University of BRISTOL

Advanced Algorithms – COMS31900

Approximation algorithms part four

Asymptotic Polynomial Time Approximation Schemes

Raphaël Clifford

Slides by Benjamin Sach

Approximation Algorithms Recap

A polynomial time algorithm A is an α-approximation for problem P if,

it always outputs a solution s with

$$\frac{\mathrm{Opt}}{\alpha} \leqslant s \leqslant \mathrm{Opt} \text{ (for a maximisation problem)}$$

$$\mathrm{Opt} \leqslant s \leqslant \alpha \cdot \mathrm{Opt} \text{ (for a minimisation problem)}$$

A poly-time approximation scheme (PTAS) is a family of algorithms:

For any constant $\epsilon > 0$, A_ϵ is a $(1+\epsilon)$-approximation for P

A PTAS is allowed to have A_ϵ which takes $O(n^{1/\epsilon})$ time

- which is polynomial in n (for any constant ϵ)

It is a (fully) FPTAS if A_ϵ takes time polynomial in both n and $1/\epsilon$ i.e. $O((n/\epsilon)^c)$

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We have seen various c-approximations with constant c

Last lecture we saw an FPTAS for SUBSETSUM

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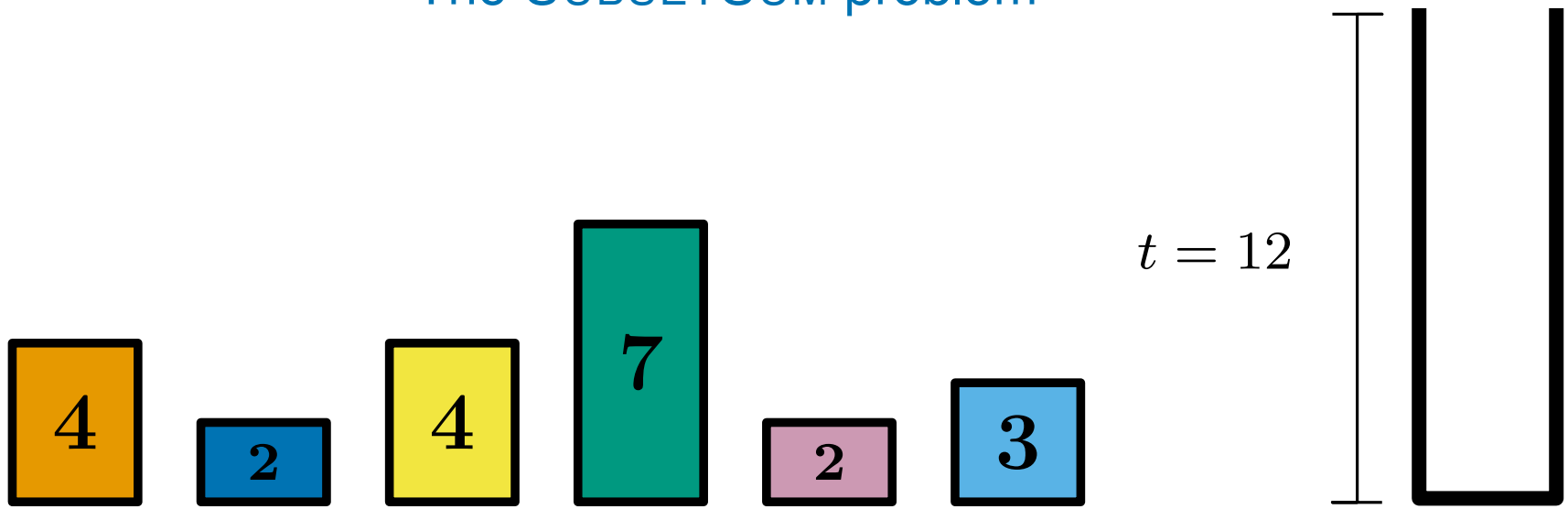

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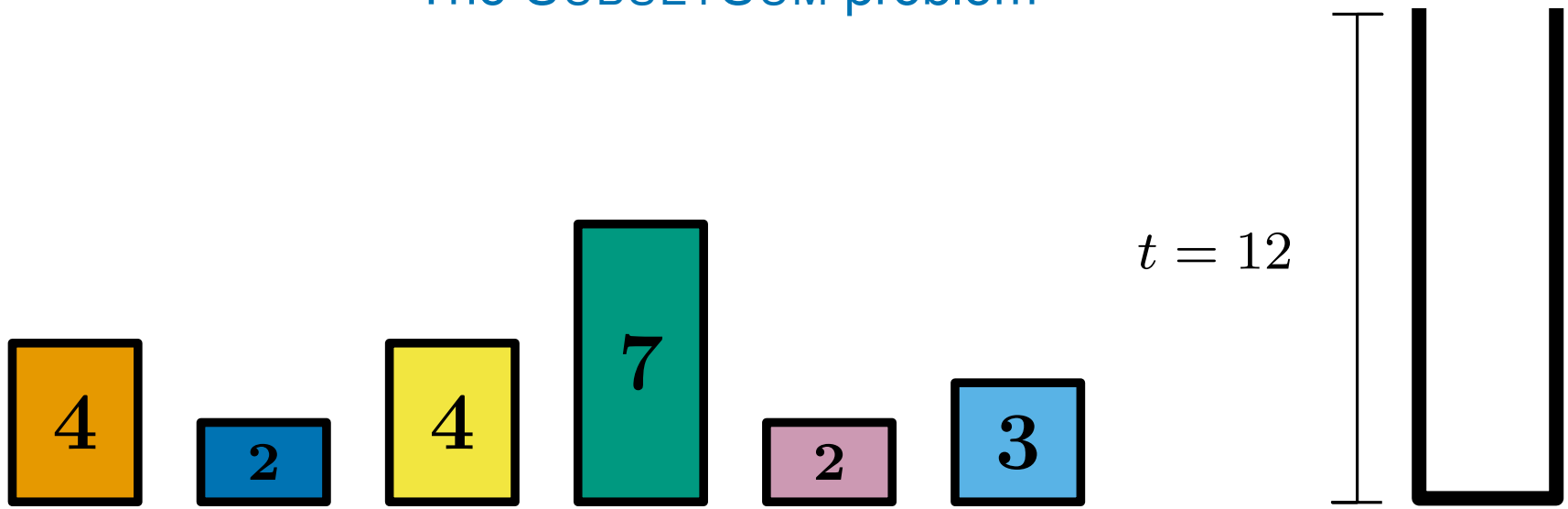

- Let S be a (multi) set of integers and t be a positive integer

here $S = \{4, 2, 4, 7, 2, 3\}$ *and* $t = 12$

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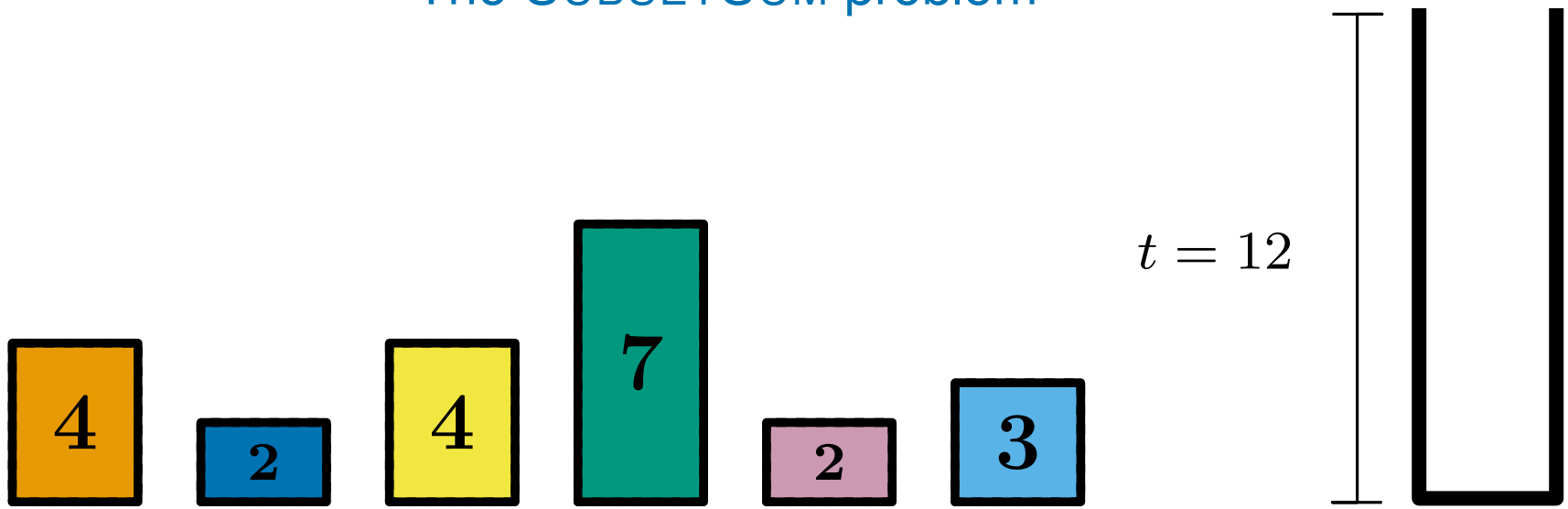

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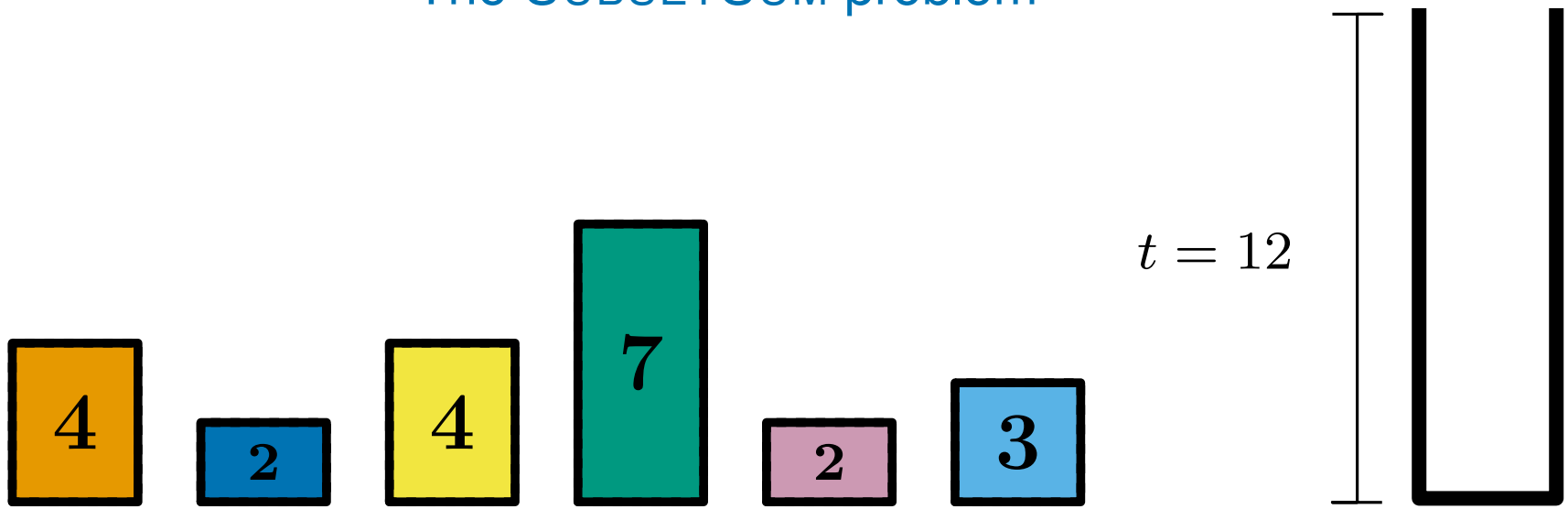

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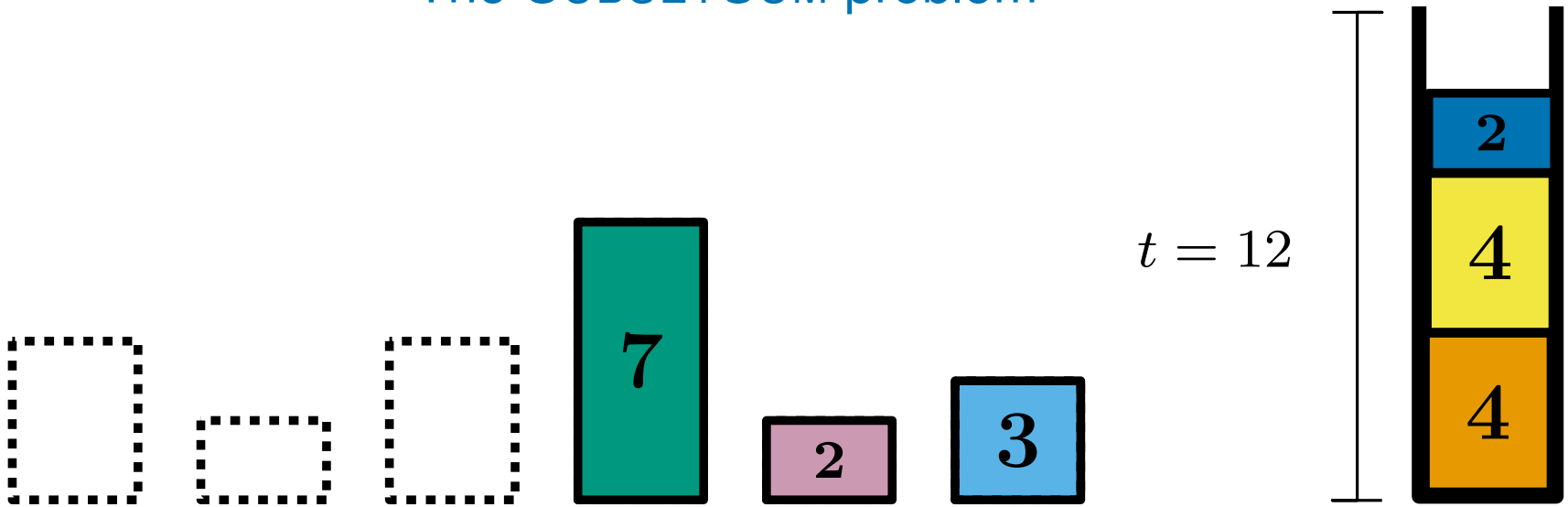

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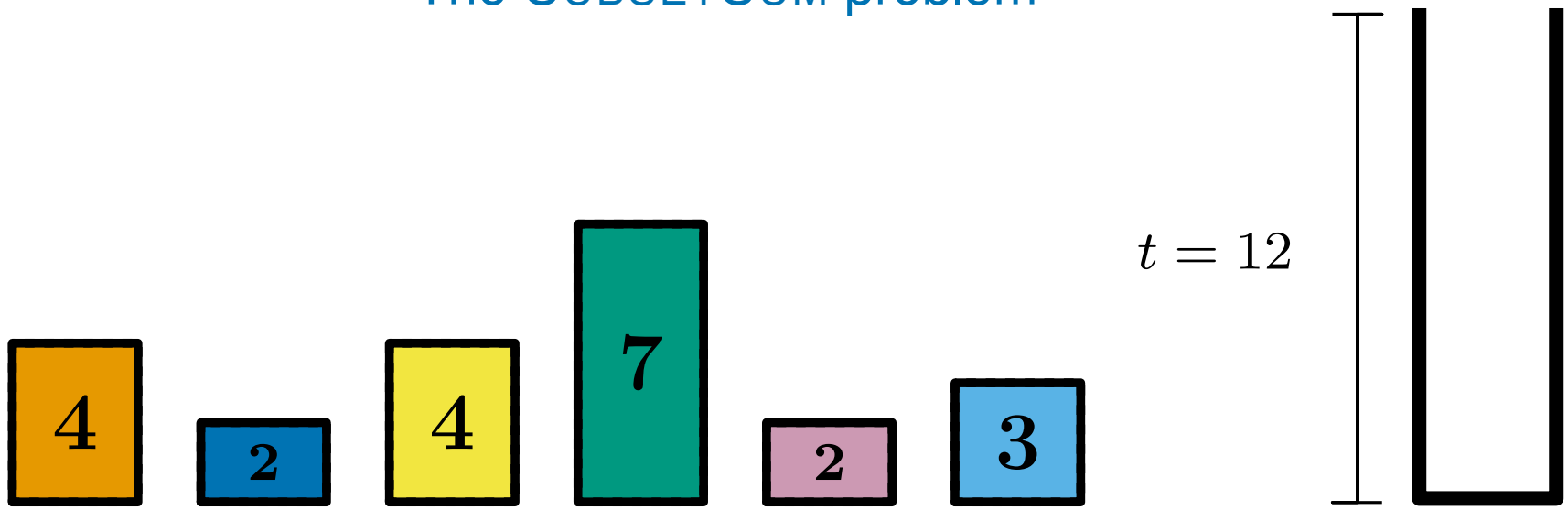

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The decision version is NP-complete

(last lecture we discussed the NP-hard optimisation version)

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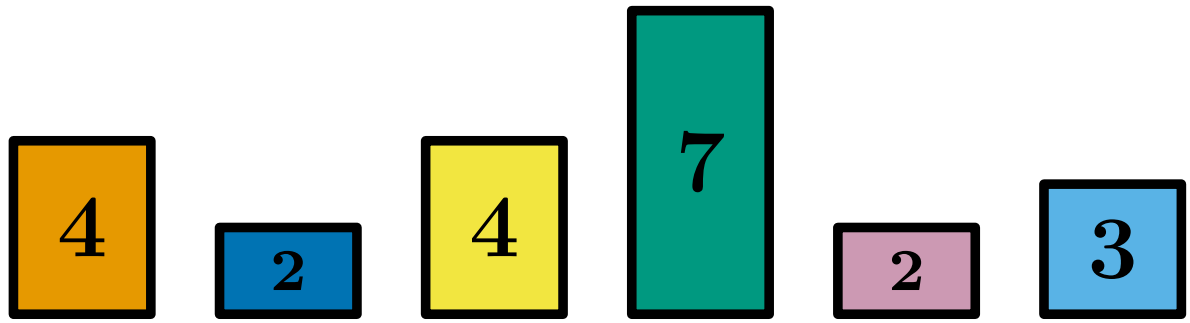

