_1(y)$.)
    $x \leftarrow y$.
  Repeat
  Give up and rehash the whole table.
      i.e. empty the table, pick two new hash functions and reinsert every key
```



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This is rather slow. . . but we will prove that it happens rarely

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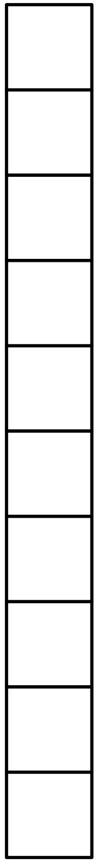
NOT ACTUALLY AN ASSUMPTION

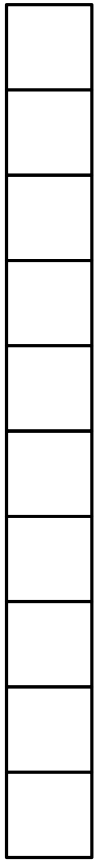
Cuckoo graph

Hash table
(size m)



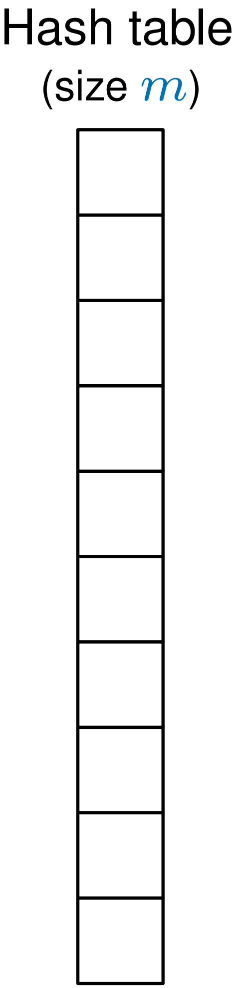
Cuckoo graph

Hash table
(size m)



The **cuckoo graph**:

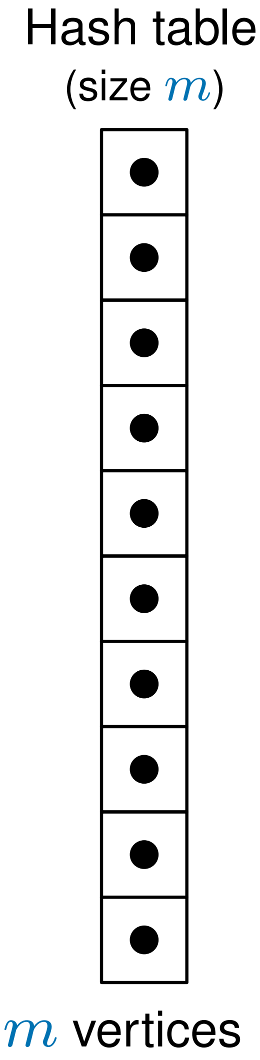
Cuckoo graph



The **cuckoo graph**:

A vertex for each position of the table.

Cuckoo graph



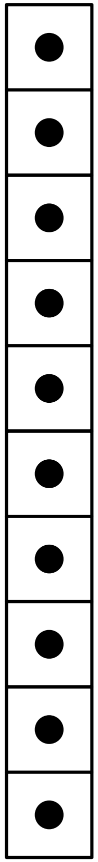
The **cuckoo graph**:

A vertex for each position of the table.

Cuckoo graph

Hash table

(size m)



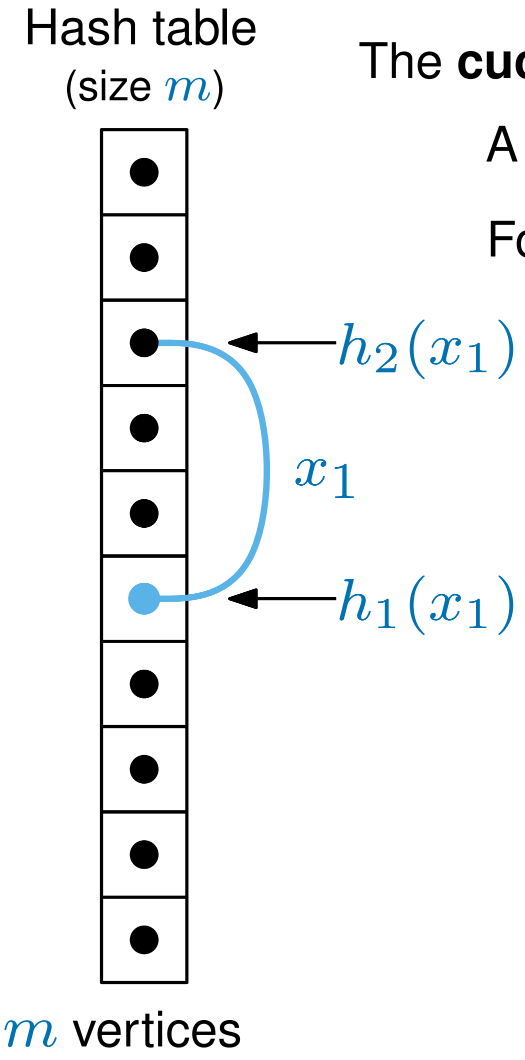
m vertices

The **cuckoo graph**:

A vertex for each position of the table.