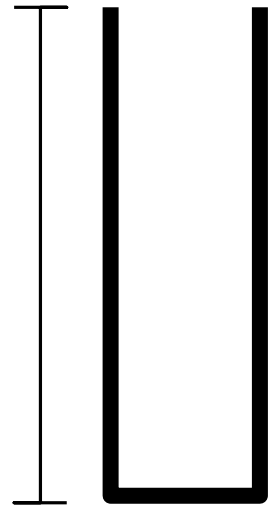

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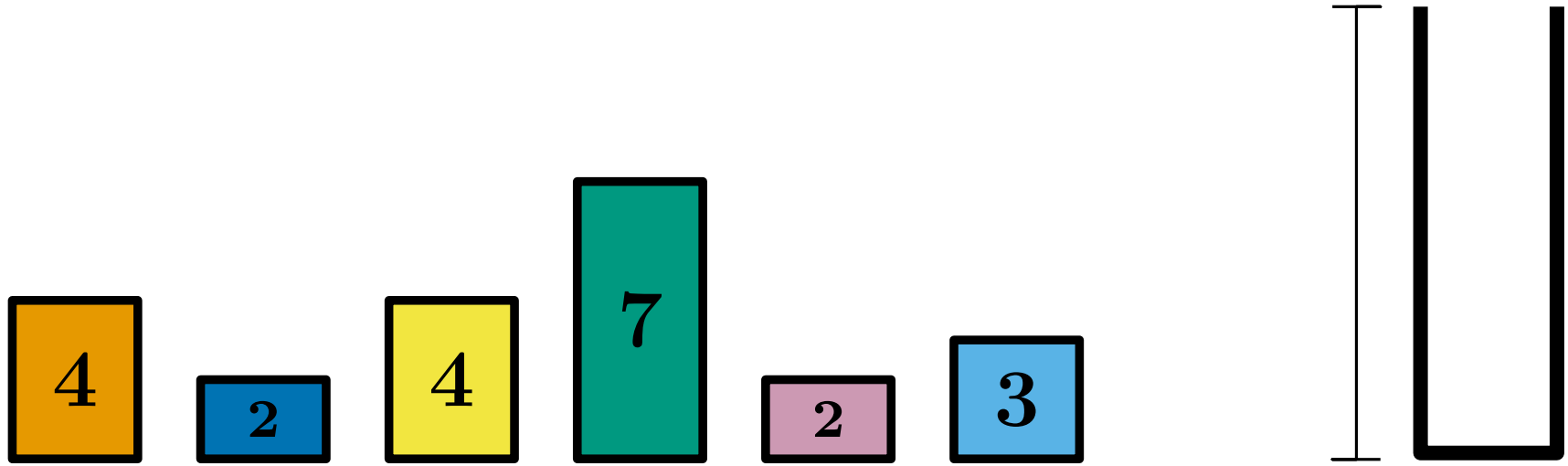

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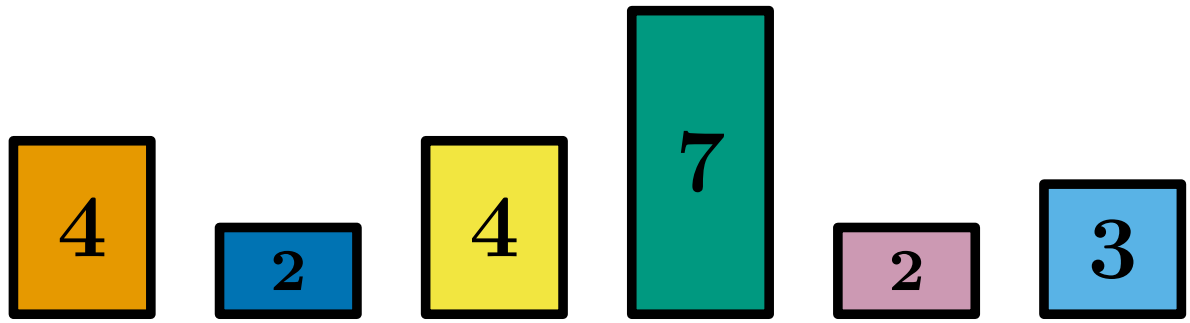

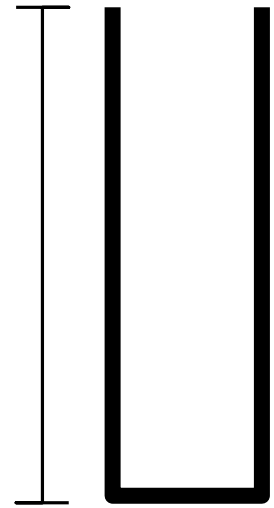

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The input S is a multi-set of integers

but t is always half the sum of item sizes

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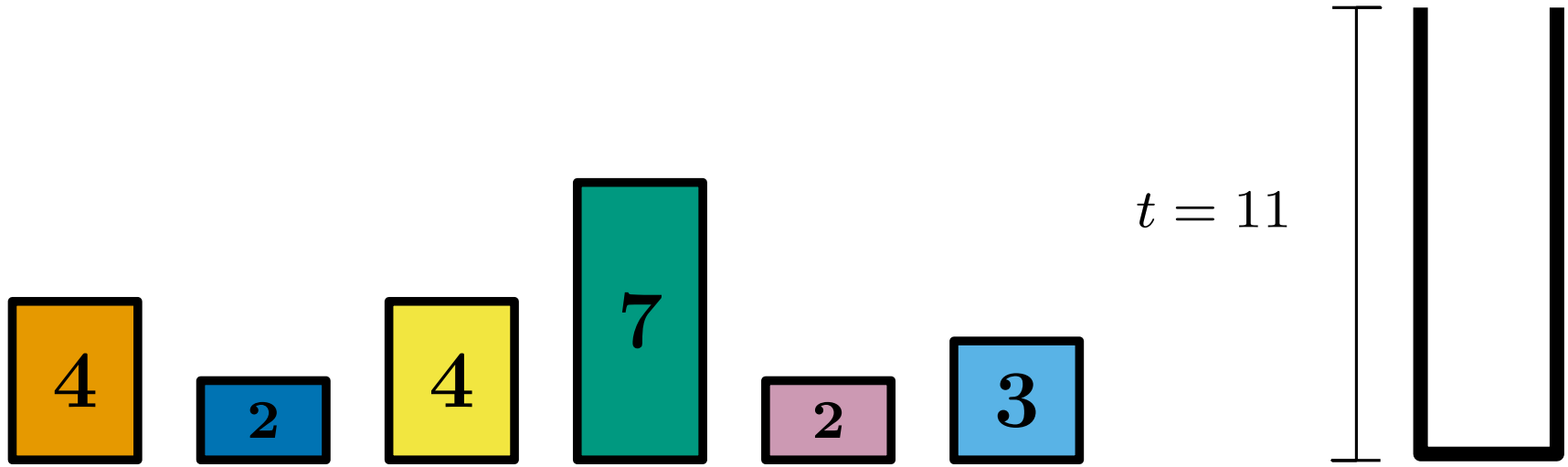

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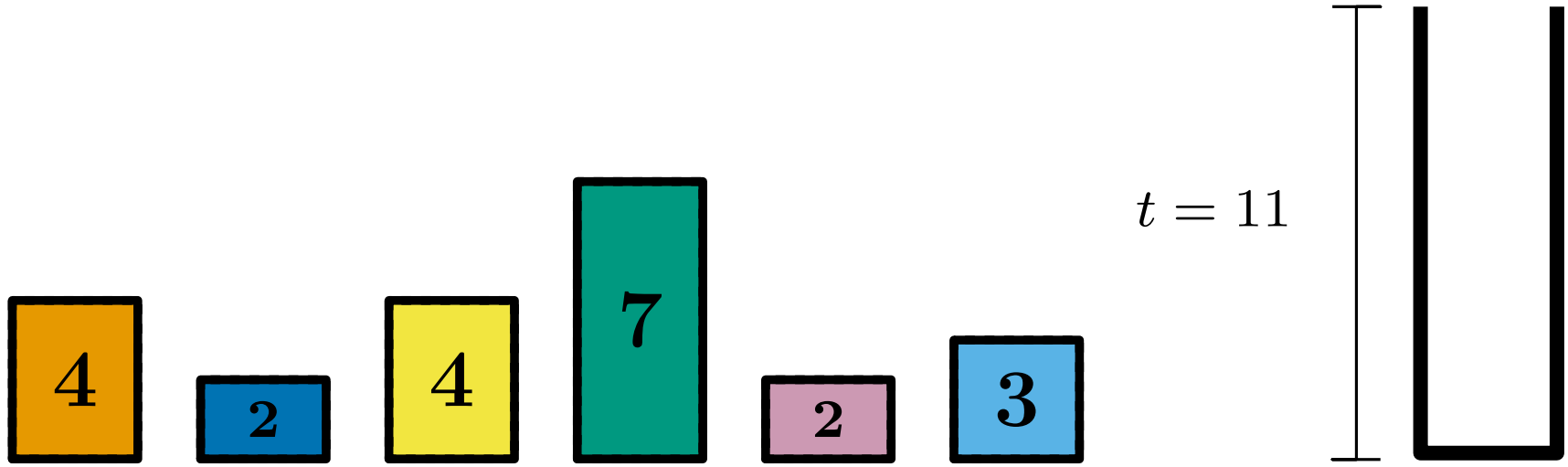

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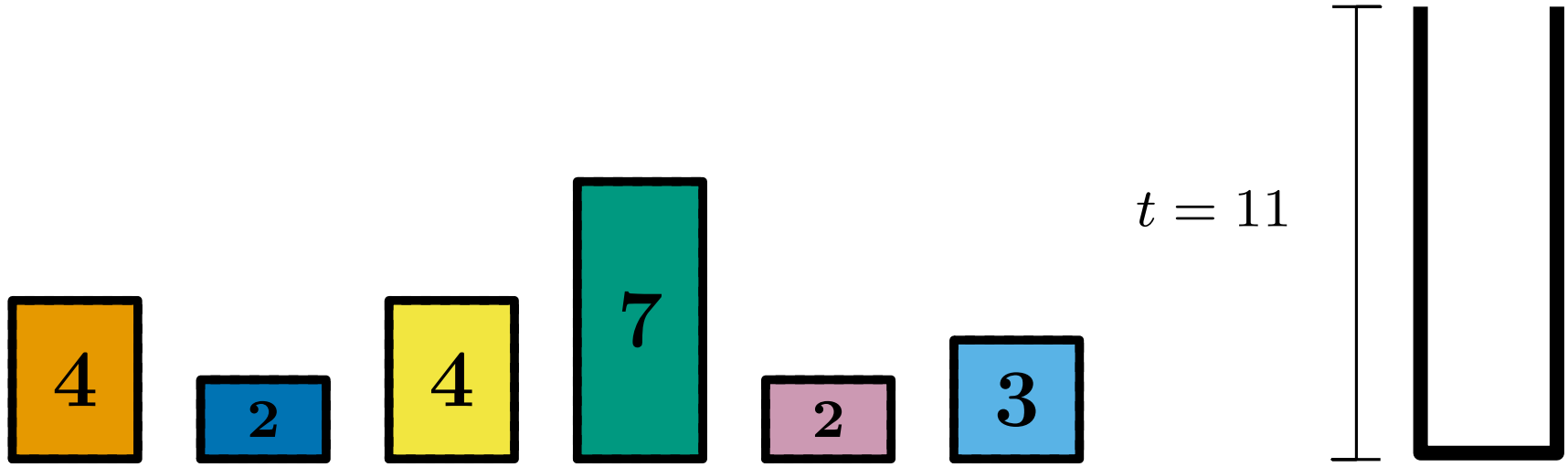

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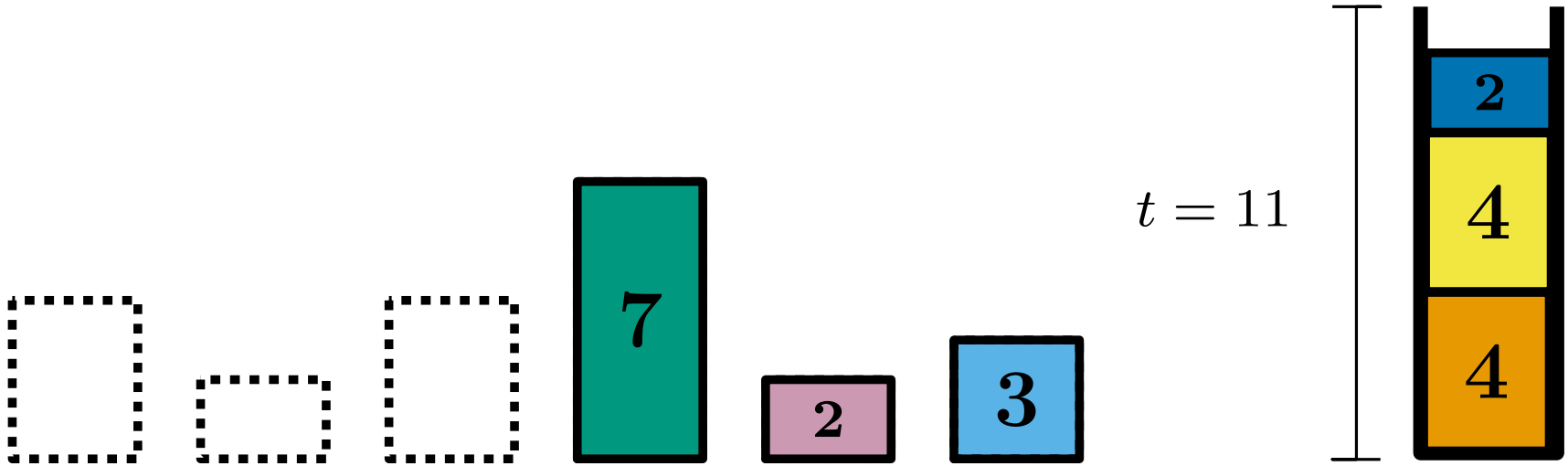

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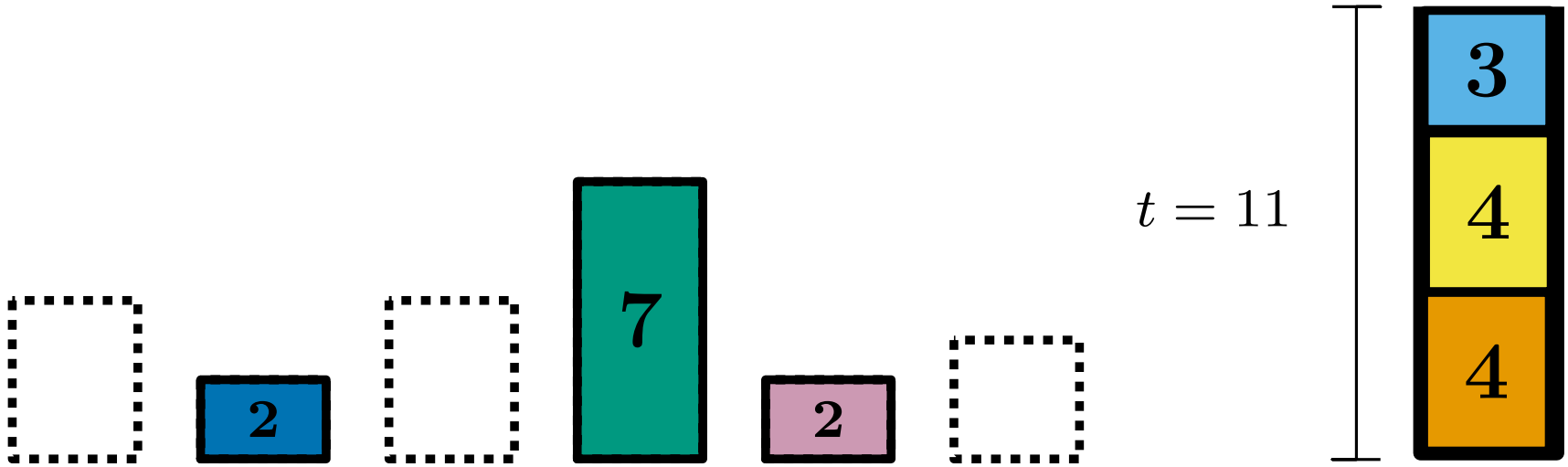

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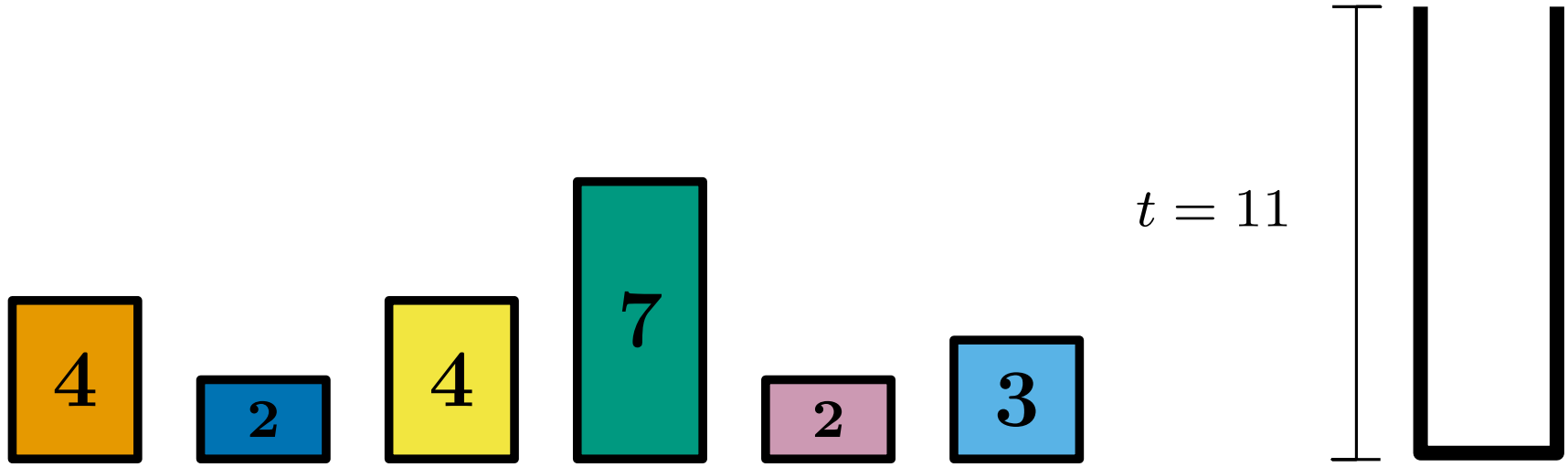