For each key x there is an undirected edge between $h_1(x)$ and $h_2(x)$.

Cuckoo graph



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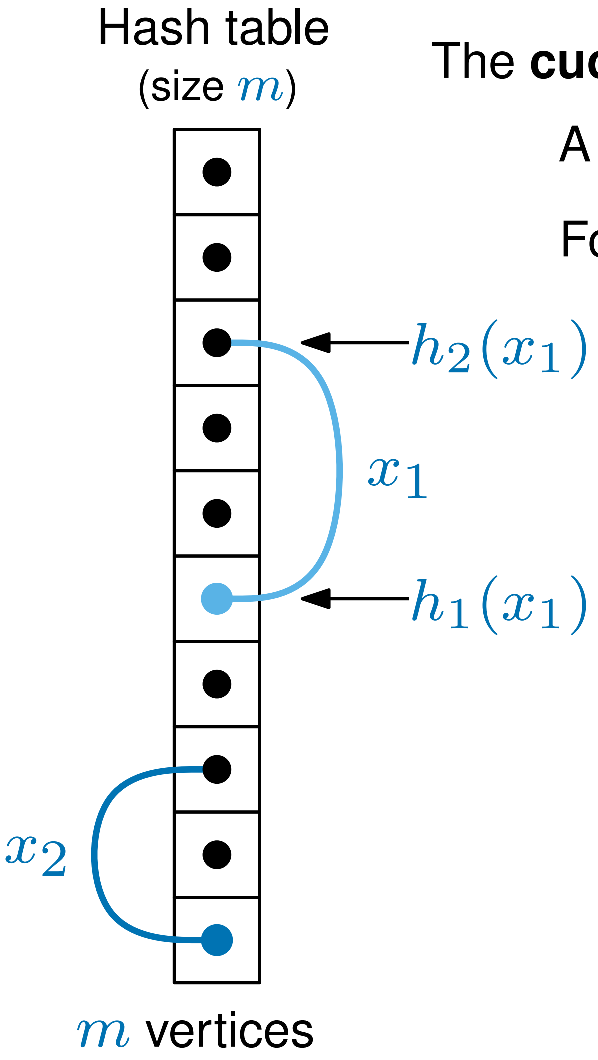

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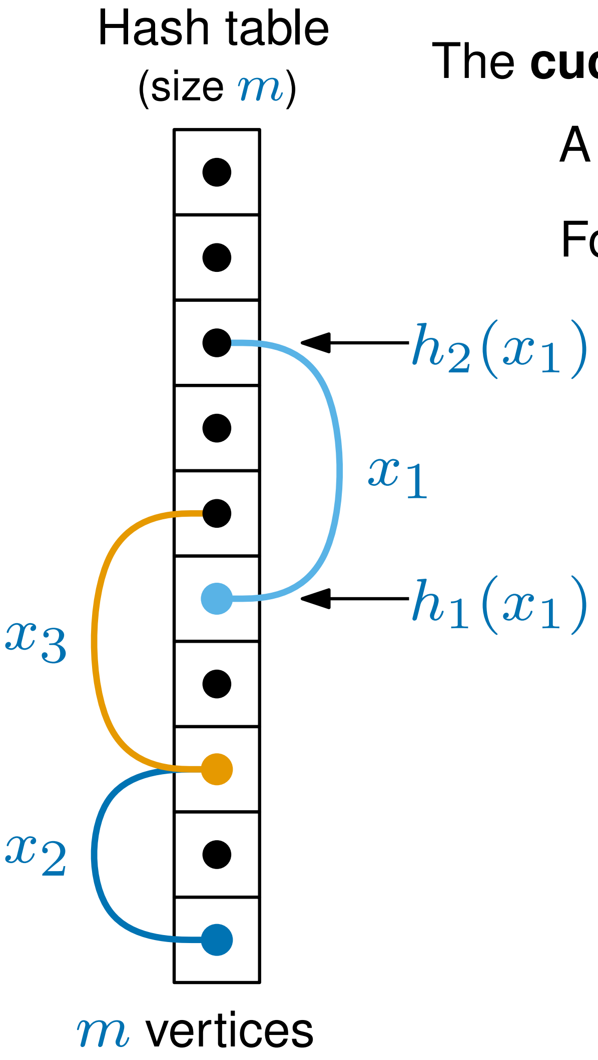
Cuckoo graph



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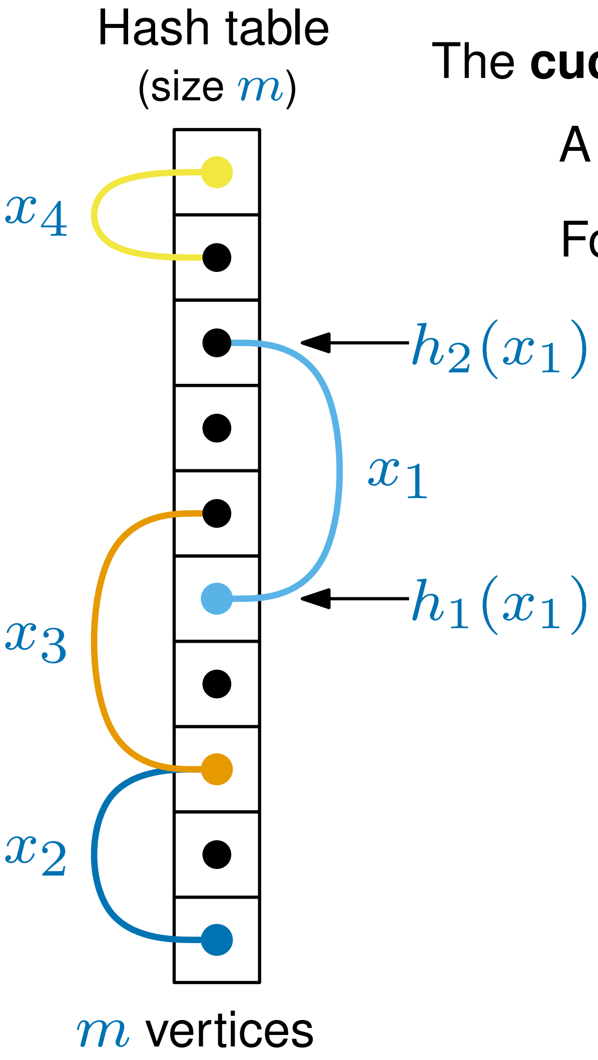

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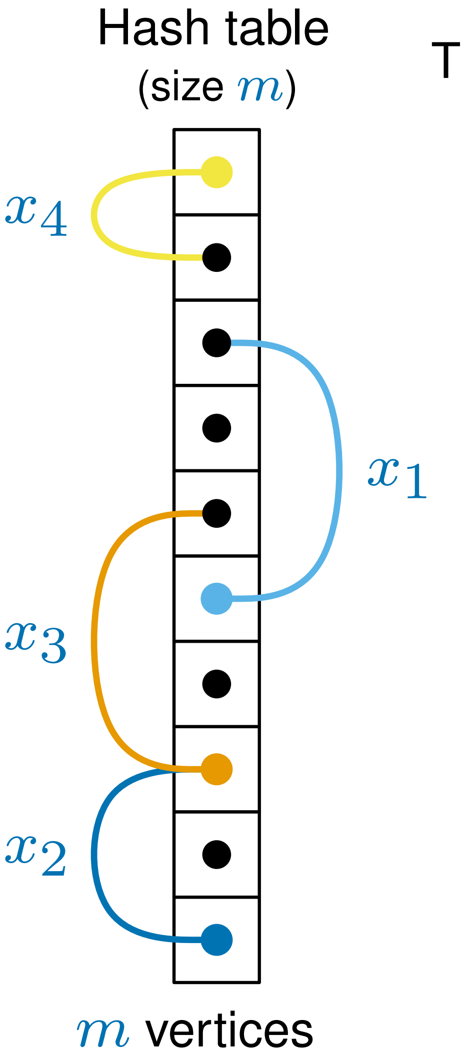

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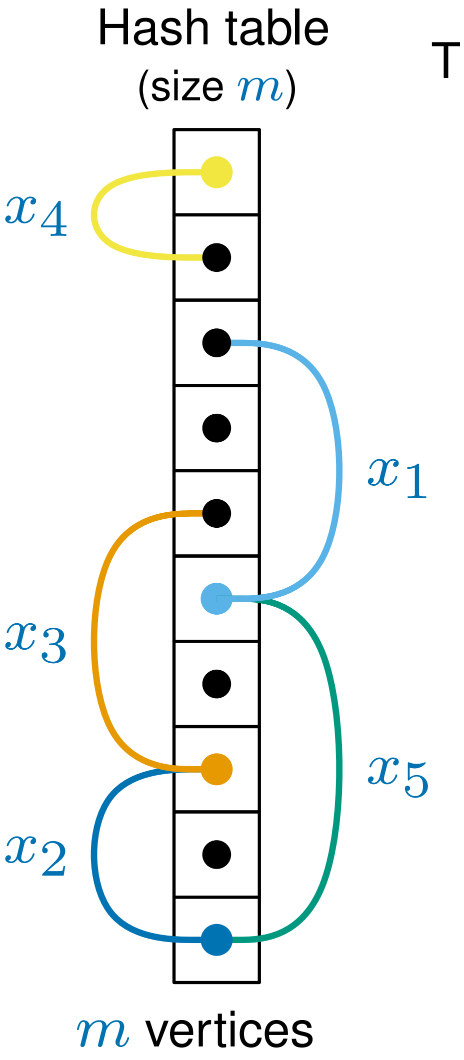

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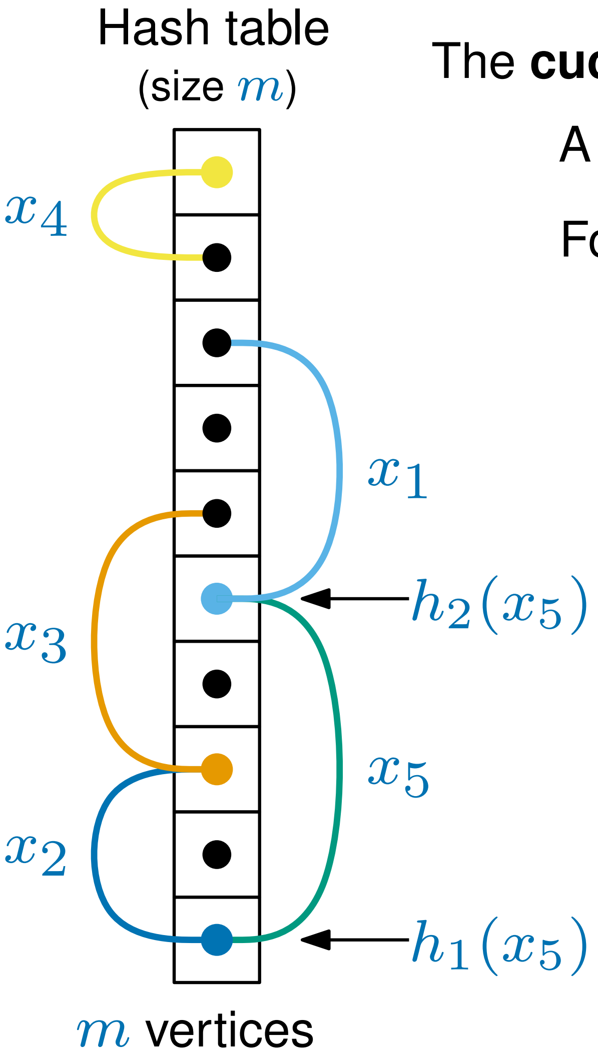

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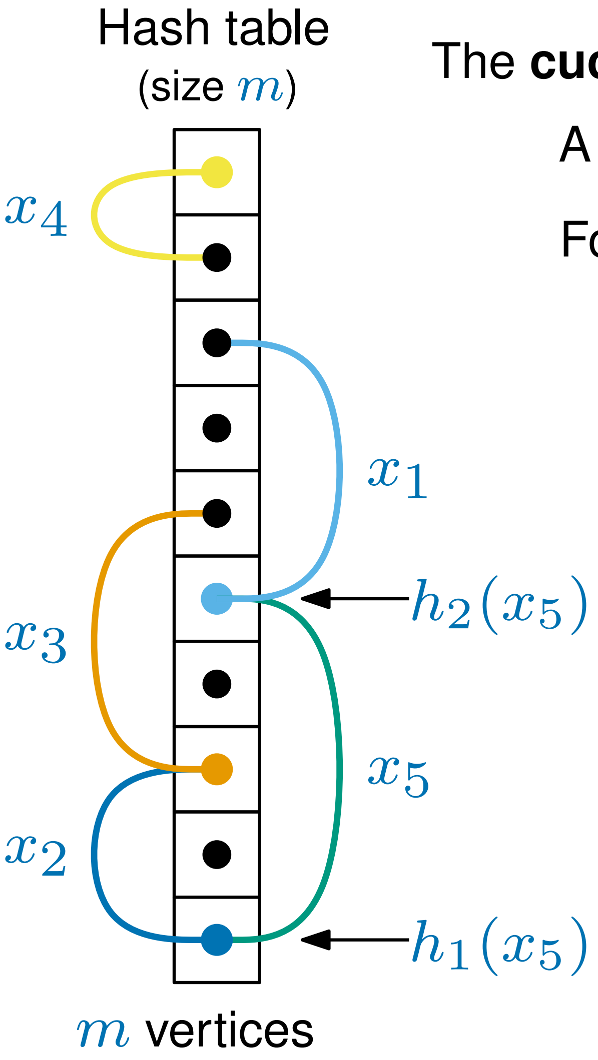
Cuckoo graph



The **cuckoo graph**:

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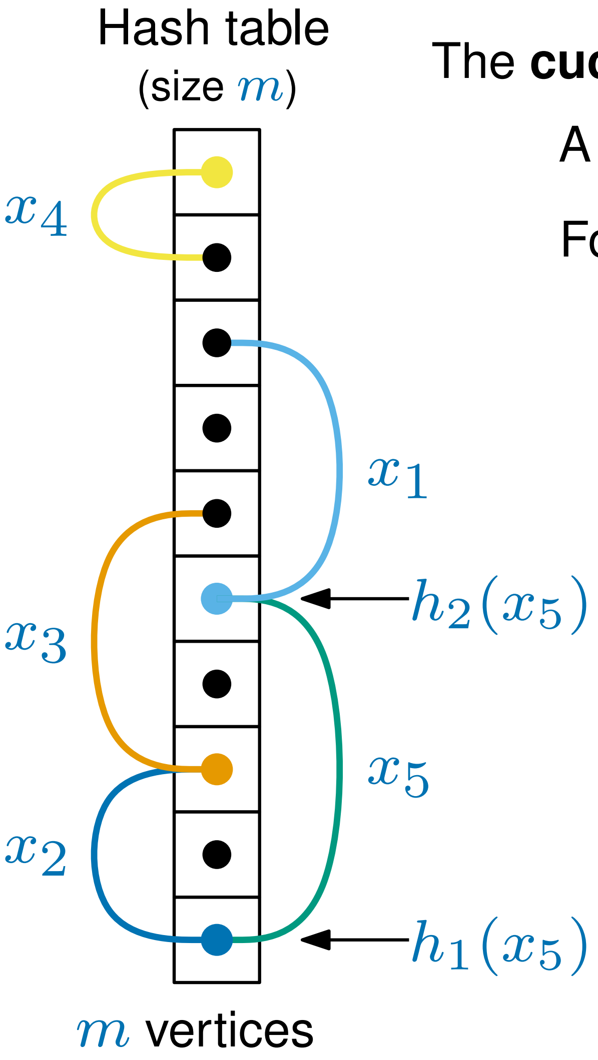
For each key x there is an undirected edge between $h_1(x)$ and $h_2(x)$.

There is no space for $x_5\ldots$

Cuckoo graph



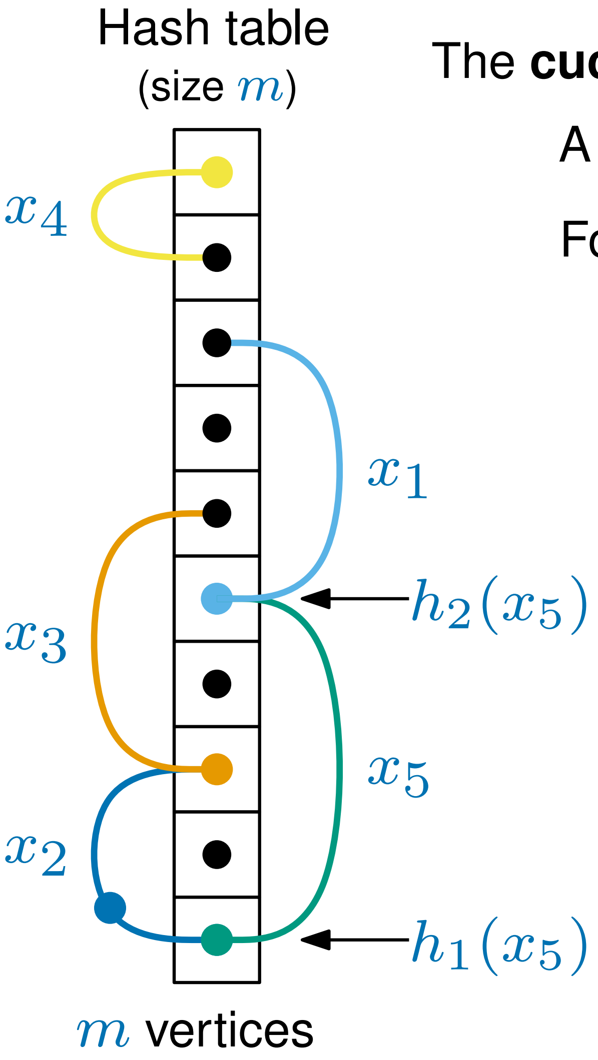
The **cuckoo graph**:

A vertex for each position of the table.

For each key x there is an undirected edge between $h_1(x)$ and $h_2(x)$.

There is no space for $x_5 \ldots$
so we make space
by moving x_2 and then x_3

Cuckoo graph



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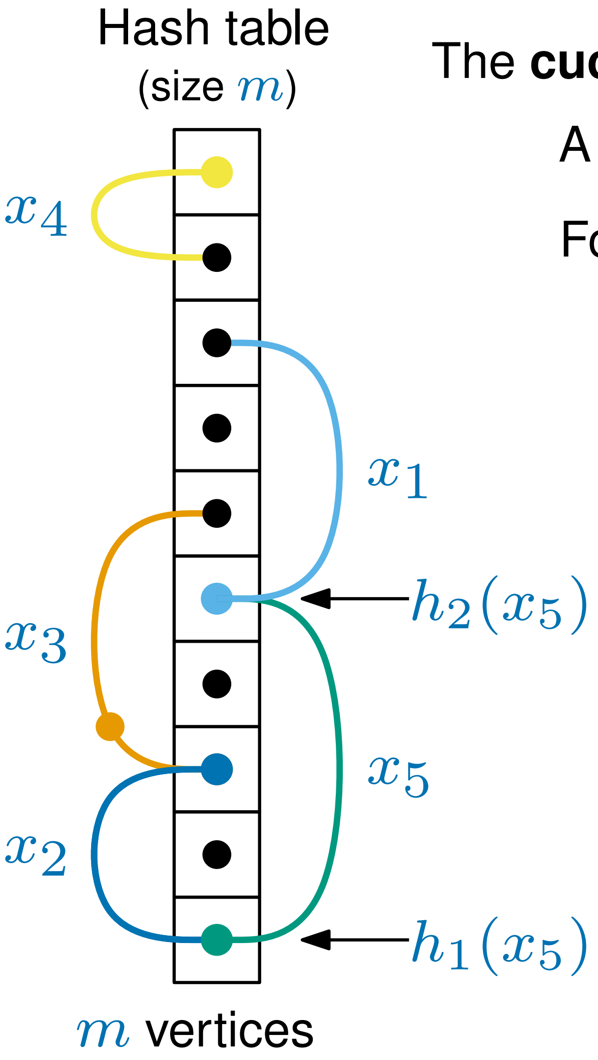

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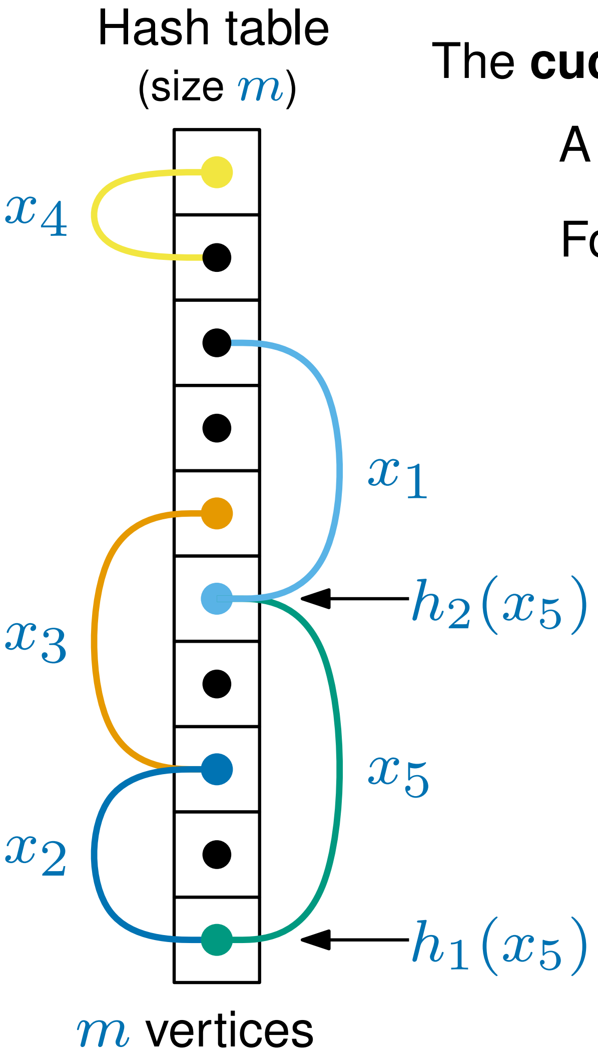

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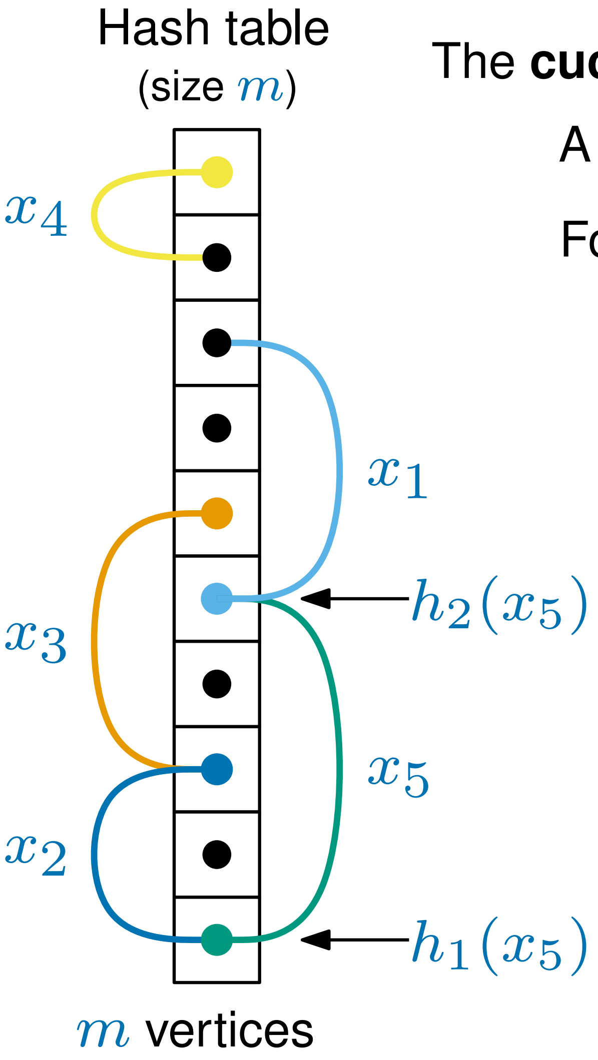

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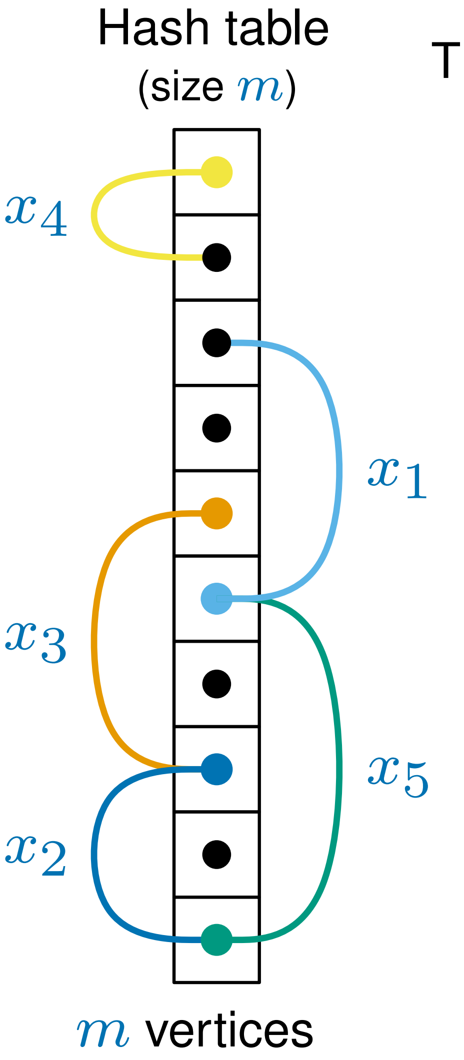
so we make space
by moving x_2 and then x_3