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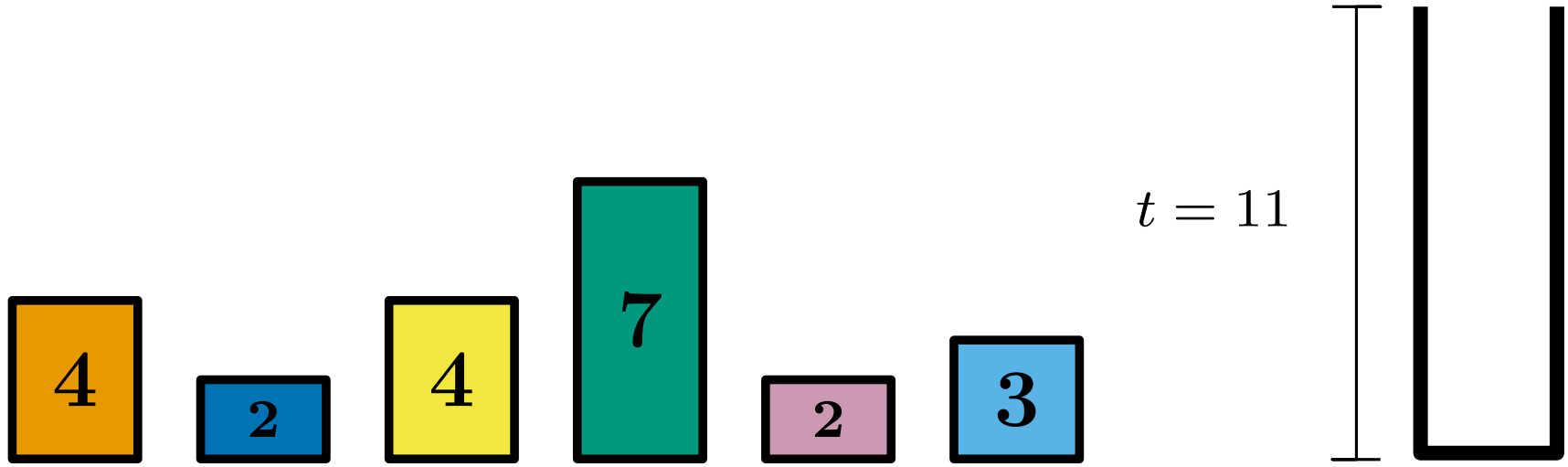

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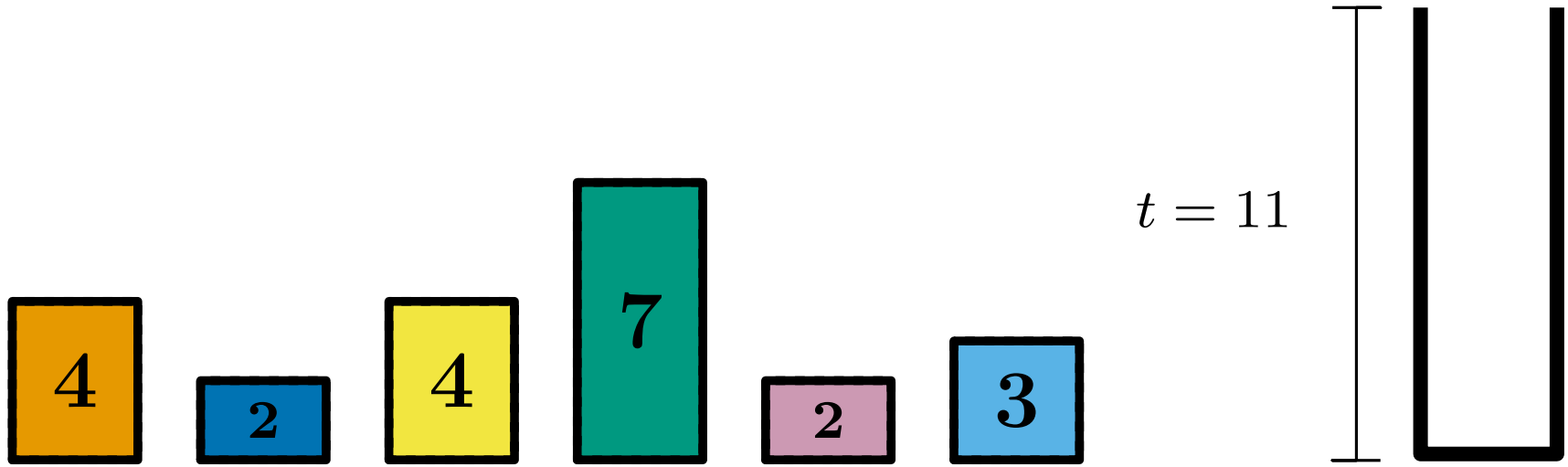

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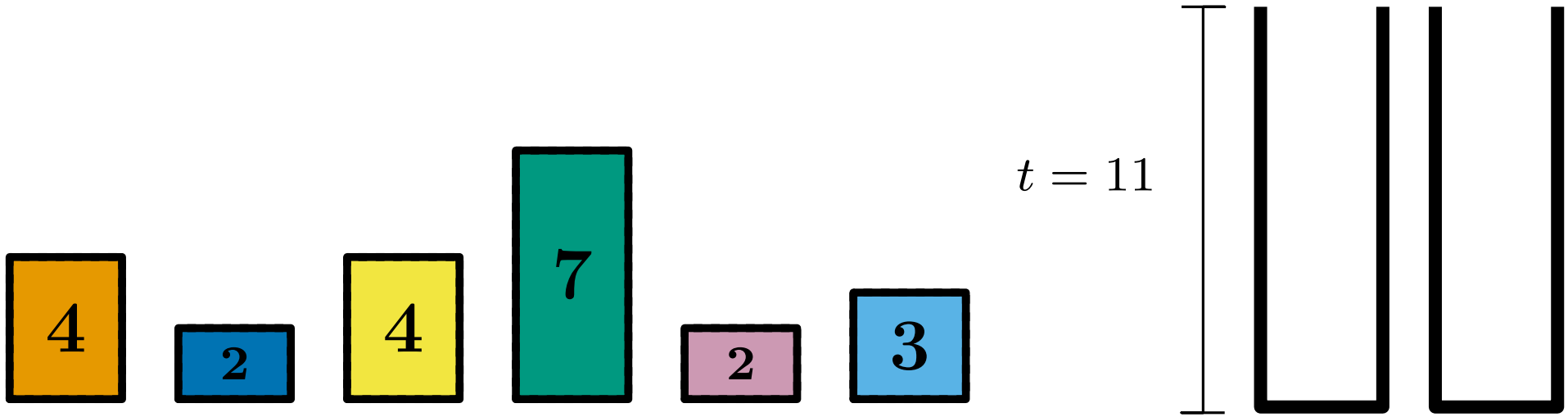

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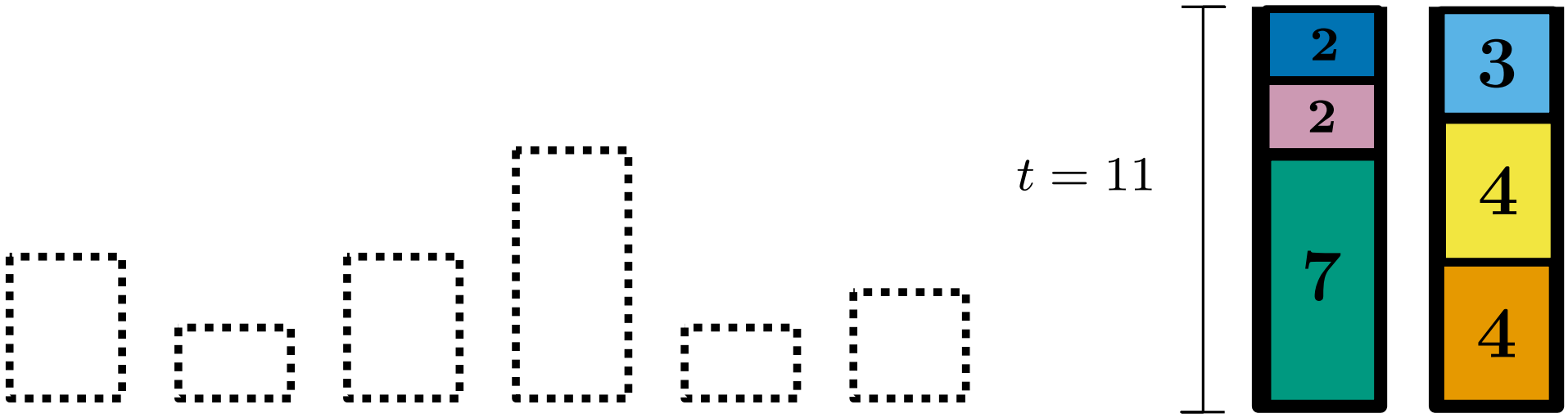

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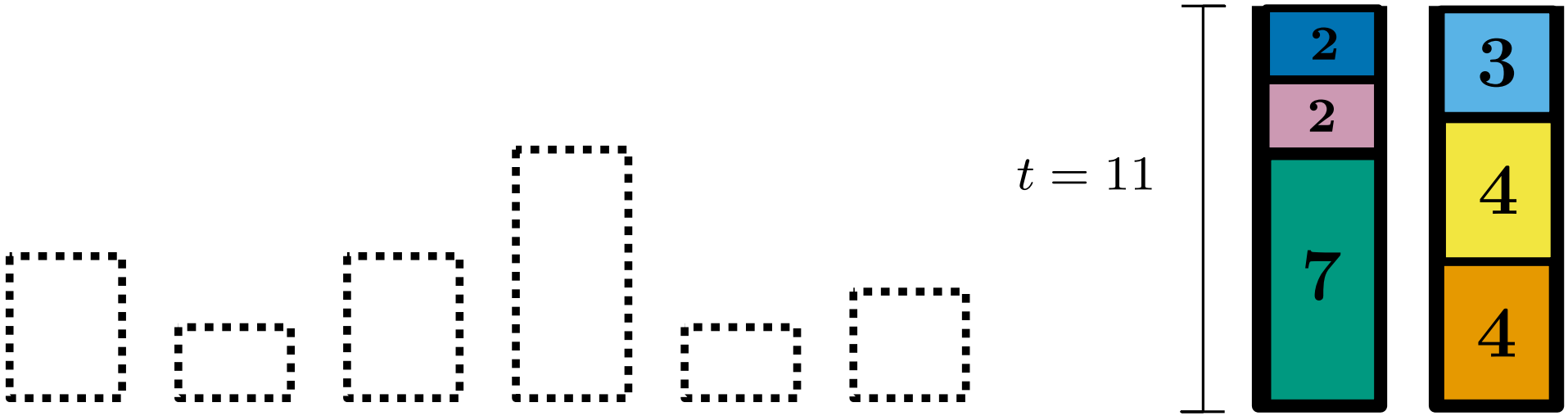

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The PARTITION problem is also NP-complete

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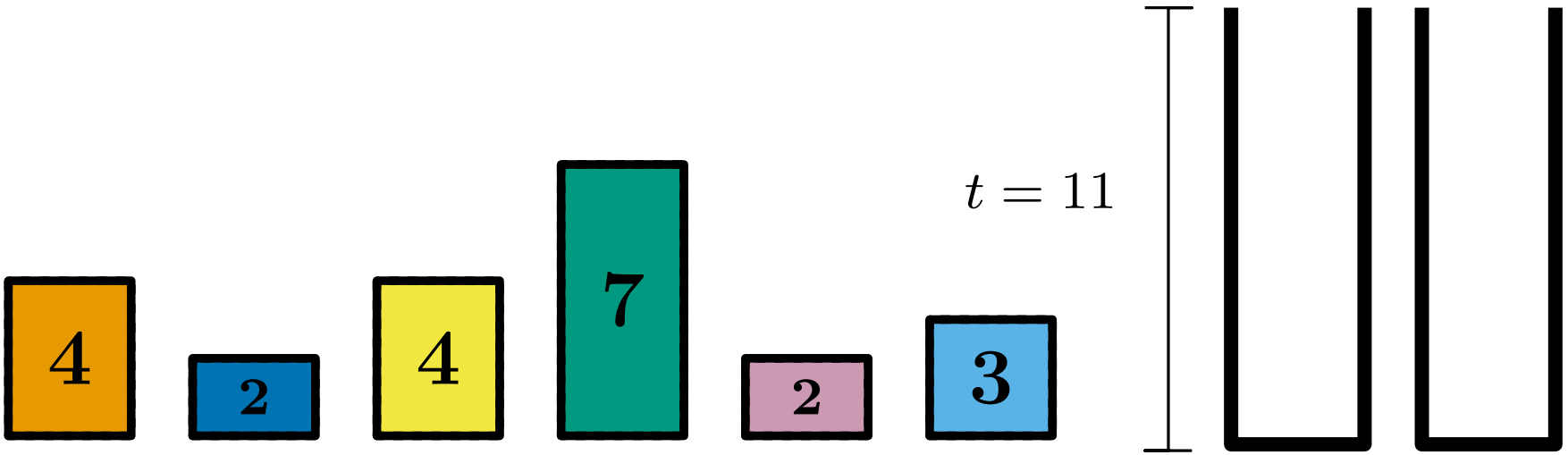

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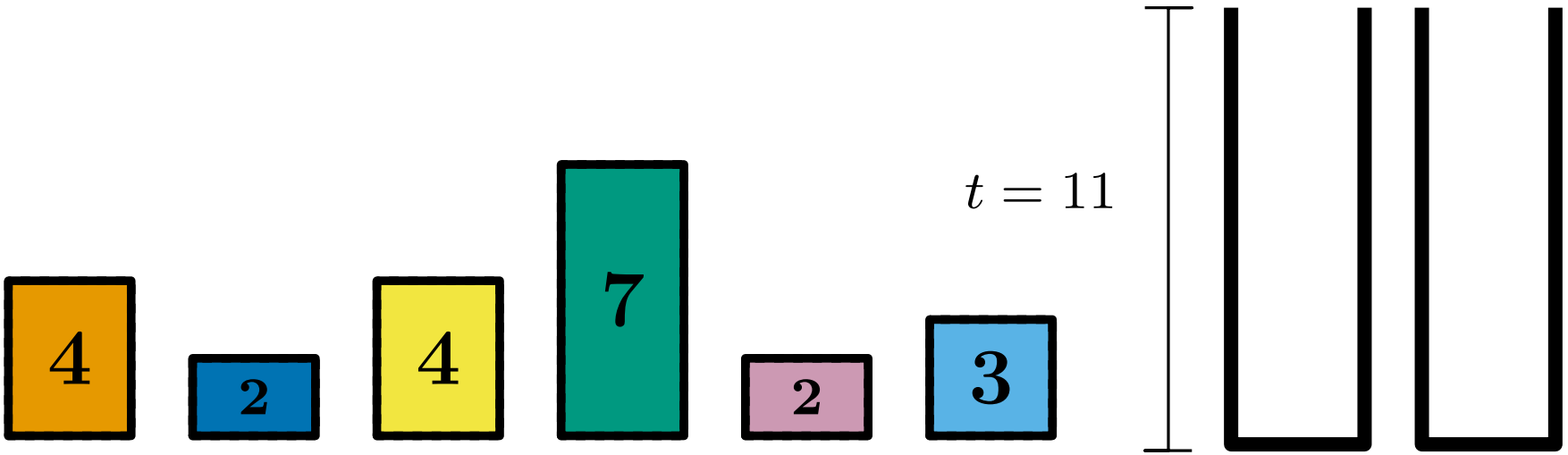

- Convert the PARTITION instance into a BINPACKING problem

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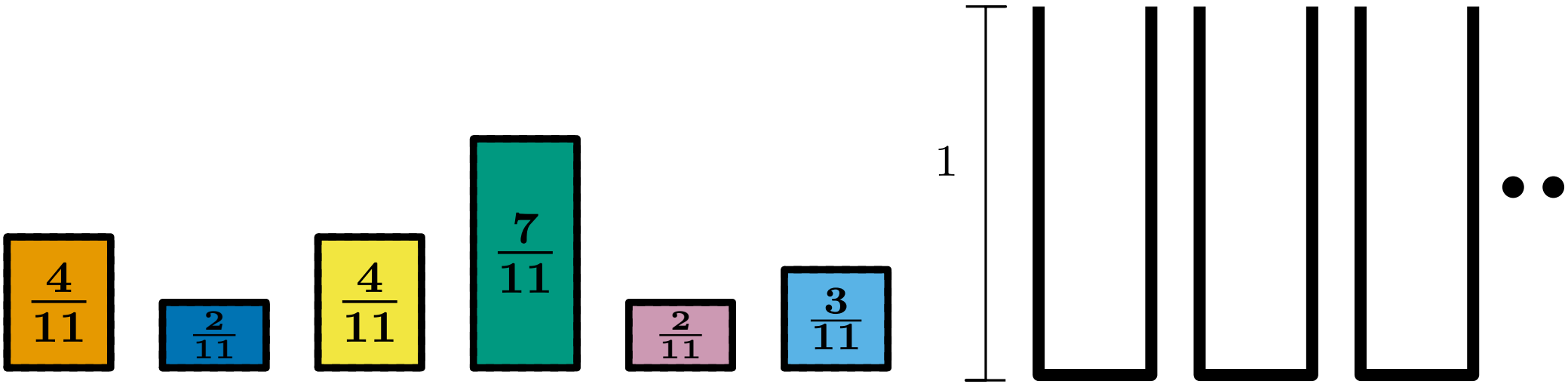