The number of moves performed while adding a key is the length of the corresponding path in the cuckoo graph

Cuckoo graph



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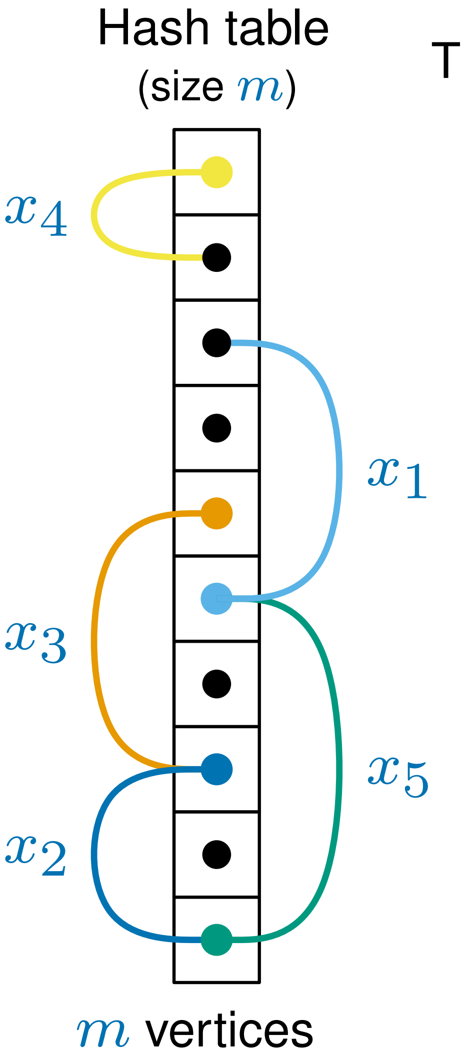

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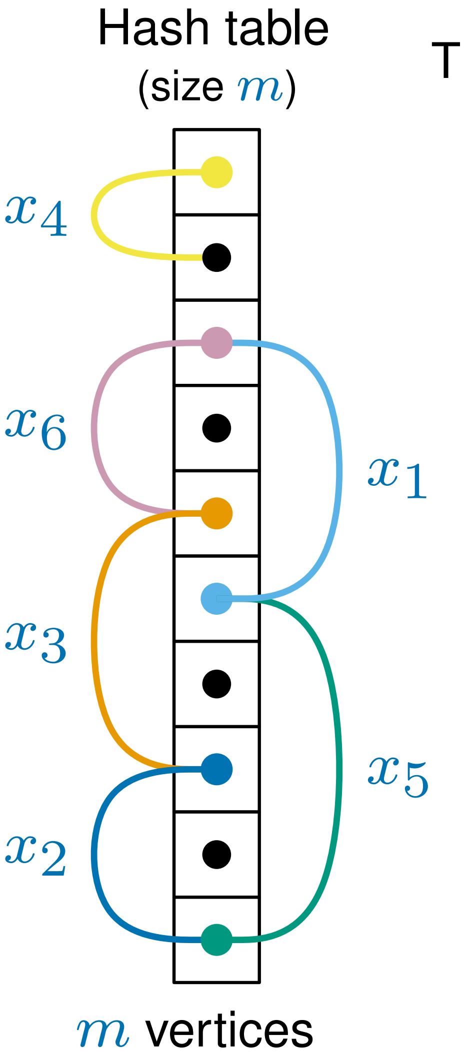

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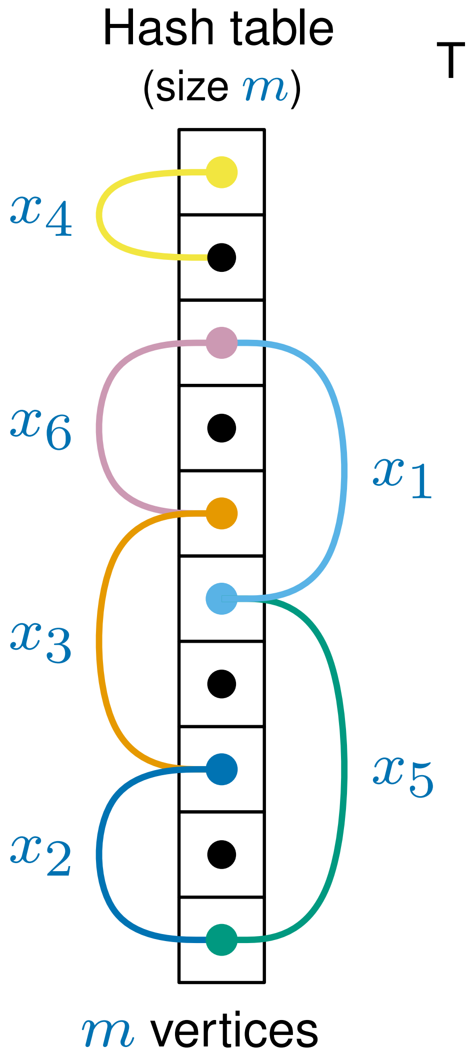
Cuckoo graph



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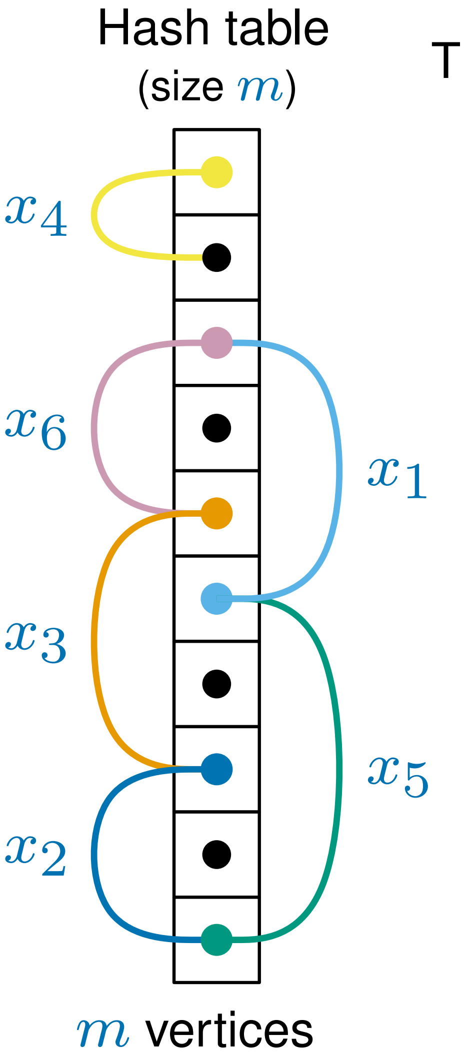
For each key x there is an undirected edge between $h_1(x)$ and $h_2(x)$.

The number of moves performed while adding a key is the length of the corresponding path in the cuckoo graph

Inserting key x_6 creates a cycle.

Cuckoo graph



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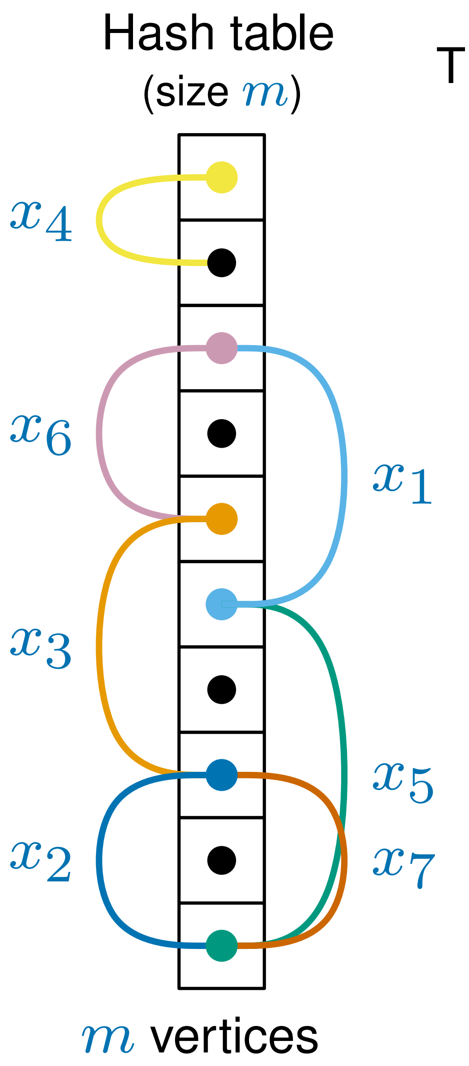
Inserting key x_6 creates a cycle.

Cycles are dangerous. . .

Cuckoo graph



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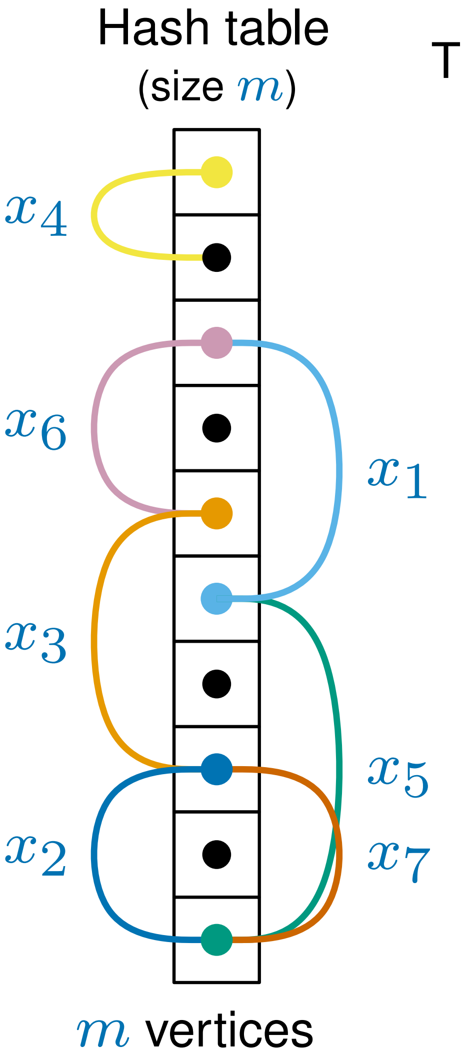

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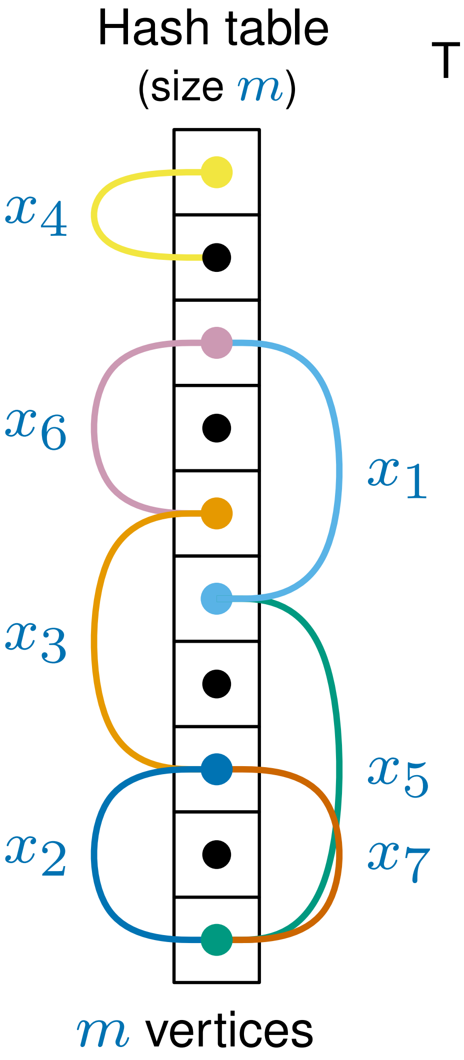
Inserting key x_6 creates a cycle.

Cycles are dangerous. . .

When key x_7 is inserted where does it go?

Cuckoo graph



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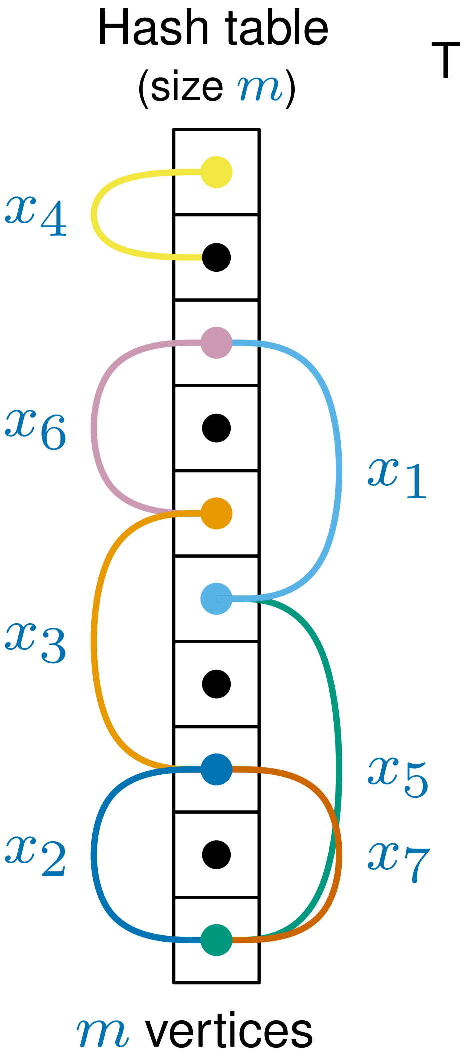
Inserting key x_6 creates a cycle.

Cycles are dangerous. . .

When key x_7 is inserted where does it go?

there are 6 keys but only 5 spaces

Cuckoo graph



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Inserting key x_6 creates a cycle.

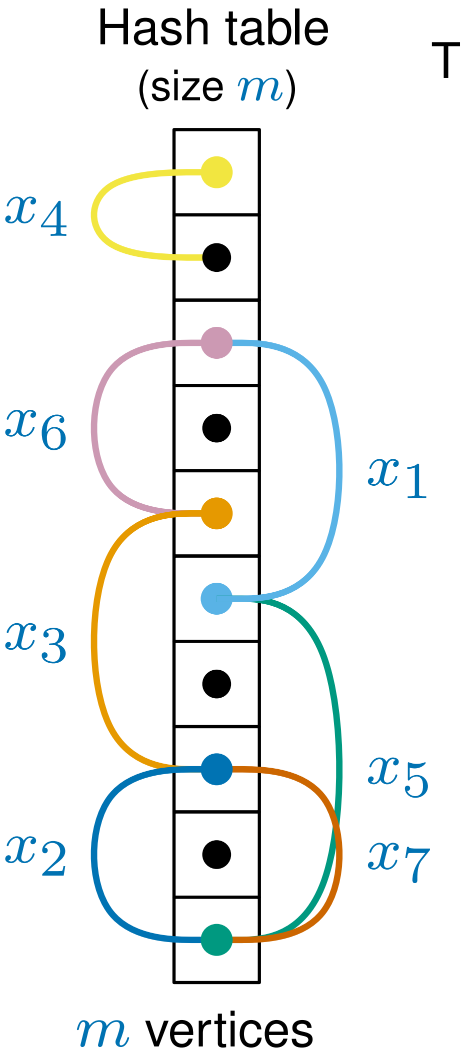
Cycles are dangerous...

When key x_7 is inserted where does it go?

there are 6 keys but only 5 spaces

The keys would be moved around in an infinite loop but we stop and rehash after n moves...

Cuckoo graph



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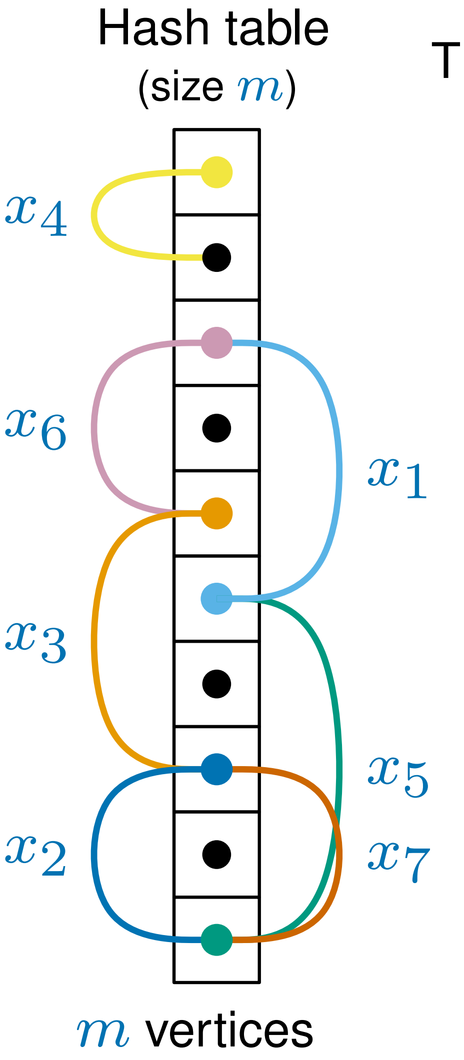
When key x_7 is inserted where does it go?