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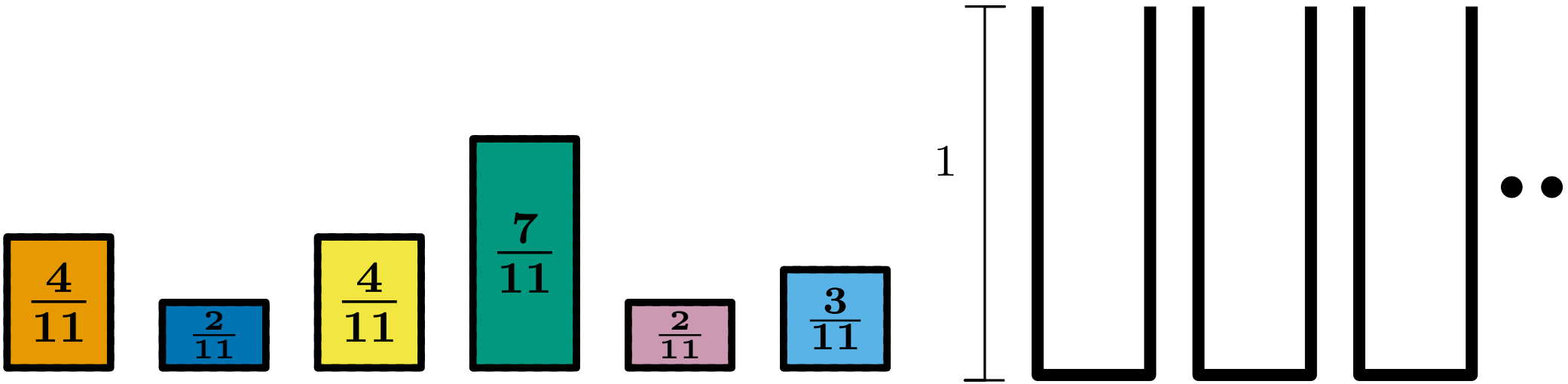

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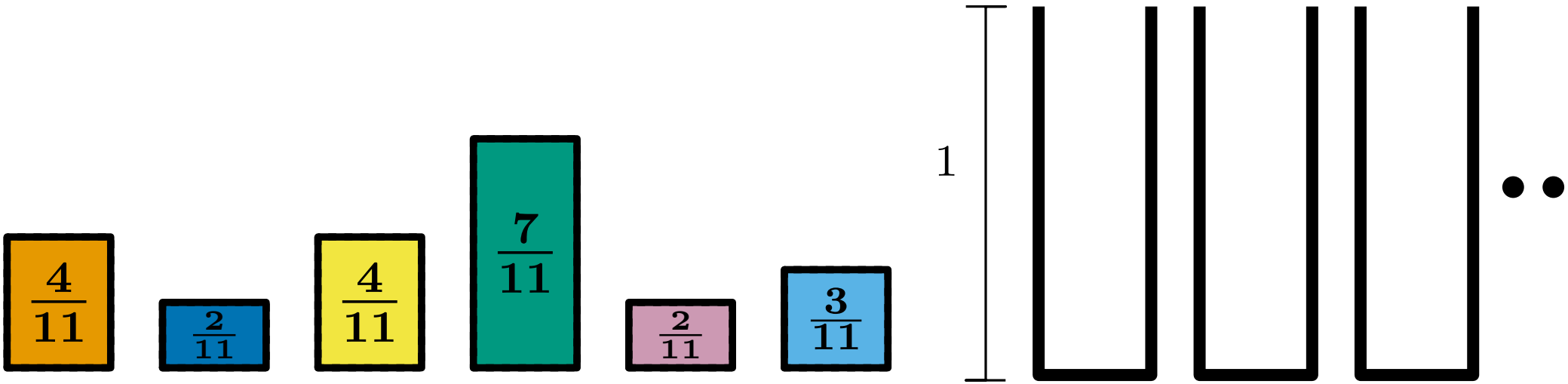

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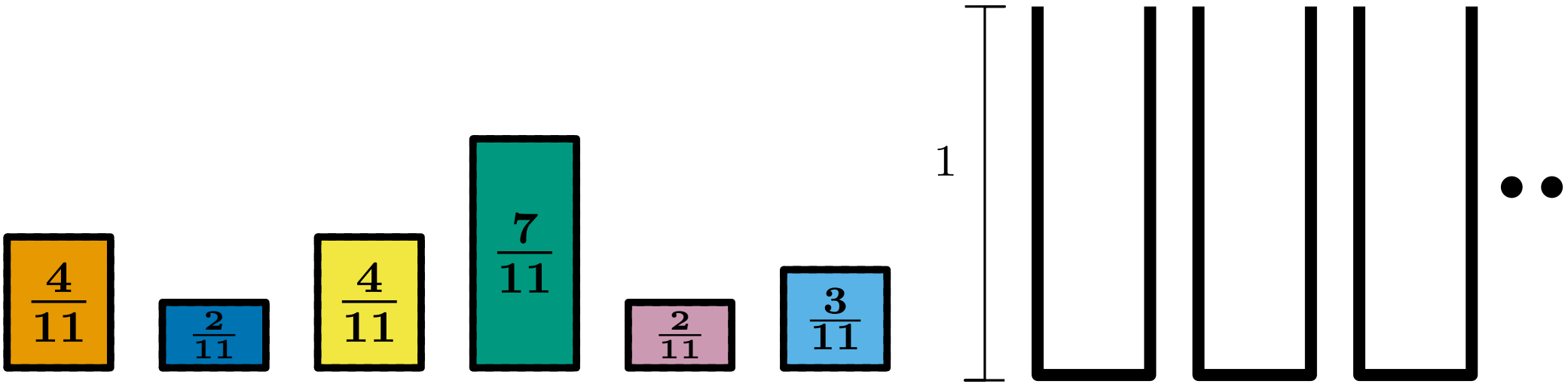

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- What does this tell us about approximating BINPACKING?

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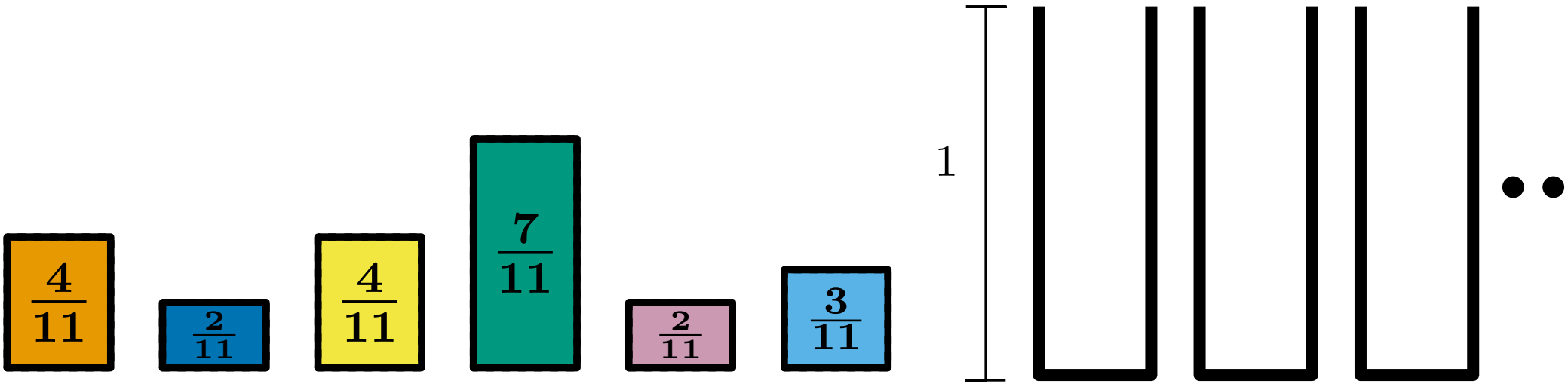

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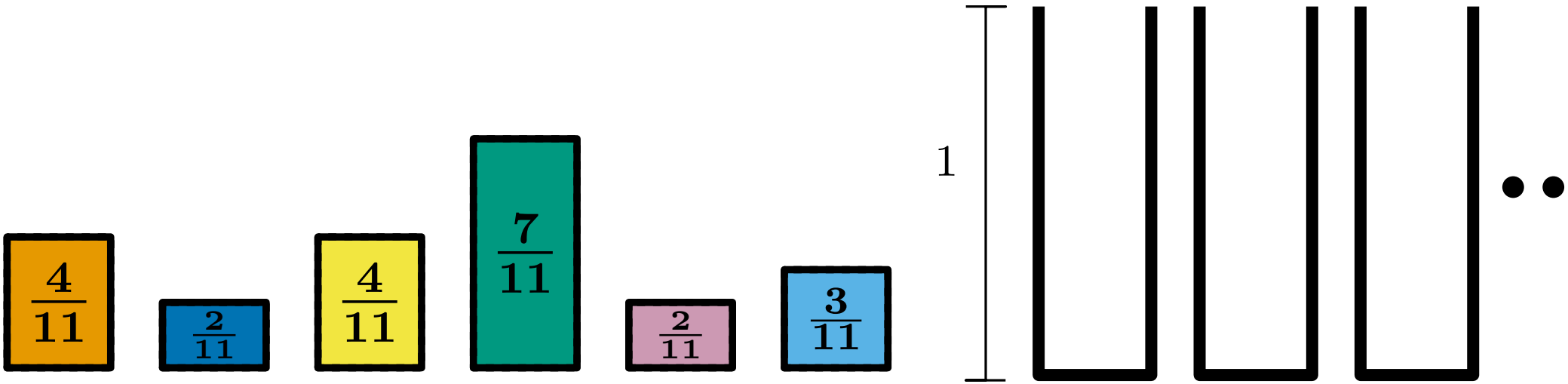

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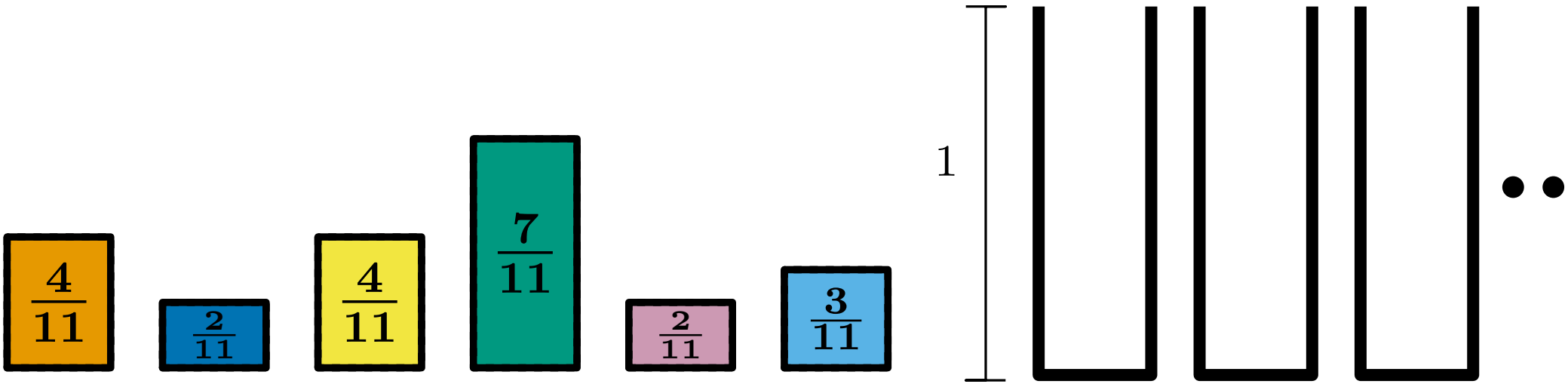

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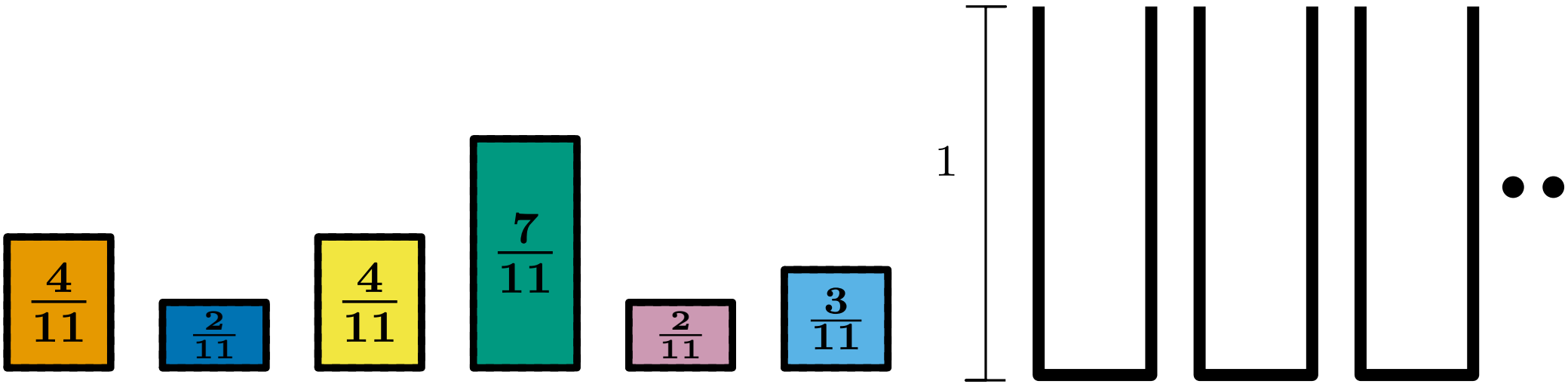

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- If $\mathrm{Opt}_b = 2$ then $2 \leqslant s \leqslant \alpha \cdot \mathrm{Opt}_b < (3/2) \cdot \mathrm{Opt}_b = 3$

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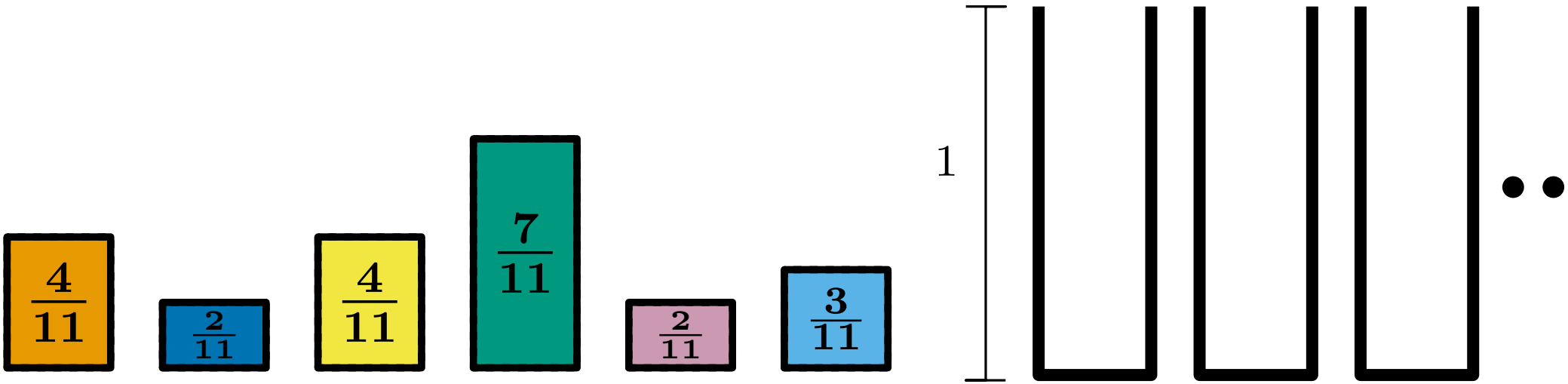

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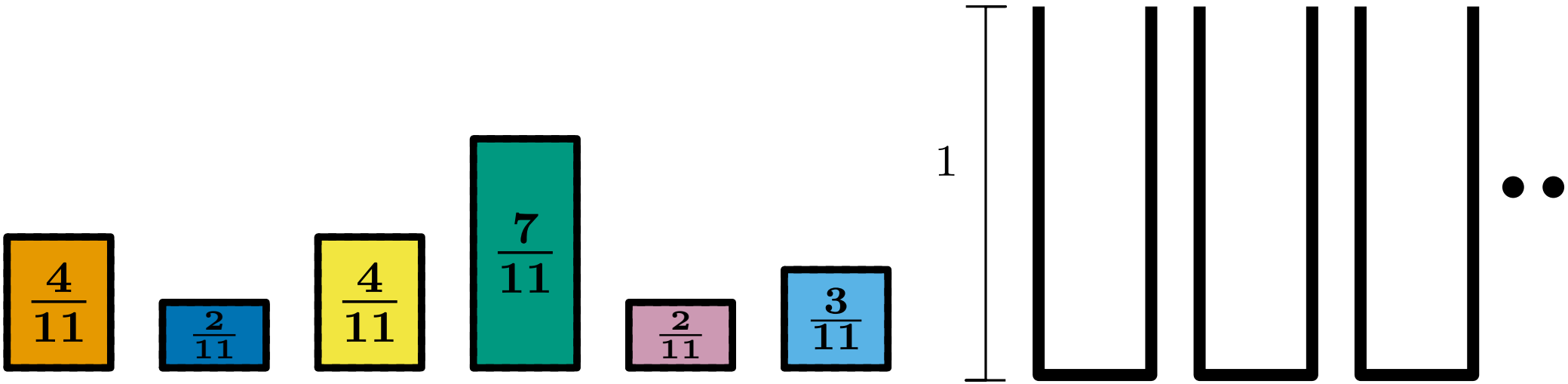

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- If $\mathrm{Opt}_b = 2$ then $2 \leqslant s \leqslant \alpha \cdot \mathrm{Opt}_b < (3/2) \cdot \mathrm{Opt}_b = 3$ so $s = 2$

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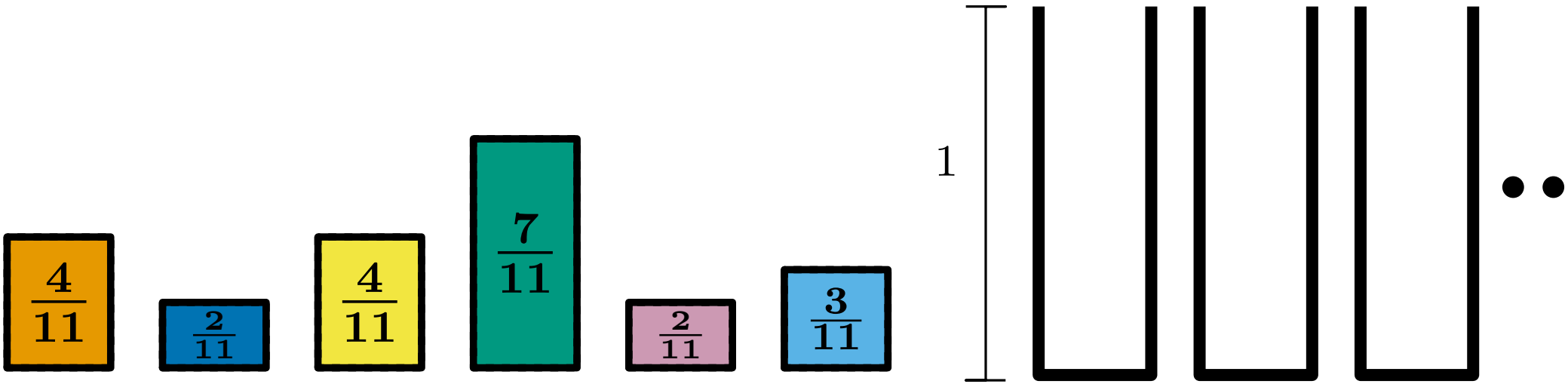

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PARTITION and BINPACKING

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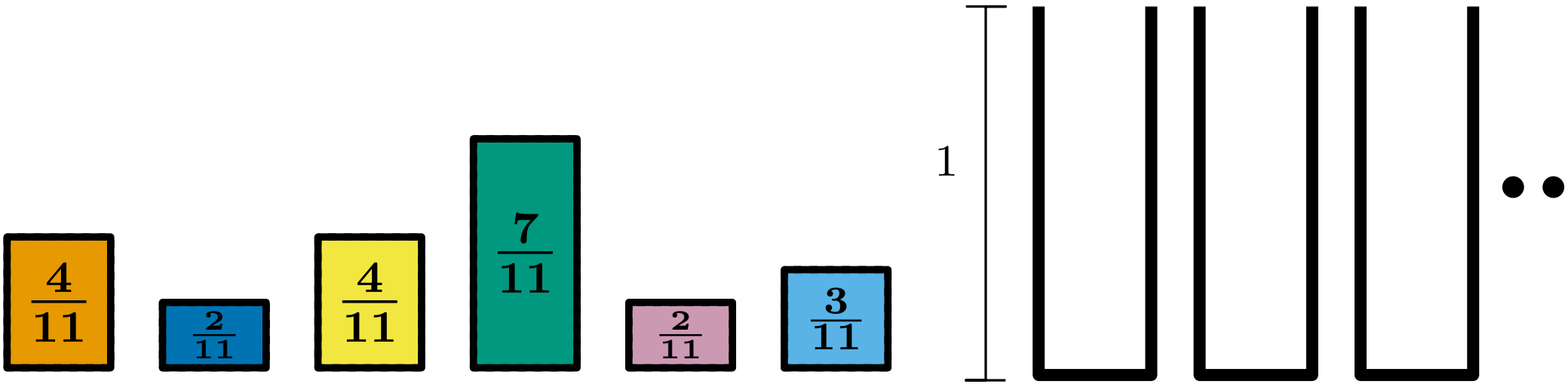

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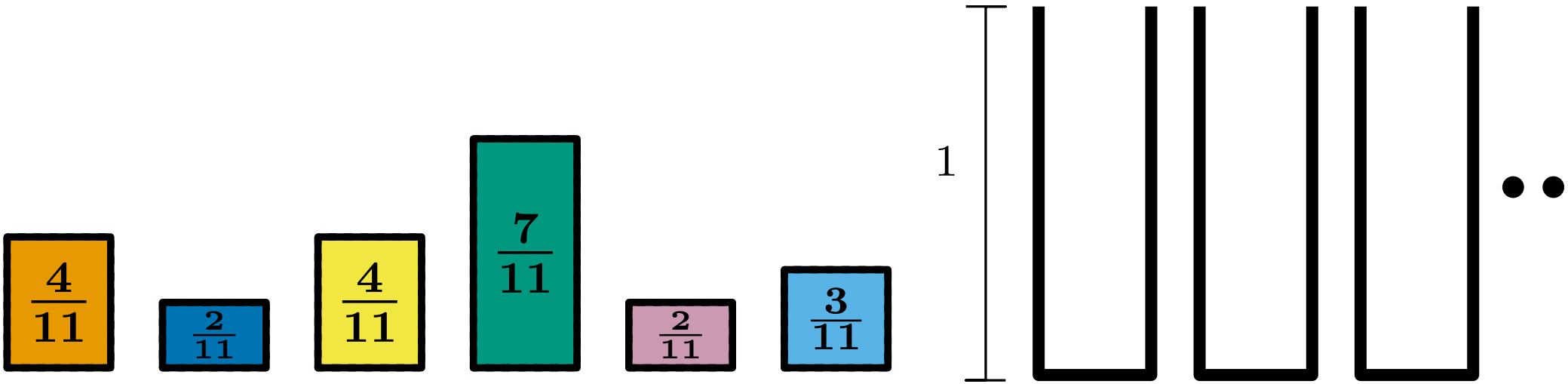

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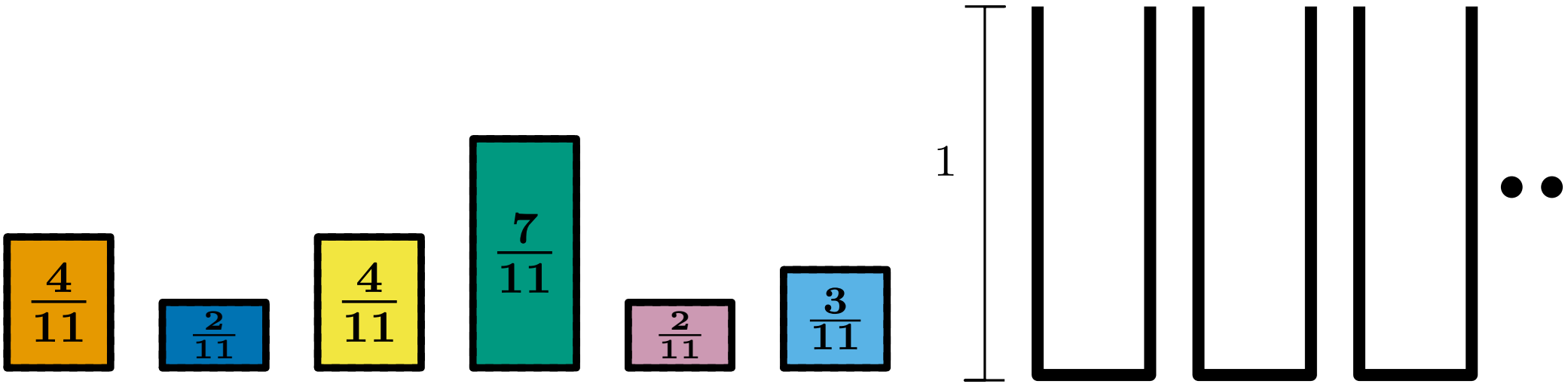

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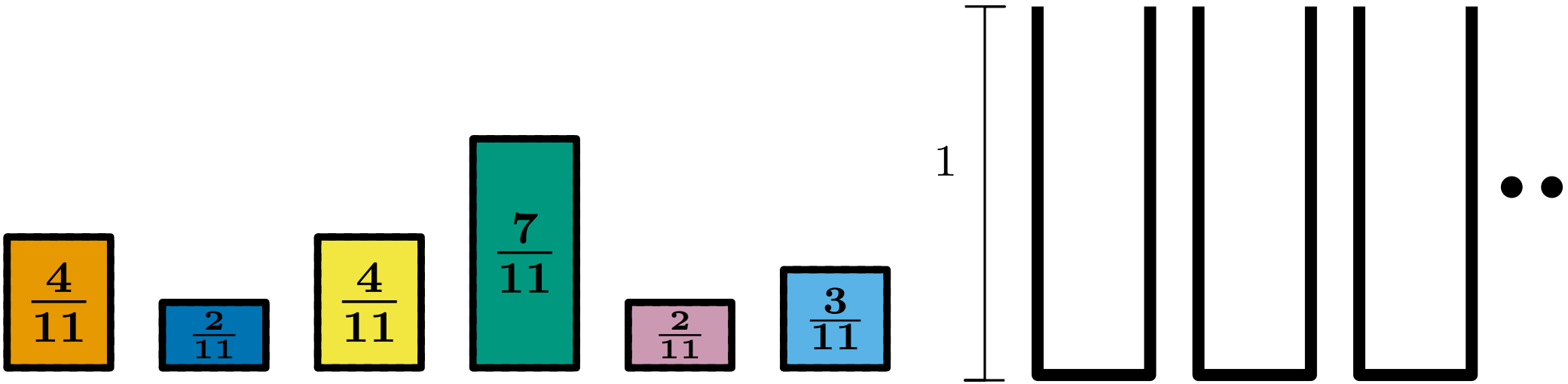

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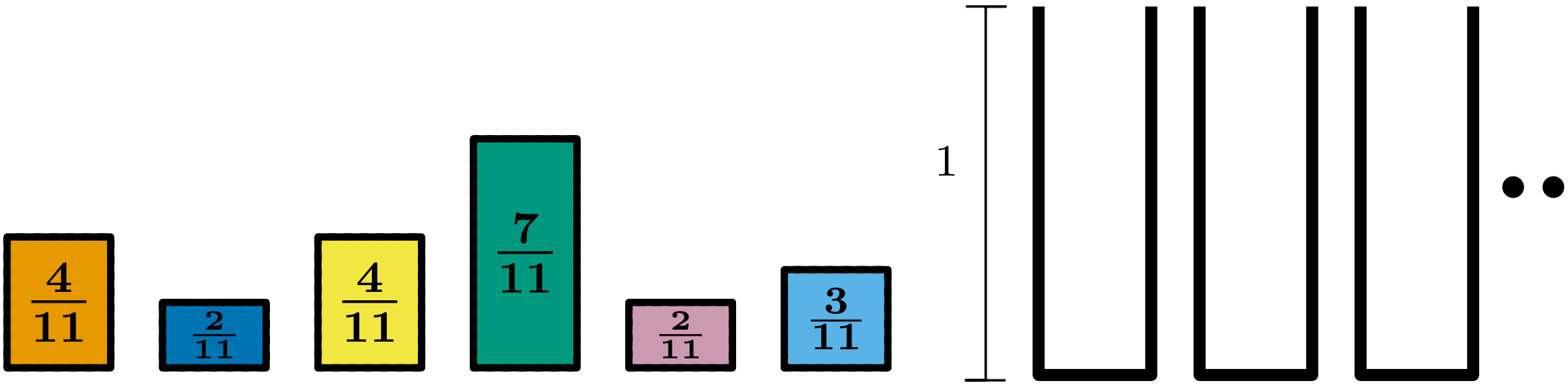

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In fact, FFD gives a solution using s bins with

$$\mathrm{Opt}_b \leqslant s \leqslant \frac{11}{9} \cdot \mathrm{Opt}_b + 1$$

Asymptotic Polynomial time approximation schemes

An asymptotic polynomial time approximation scheme (APTAS)

for problem P is a family of algorithms:

There is a constant $c > 0$ such that

For any constant $\epsilon > 0$, A_ϵ runs in polynomial time (poly-time)

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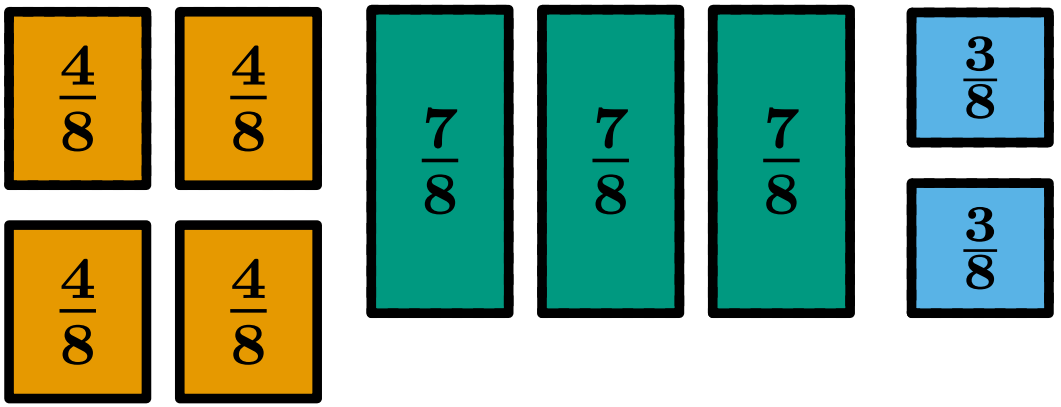
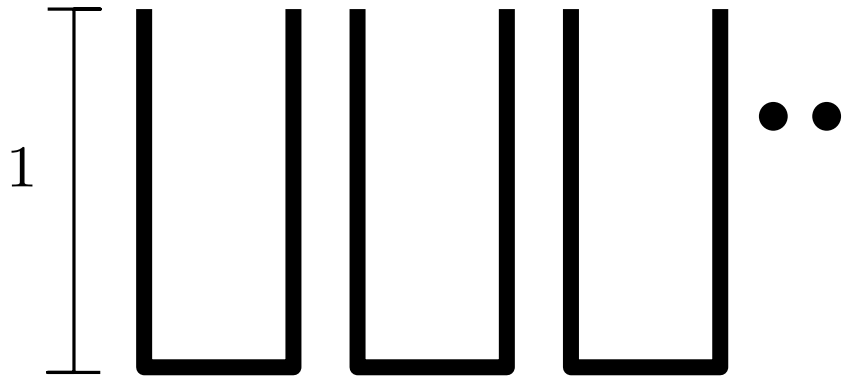
We will see an APTAS for BinPacking

which uses at most $(1+\epsilon) \cdot \mathrm{Opt} + 1$ bins

A special case of BINPACKING

We will now see a special case of BINPACKING which we can solve optimally in polynomial time





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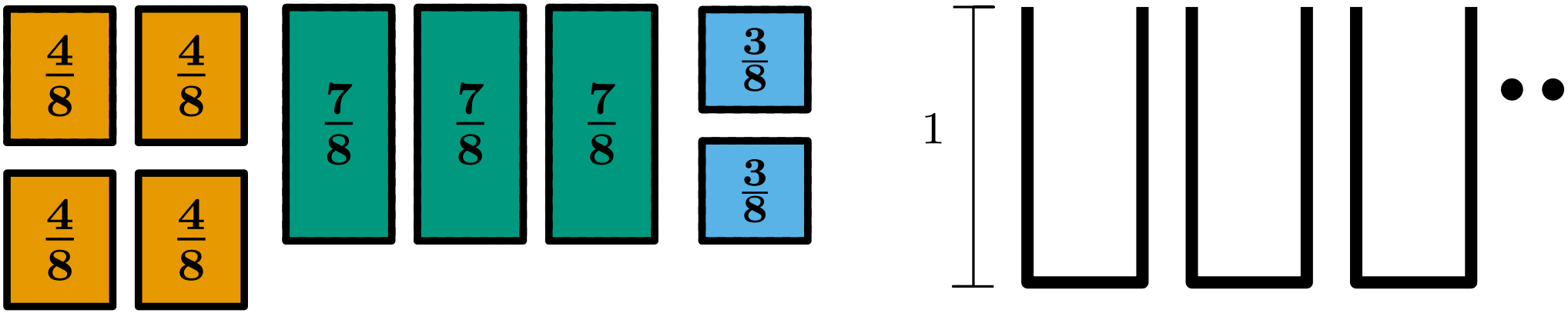

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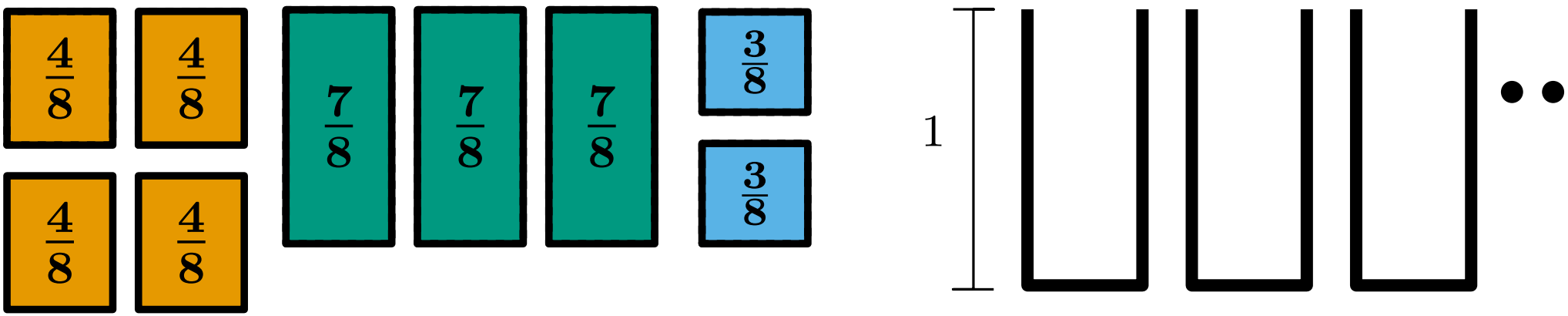