there are 6 keys but only 5 spaces

The keys would be moved around in an infinite loop but we stop and rehash after n moves. . .

Inserting a key into a cycle **always** causes a rehash

Cuckoo graph



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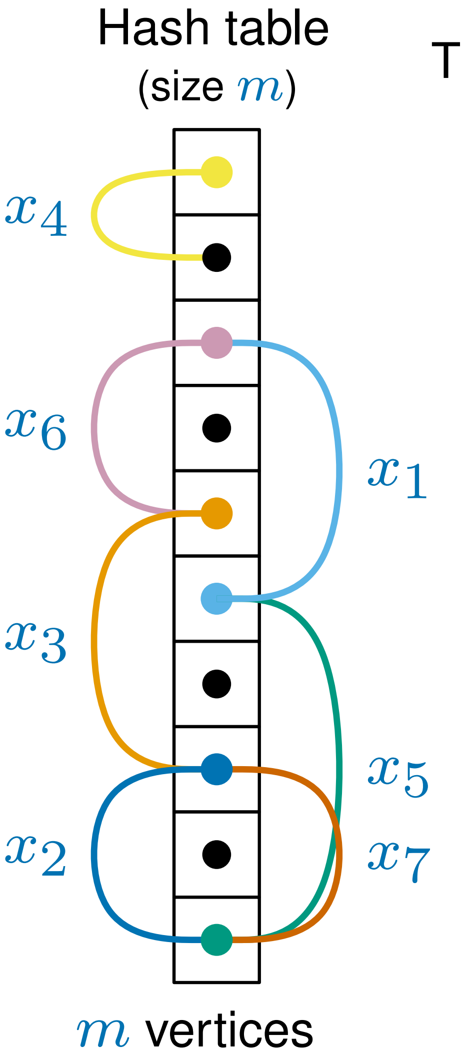
Cuckoo graph



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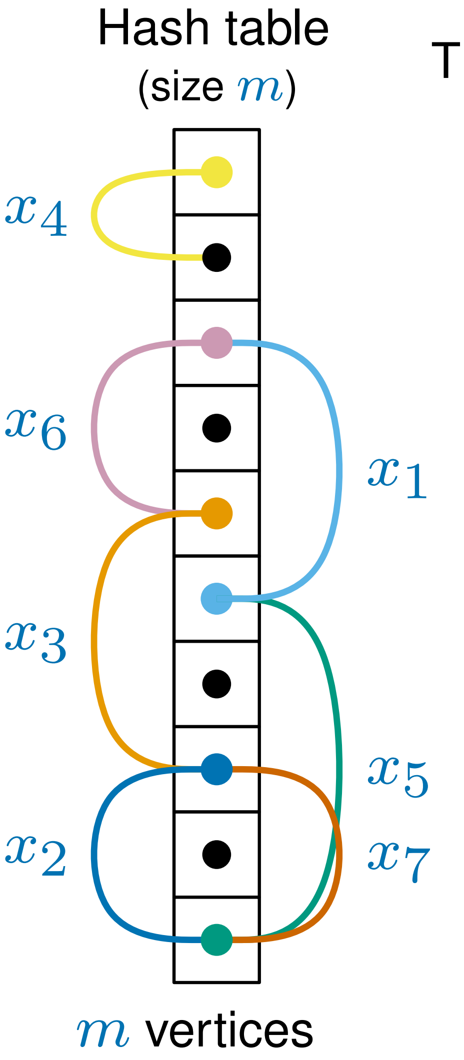
Cuckoo graph



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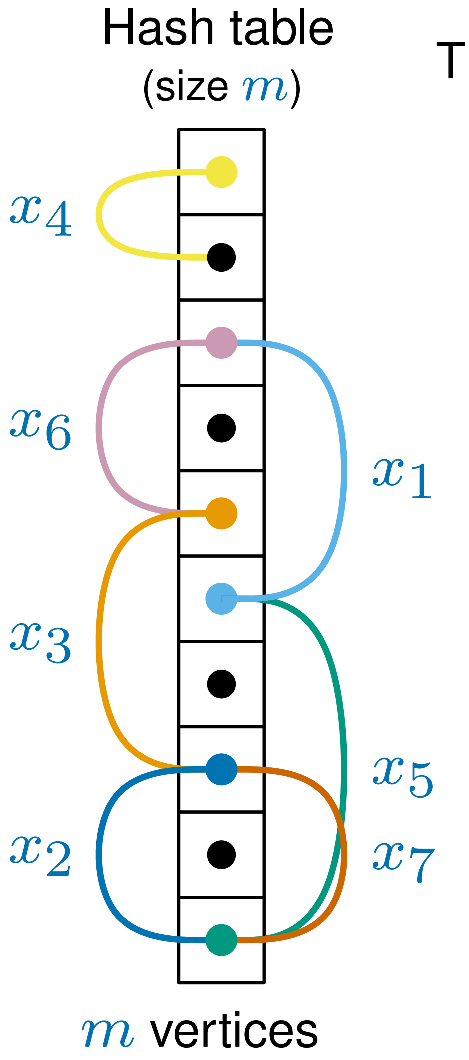
Inserting a key into a cycle **always** causes a rehash

This is the only way a rehash can happen

Cuckoo graph



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The number of moves performed while adding a key is the length of the corresponding path in the cuckoo graph

Inserting a key into a cycle **always** causes a rehash

This is the only way a rehash can happen

We will analyse the probability of either a cycle or a long path occuring in the graph while inserting any n keys.

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

What does this say?

table size is m

Paths in the cuckoo graph

n keys

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For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

table size is m

Paths in the cuckoo graph

n keys

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For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

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For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

Probability of a shortest path of length *1* is at most $\frac{1}{2 \cdot m}$

table size is m

Paths in the cuckoo graph

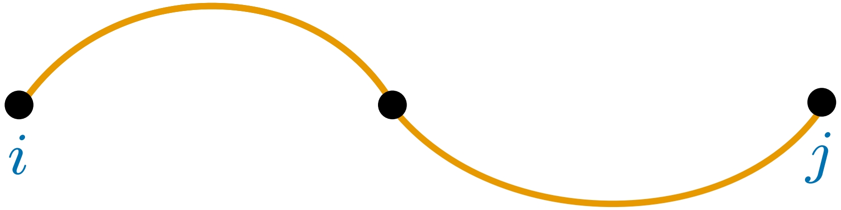
n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?



Probability of a shortest path of length *2* is at most $\frac{1}{4 \cdot m}$

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

Probability of a shortest path of length *3* is at most $\frac{1}{8 \cdot m}$

table size is m

Paths in the cuckoo graph

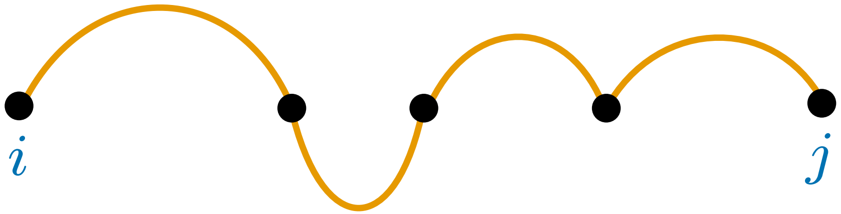
n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?



Probability of a shortest path of length *4* is at most $\frac{1}{16 \cdot m}$

table size is m

Paths in the cuckoo graph

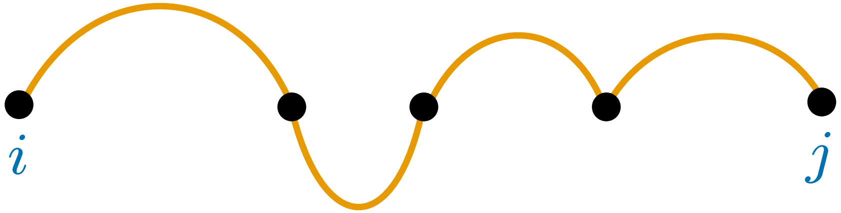
n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?



Probability of a shortest path of length *4* is at most $\frac{1}{16 \cdot m}$

How likely is it that there even is a path?

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

How likely is it that there even is a path?

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

How likely is it that there even is a path?

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^{\ell} \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

How likely is it that there even is a path?

If a path exists from i to j, there must be a shortest path (from i to j)

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

How likely is it that there even is a path?

If a path exists from i to j, there must be a shortest path (from i to j)

Therefore the probability of a path from i to j existing is at most...

$$\sum_{\ell=1}^{\infty} \frac{1}{c^\ell \cdot m}$$

(using the union bound over all possible path lengths.)

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

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What does this say?

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(using the union bound over all possible path lengths.)

table size is m

Paths in the cuckoo graph

n keys

LEMMA

For any positions i and j, and any constant $c > 1$, if $m \geqslant 2cn$ then the probability that there exists a shortest path in the cuckoo graph from i to j with length $\ell \geqslant 1$, is at most $\frac{1}{c^\ell \cdot m}$.

(let $c = 2$ for simplicity)

What does this say?

How likely is it that there even is a path?