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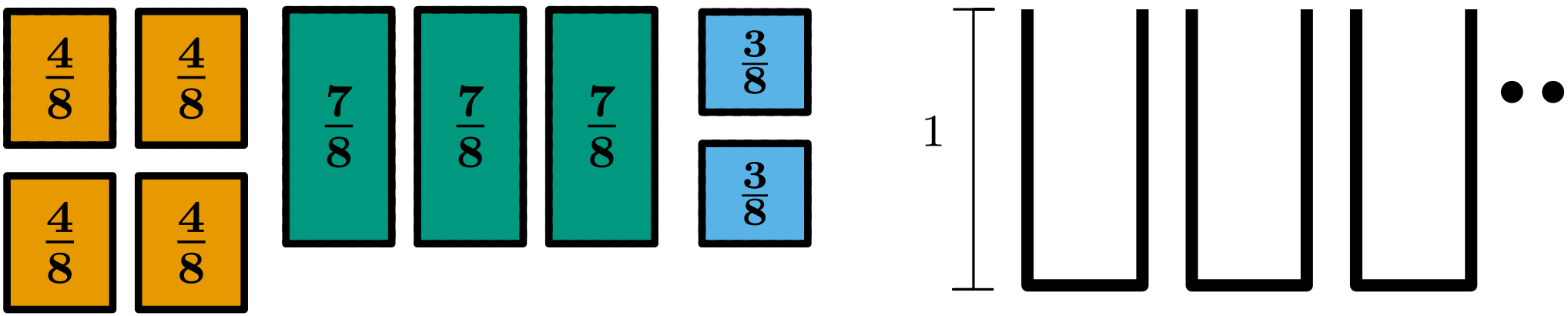

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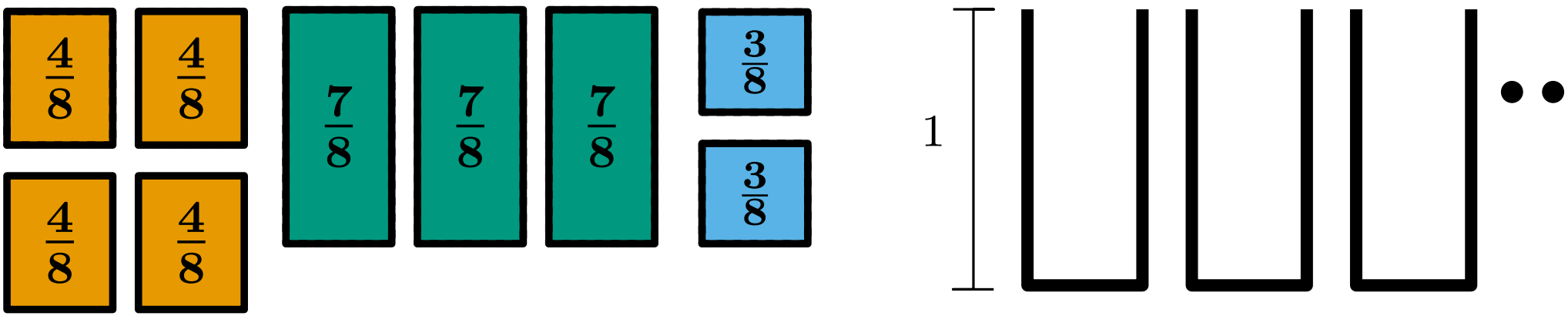

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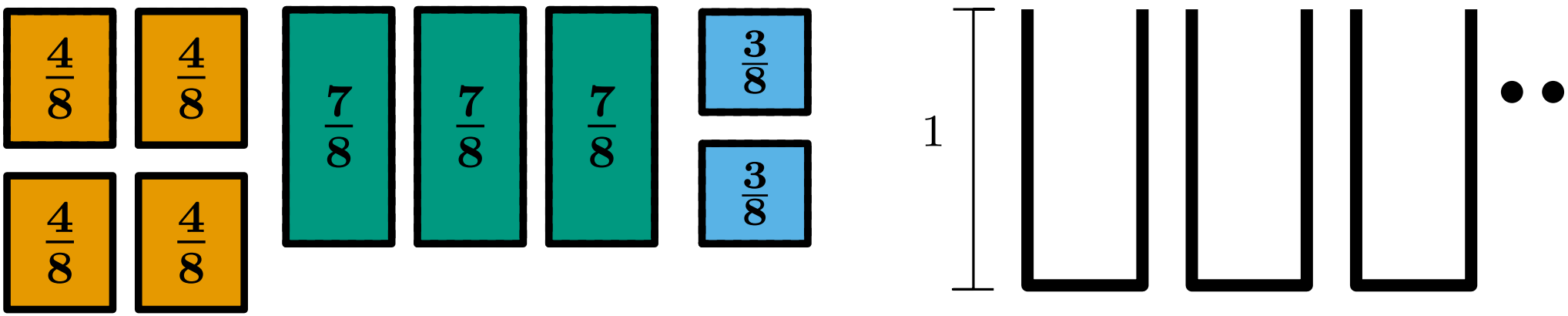

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How many different ways are there to fill a single bin?

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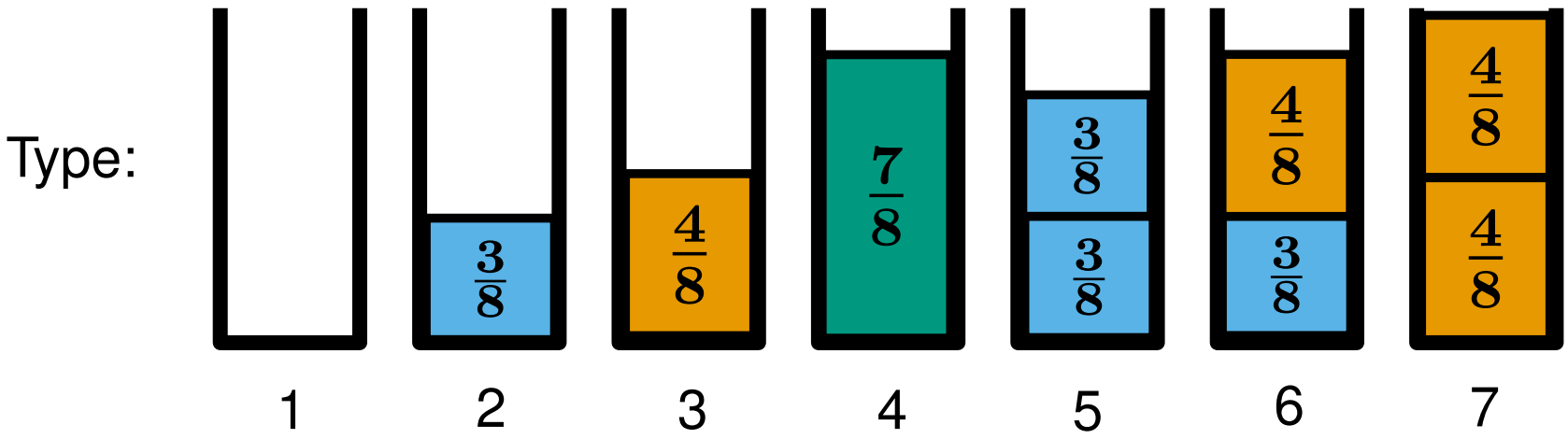

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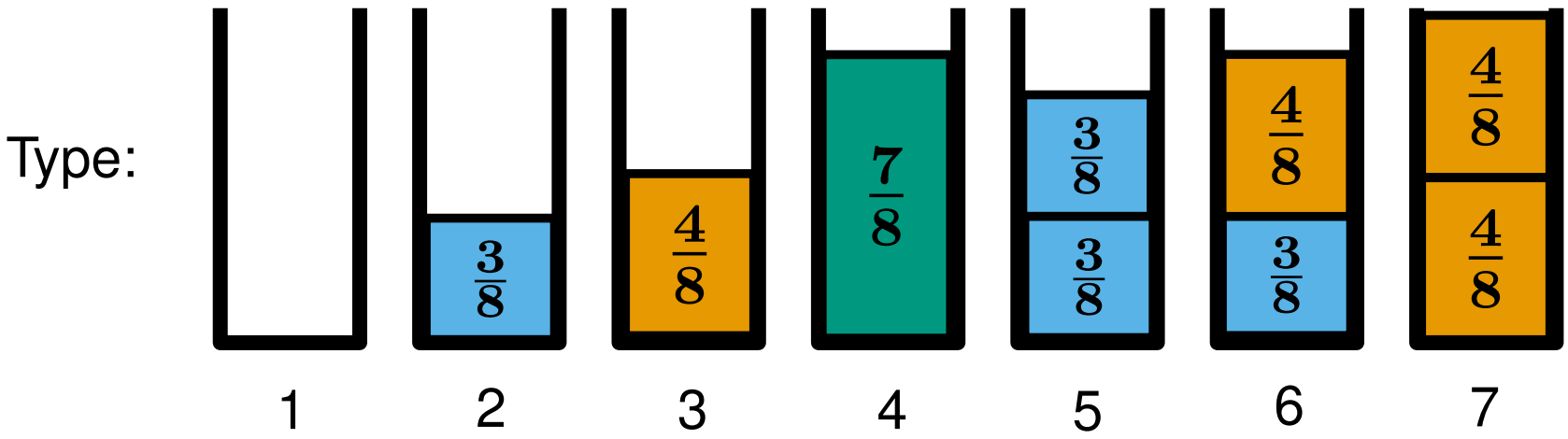

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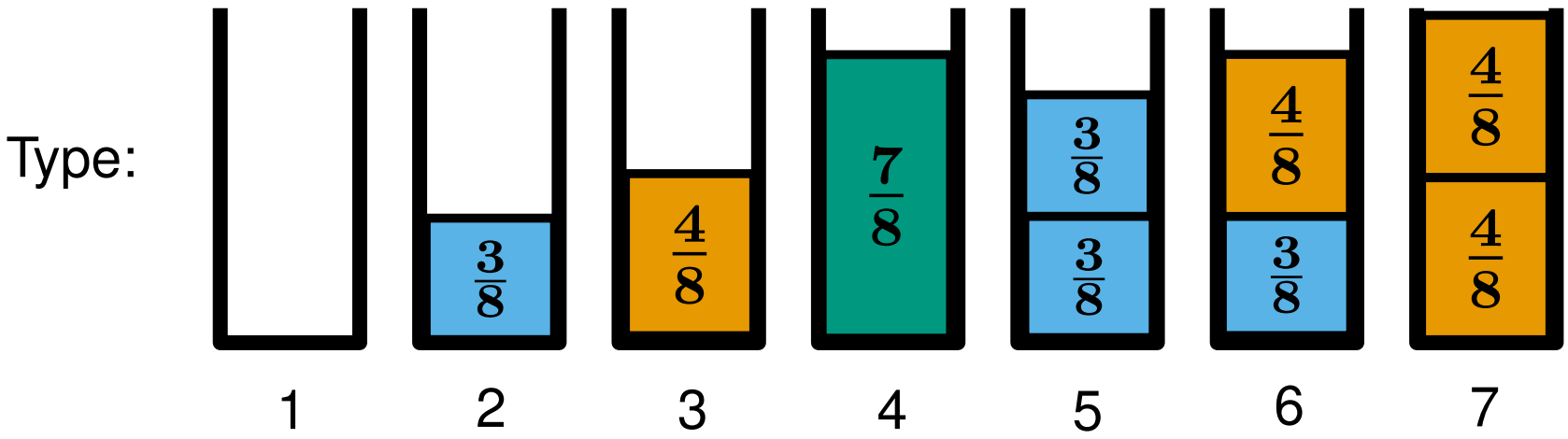

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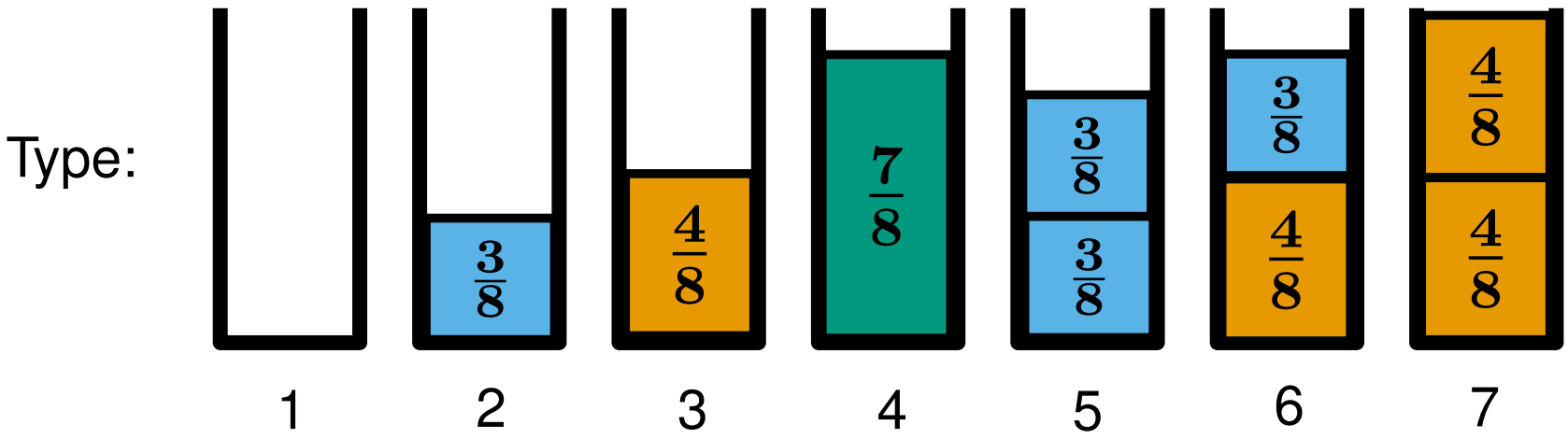

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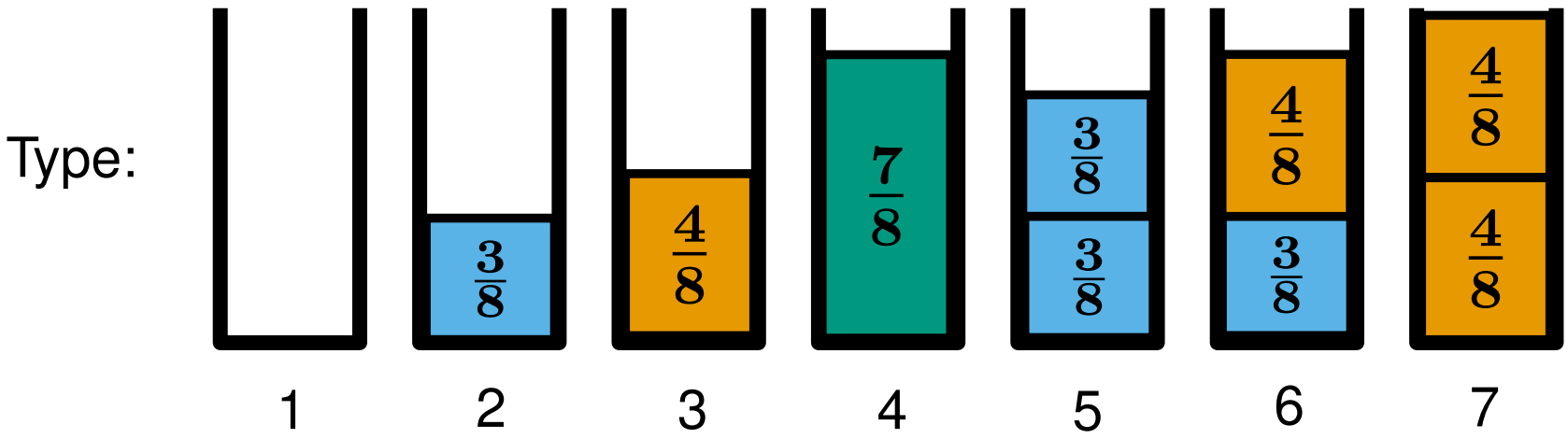

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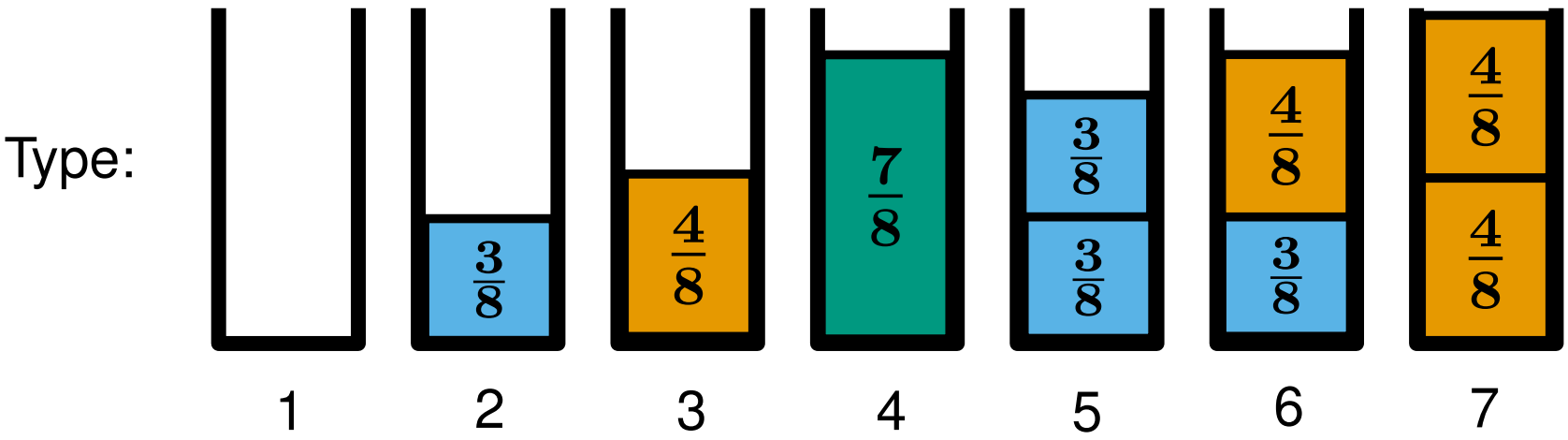