If a path exists from i to j, there must be a shortest path (from i to j)

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So a path from i to j is rather unlikely to exist

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The proof is in the directors cut of the slides (see notes)

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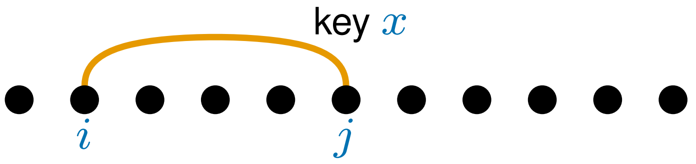
The proof is by induction on the length ℓ:

Base case: $\ell = 1$.

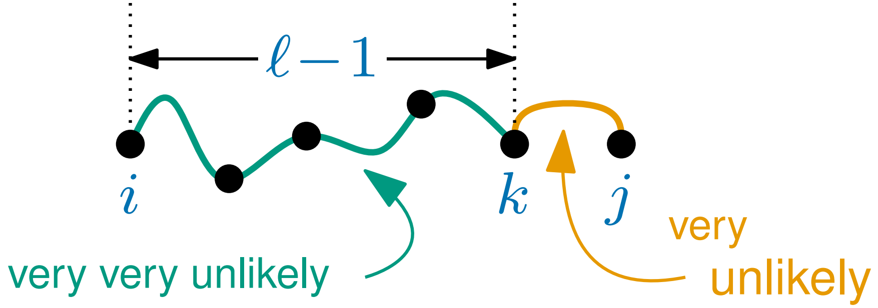


Argue that each key has prob $\frac{2}{m^2}$ to create an edge (i,j)

Union bound over all n keys

Inductive step:

Pick a third point k to split the path



Union bound over all k *then* all keys

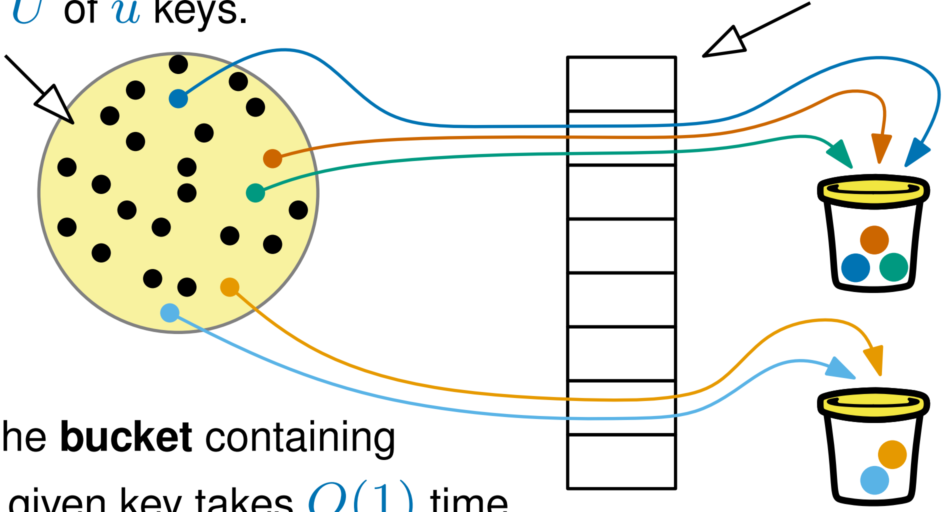
table size is m

Back to the start (again) (again)

n keys

- A **dynamic dictionary** stores (*key*, *value*)-pairs and supports:

  add(*key*, *value*), lookup(*key*) (which returns *value*) and delete(*key*)



Universe U of u keys.

Hash table T of size $m \geqslant n$.

Collisions are fixed by **~~chaining~~**

bucketing

We require that we can recover

any key from its **bucket** in $O(s)$ time

where s is the number of keys in the ***bucket***

Locating the **bucket** containing

a given key takes $O(1)$ time

n *arbitrary operations arrive online, one at a time*.

If our construction has the property that,

for any two keys $x, y \in U$ (with $x \neq y$),

the probability that x and y are in the same bucket is $O\left(\frac{1}{m}\right)$

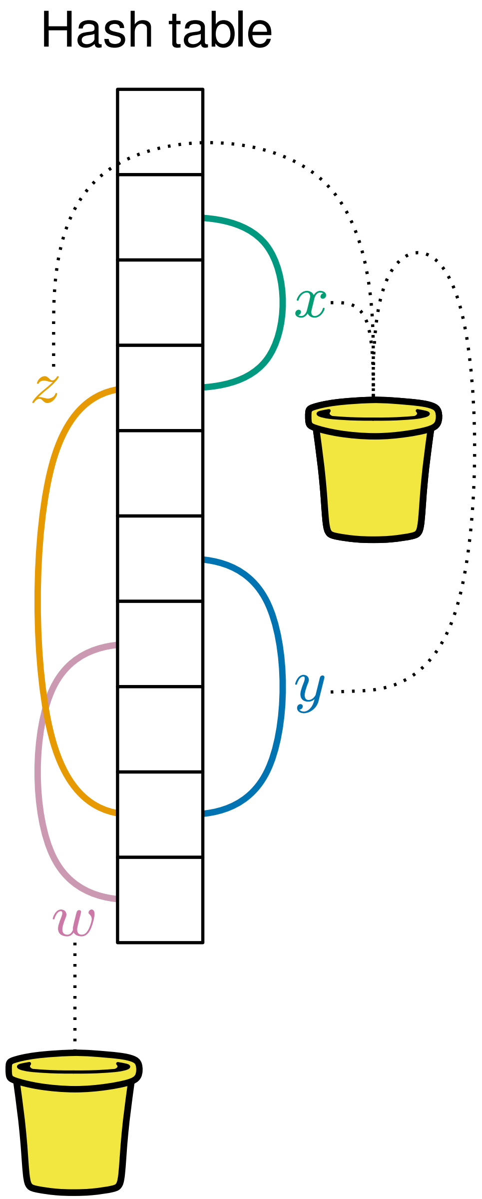
For *any* n operations, the *expected* run-time is $O(1)$ per operation.

table size is m

Don't put all your eggs in one bucket

n keys

Hash table



We say that two keys x, y are in the same **bucket** (conceptually)
iff there is a path between $h_1(x)$ and $h_1(y)$
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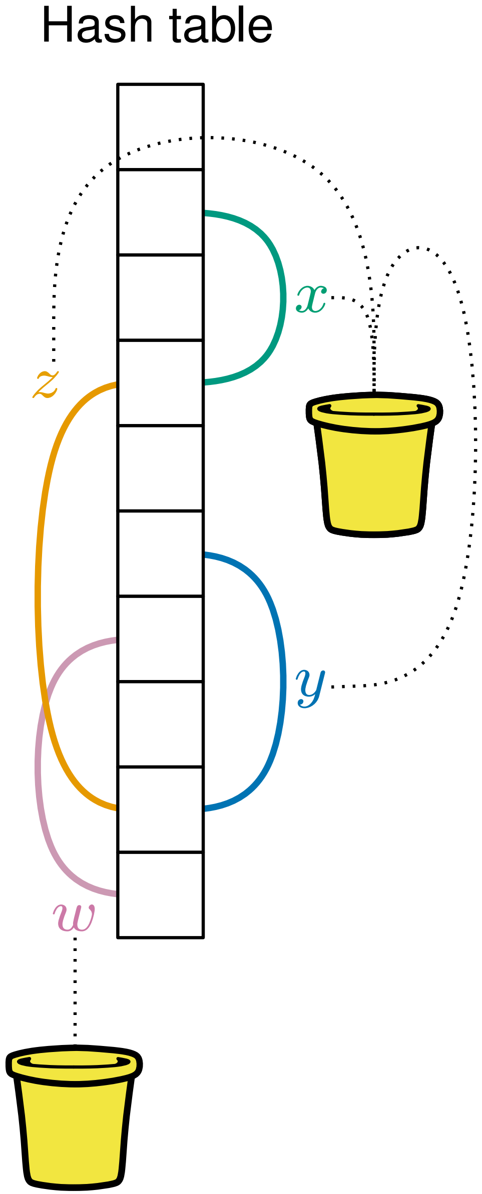

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$$\sum_{\ell=1}^{\infty} \frac{4}{c^\ell \cdot m} = \frac{4}{m} \cdot \sum_{\ell=1}^{\infty} \frac{1}{c^\ell} = \frac{4}{m(c-1)} = O\left(\frac{1}{m}\right)$$

where $c > 1$ is a constant.

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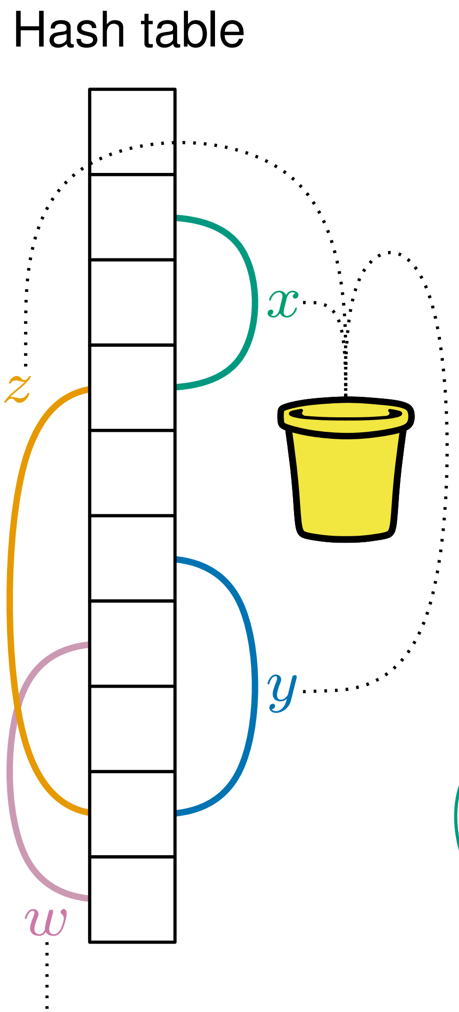

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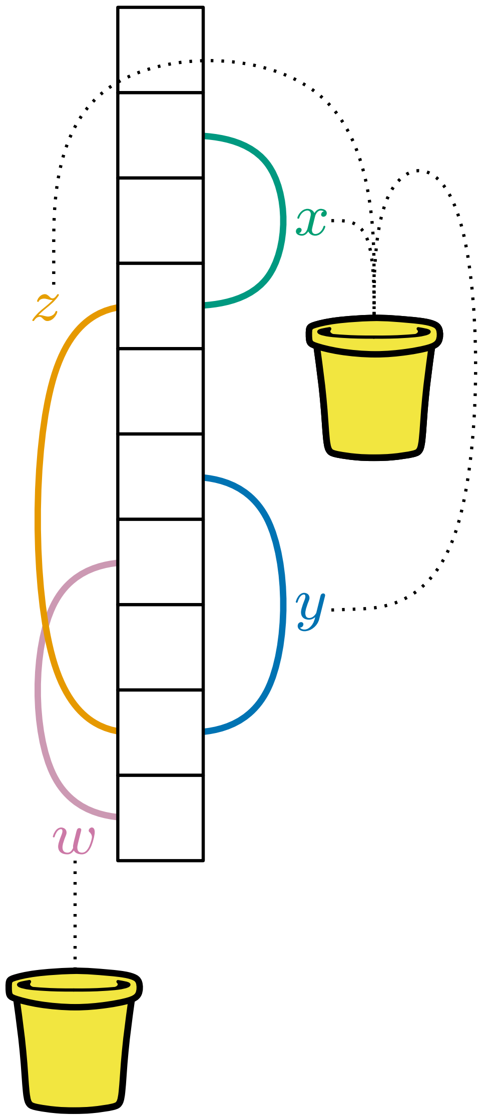

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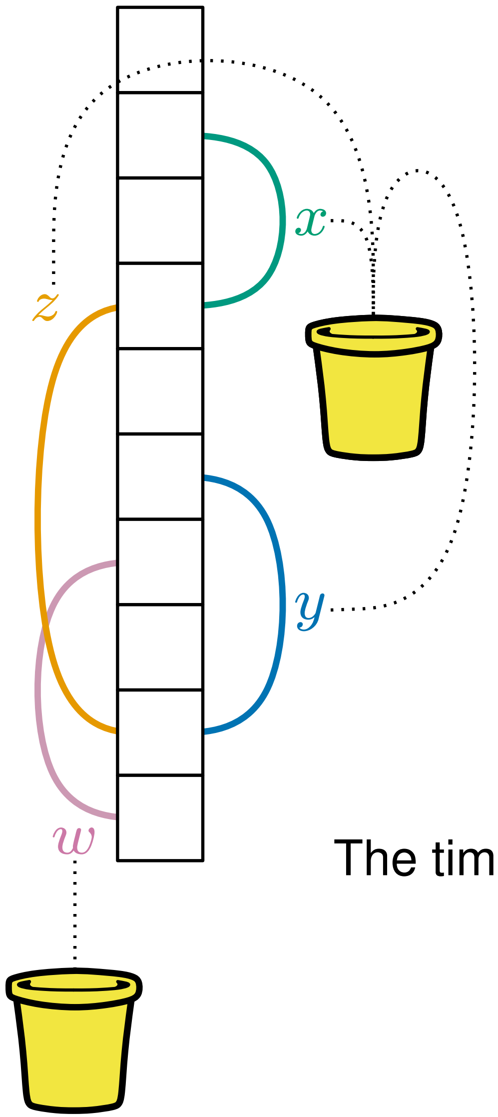

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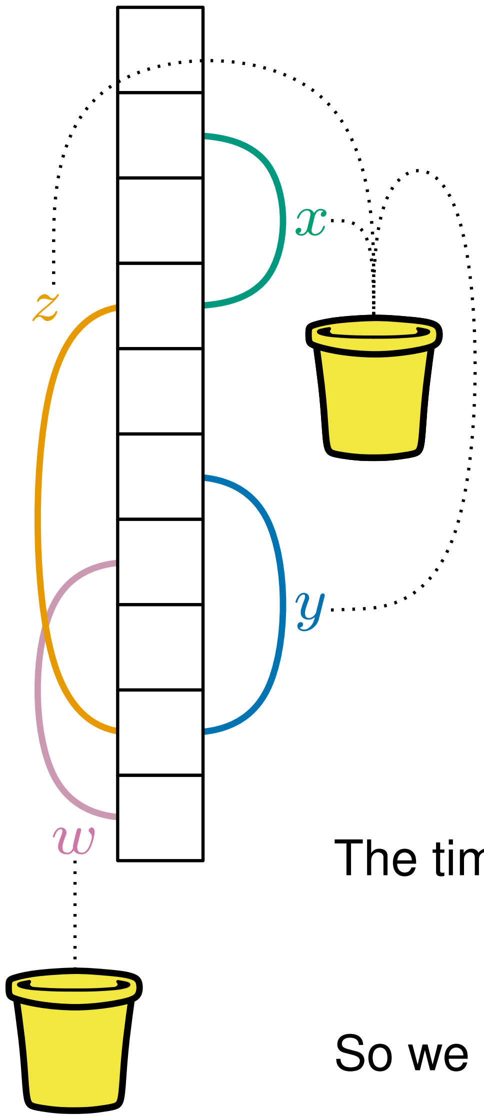

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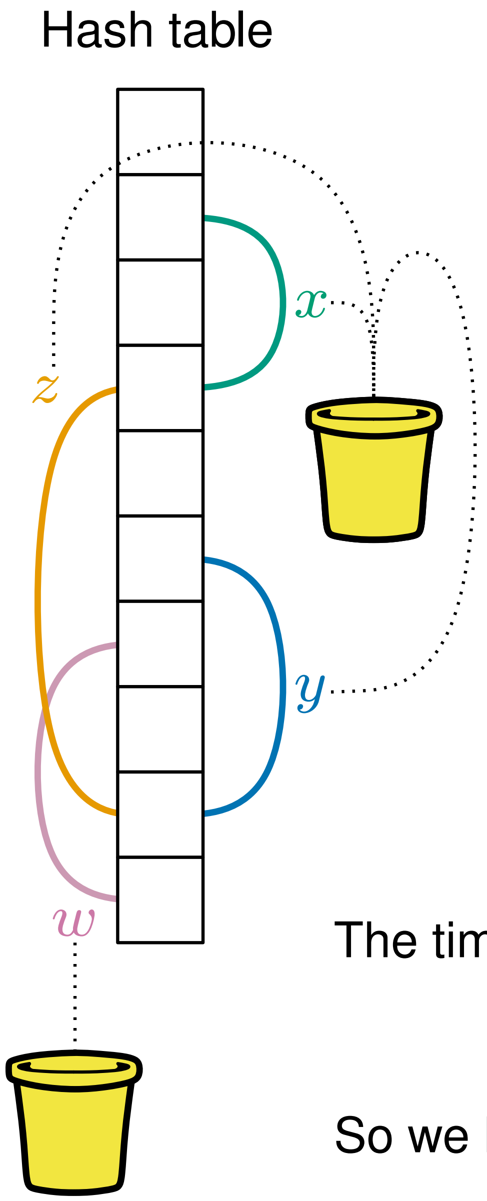

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Further, lookups take $O(1)$ time in the *worst case*.

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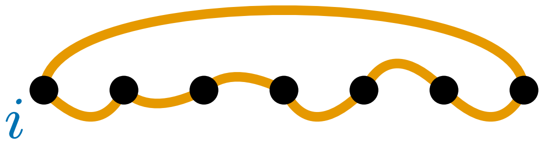

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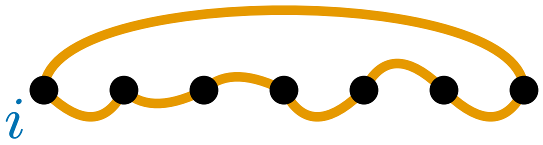

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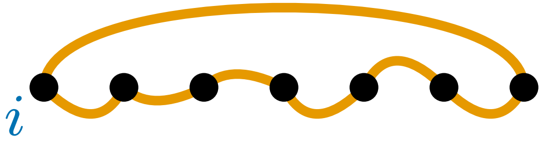

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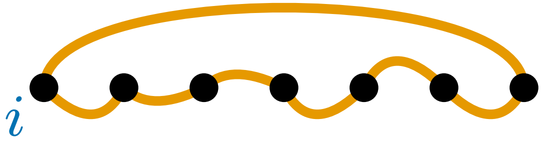

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So the expected number of rehashes during n insertions is at most $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i} = 1$.

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If there is no cycle, insert all the elements,

this takes $O(n)$ time in expectation (as we have seen).

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such that $h(x)$ can be computed in $O(1)$ time

By changing the cuckoo hashing algorithm to perform a rehash after $\log n$ moves

it can be shown (via a similar but harder proof) that the results still hold

THEOREM

In the **Cuckoo hashing** scheme:

- Every lookup and every delete takes $O(1)$ *worst-case* time,
- The space is $O(n)$ where n is the number of keys stored
- An insert takes *amortised expected* $O(1)$ time