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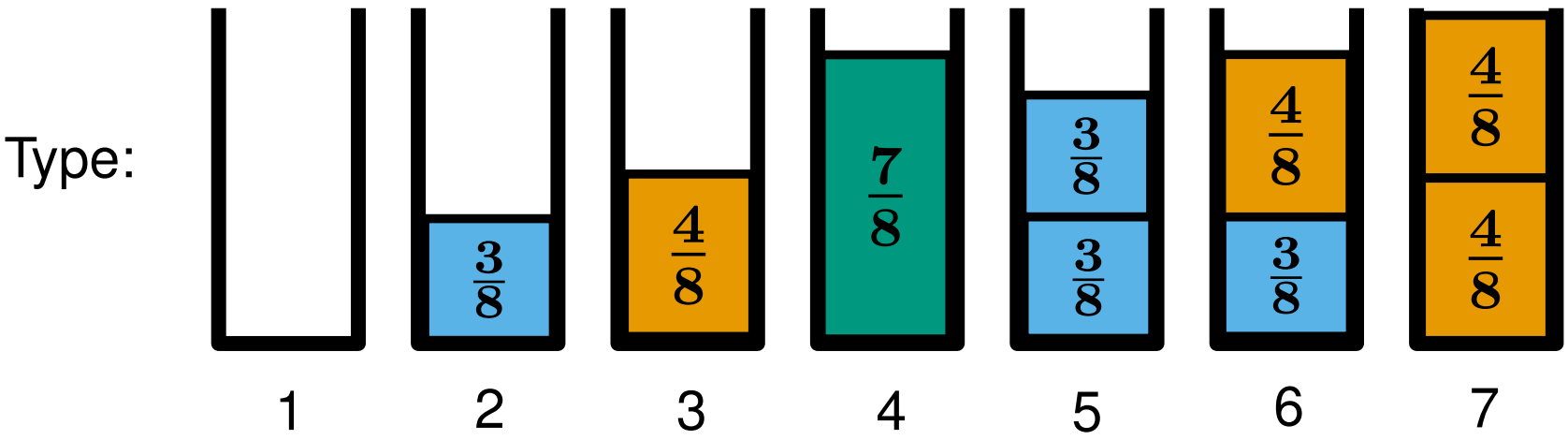

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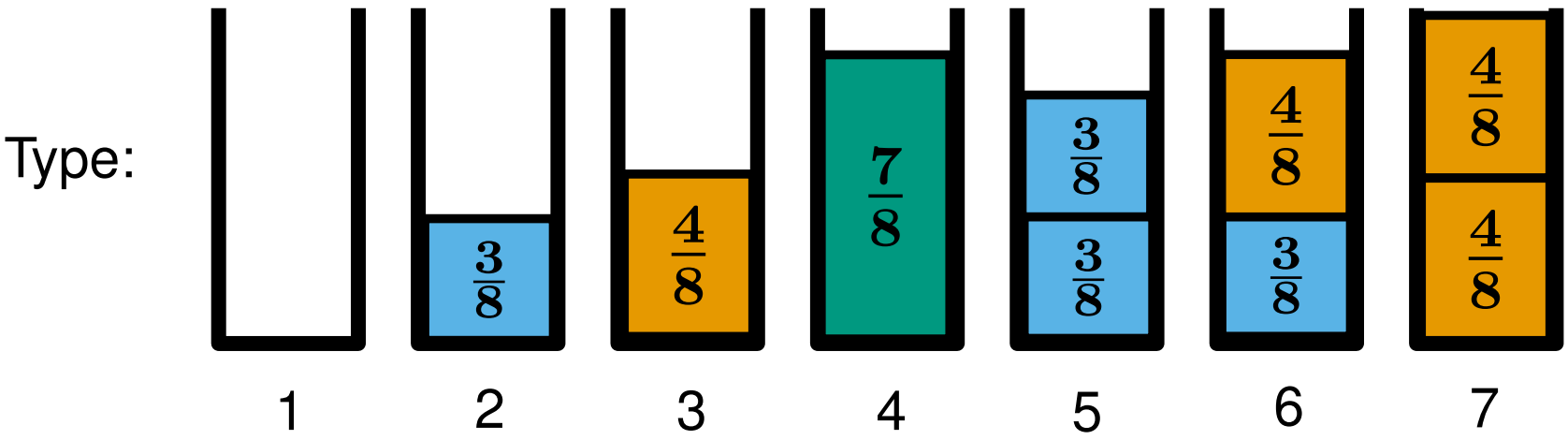

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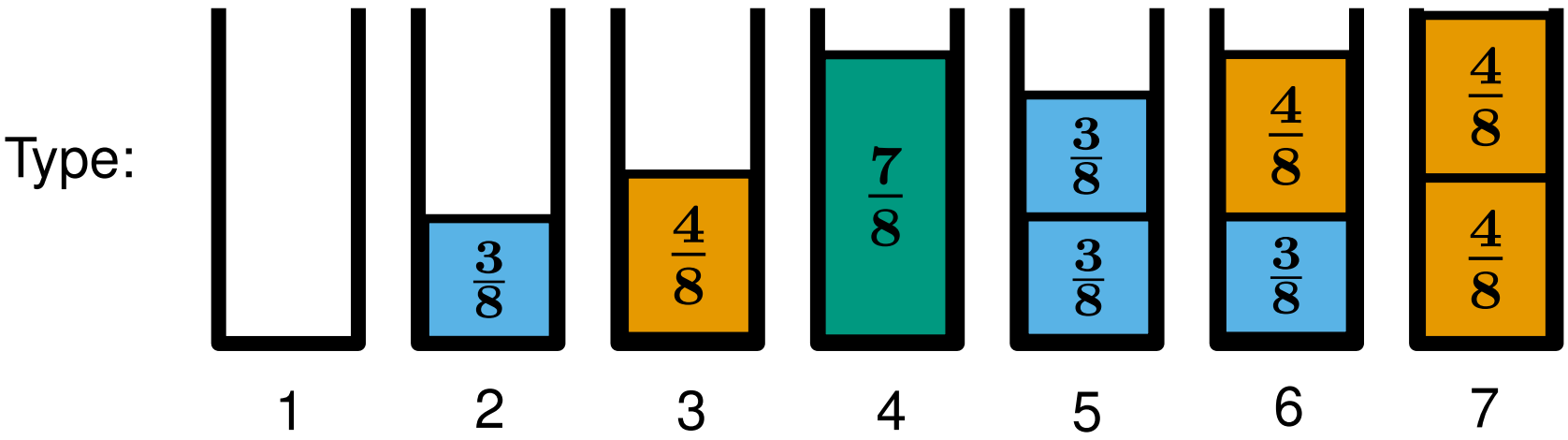

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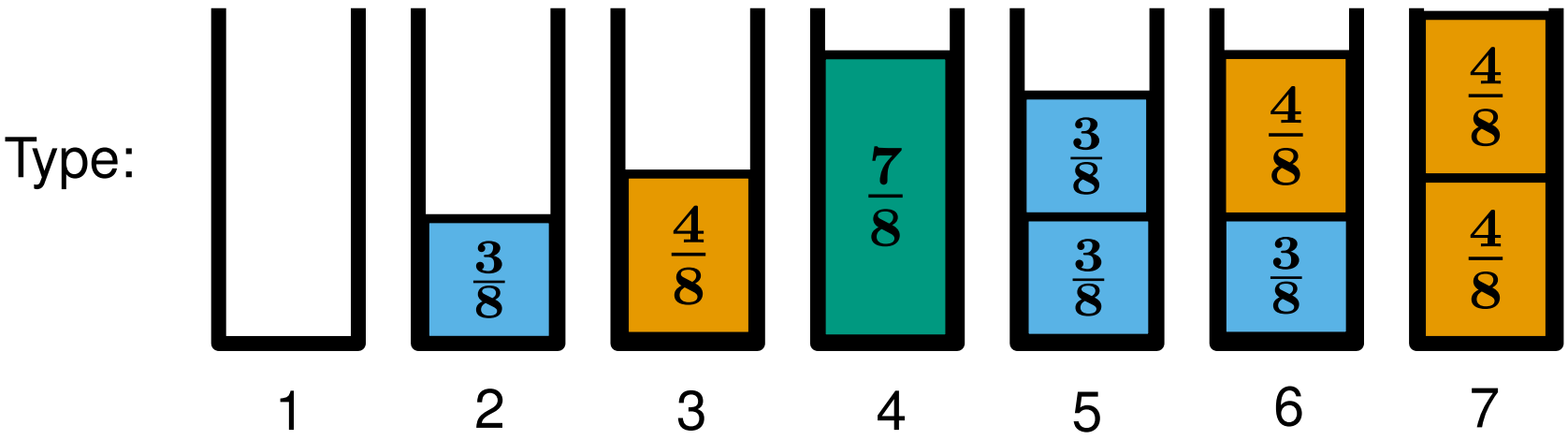

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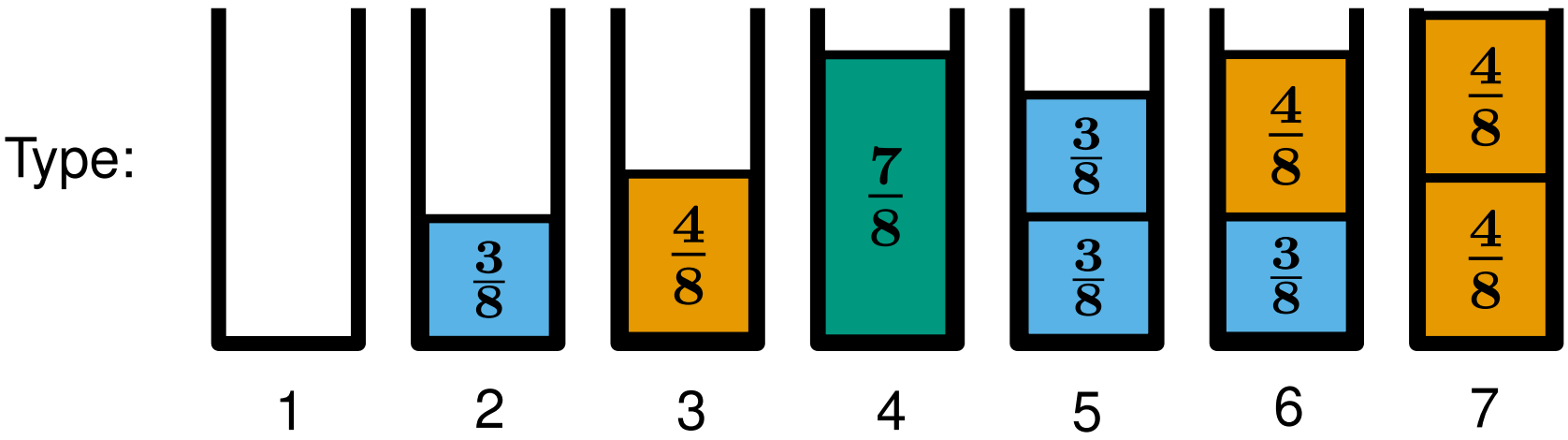

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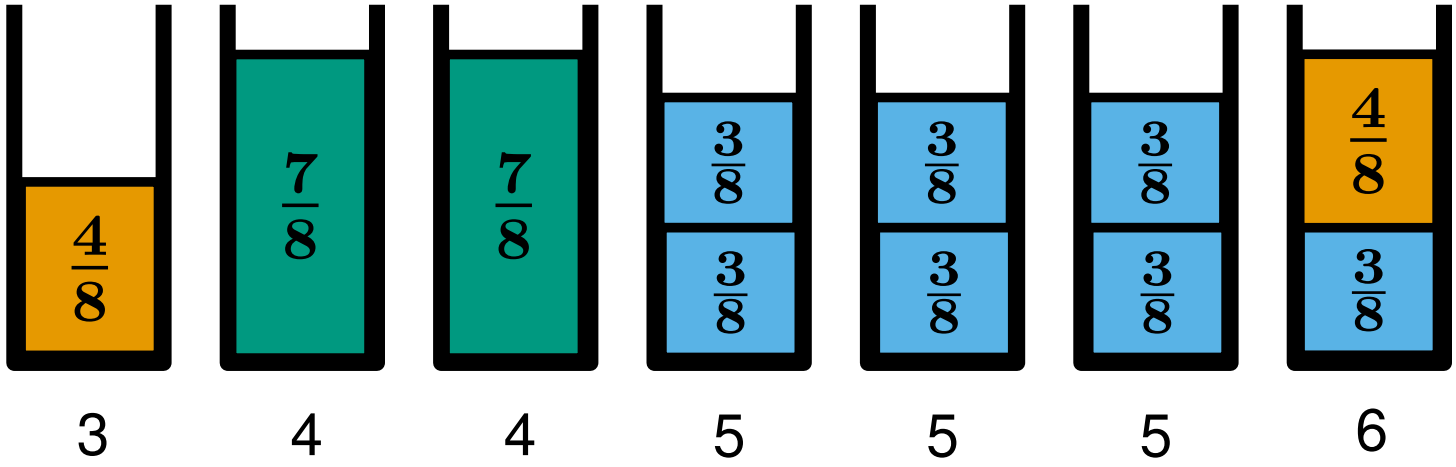

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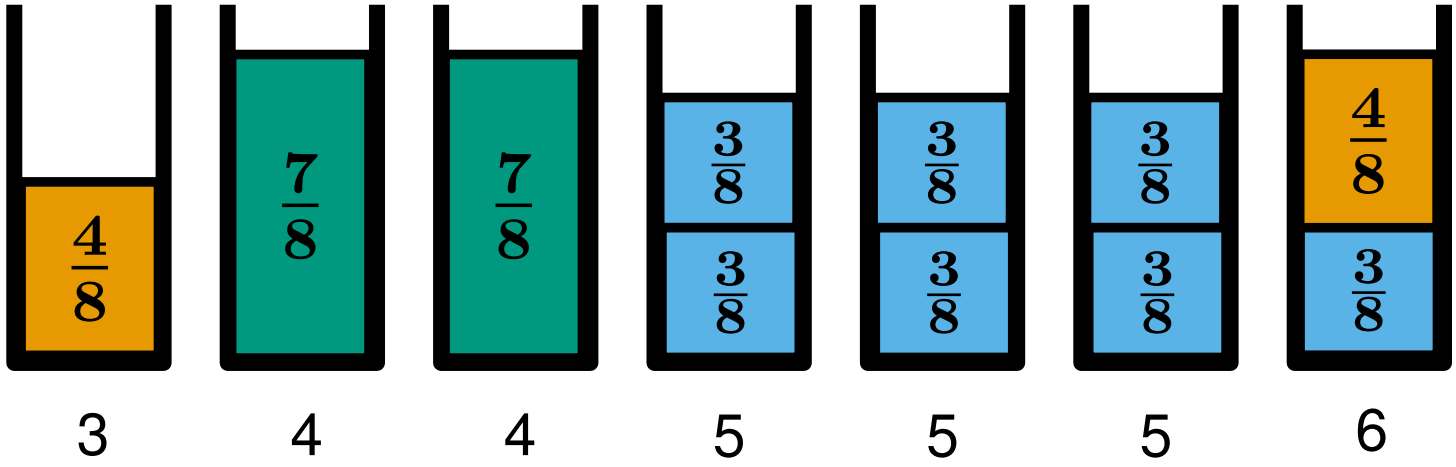

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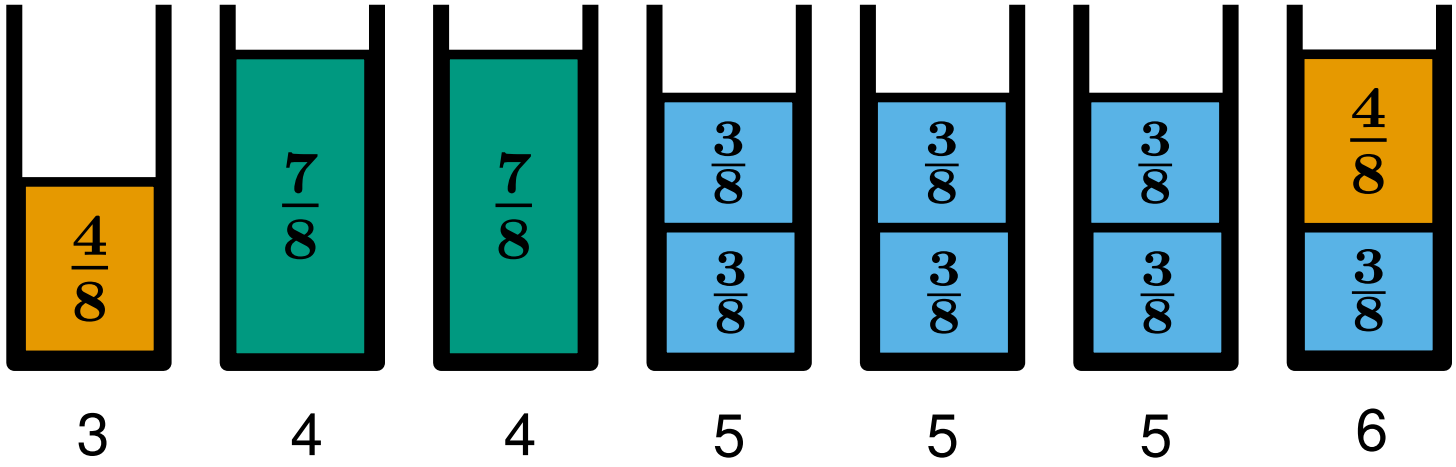

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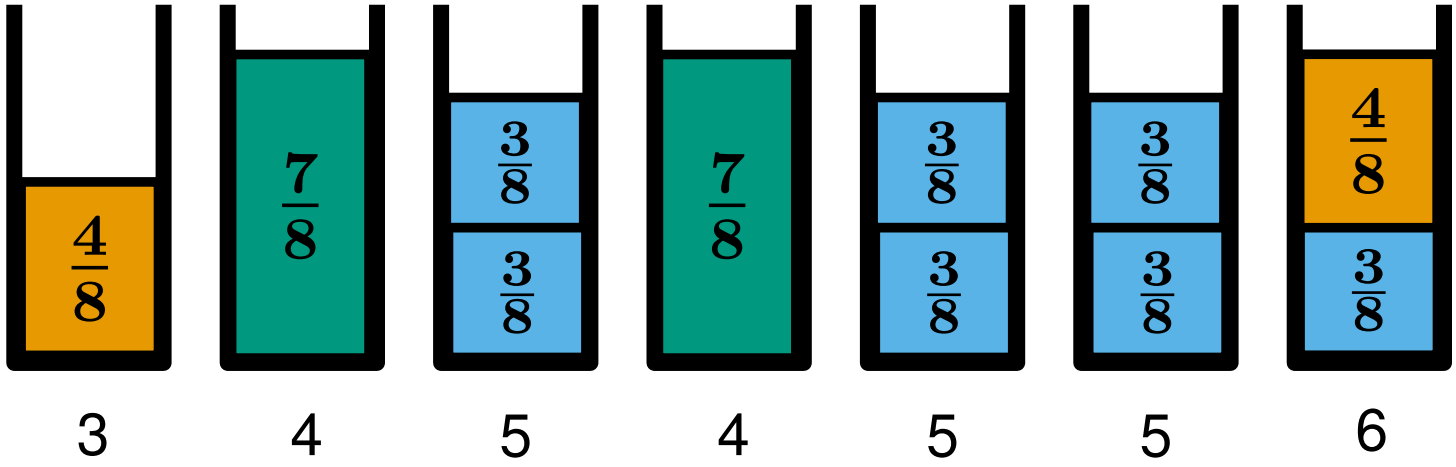

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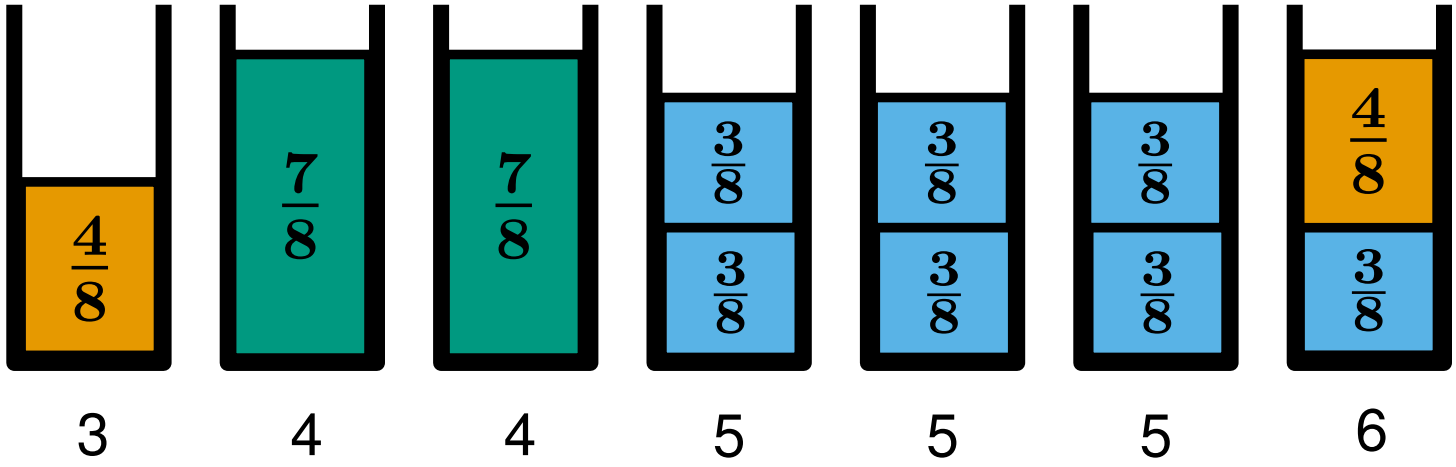

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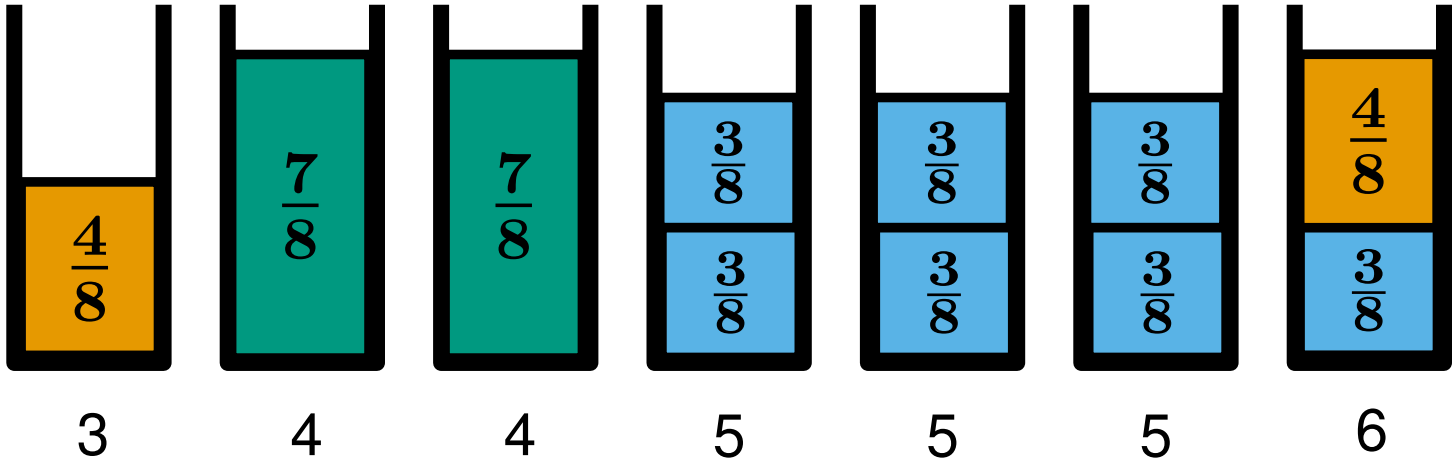

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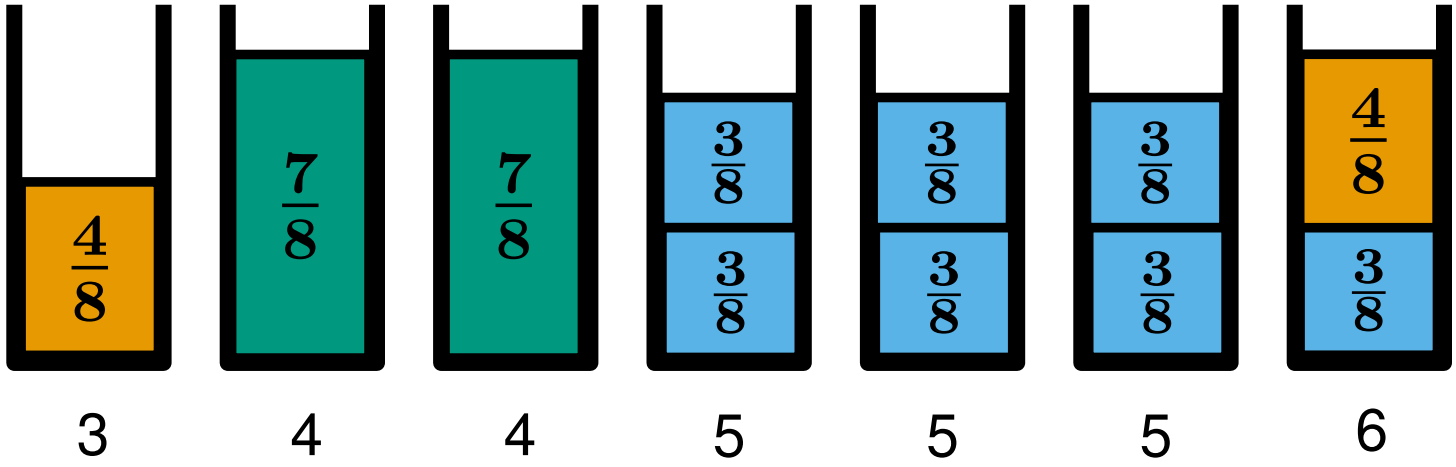

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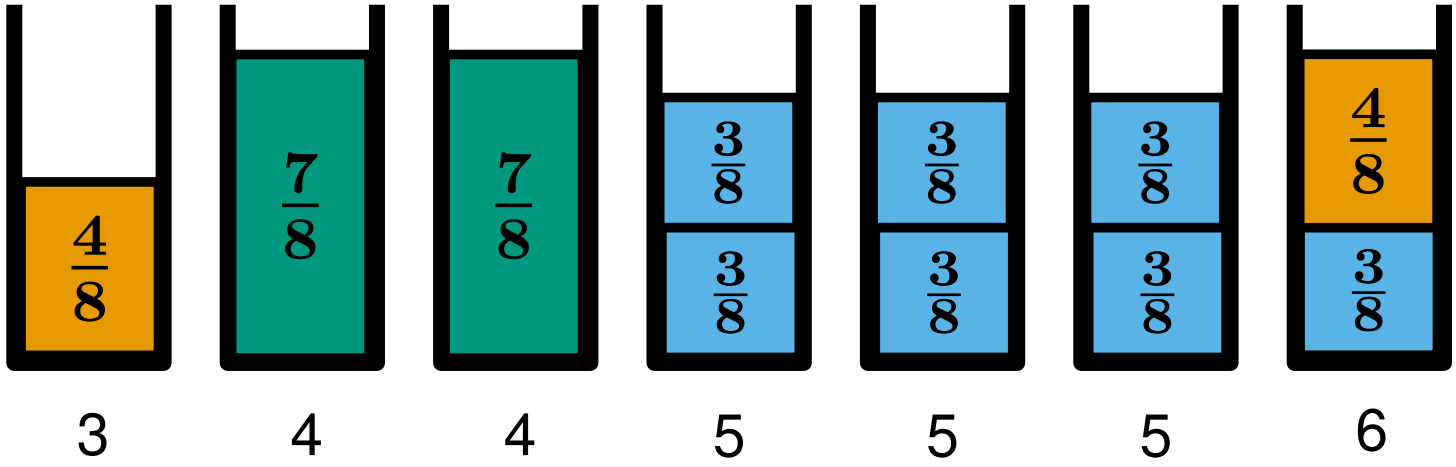

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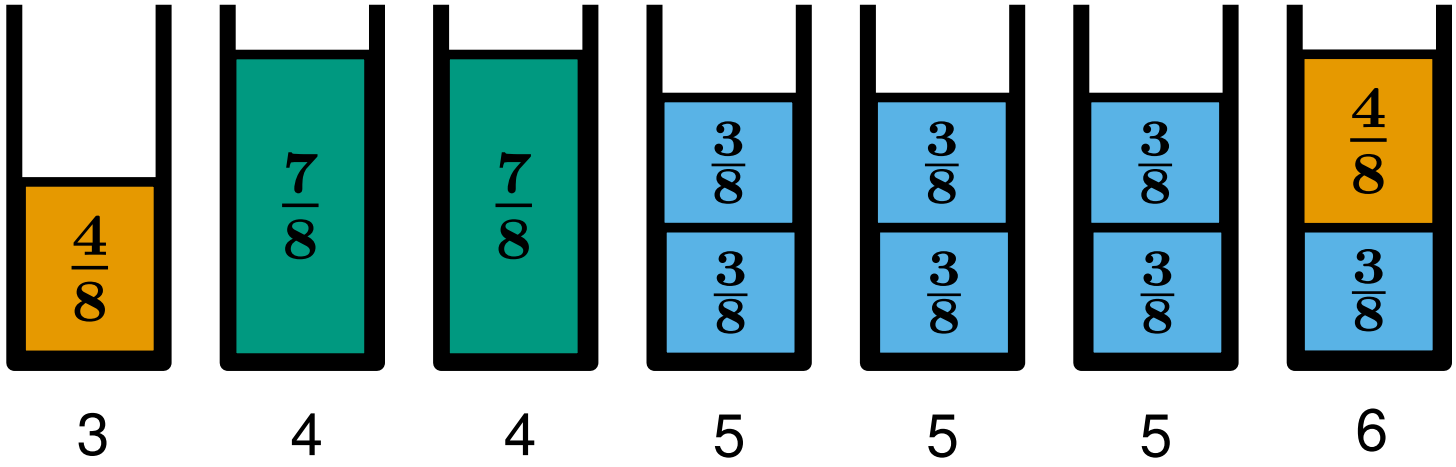
A special case of BINPACKING

c_s diff. item sizes



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$2\times$ Type 4
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- We can completely describe any packing of S into $b \leqslant n$ bins

  by the number of bins of each type used

  we ignore rearrangement of bins

- How *different* packings can there be?

  There are between 0 and n bins of any type

  There are at most $(c_b + 1)^{c_s}$ different types

  number of packings $\leqslant$

c_b items fit in each bin

A special case of BinPacking

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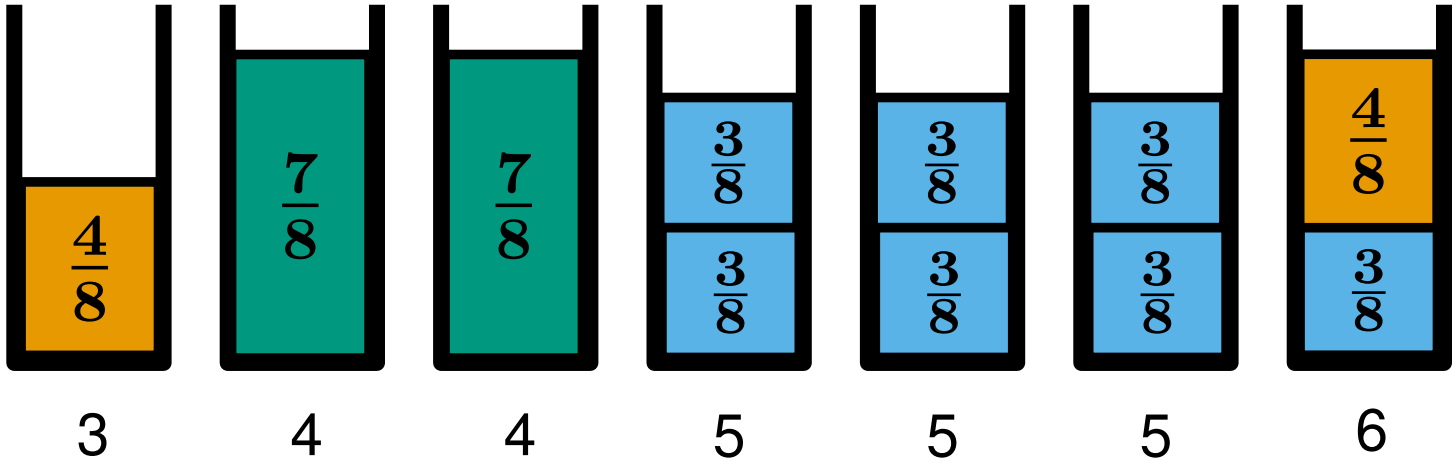

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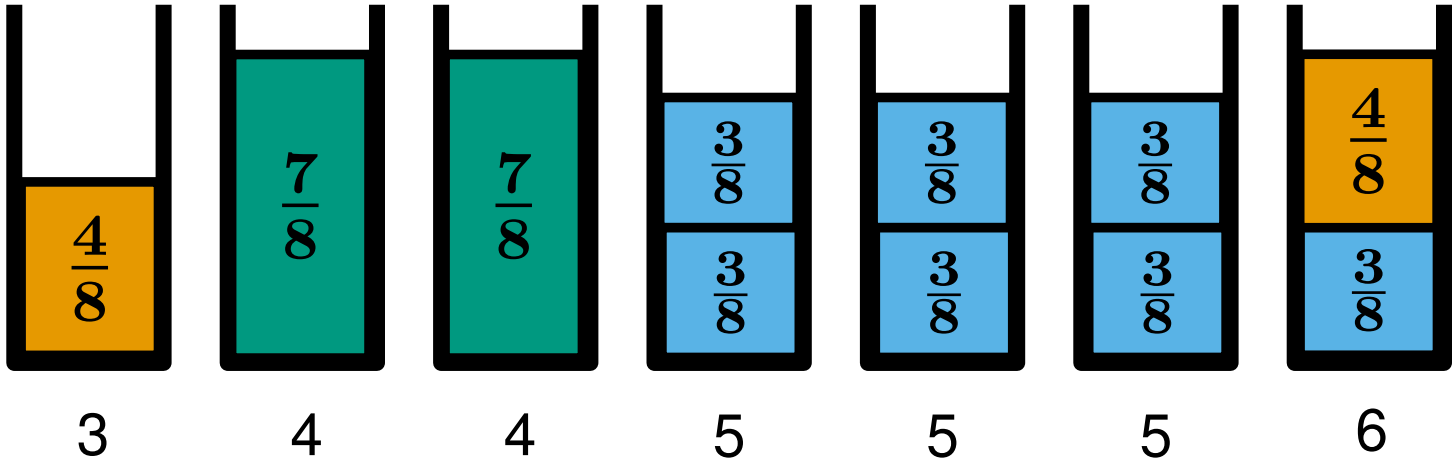

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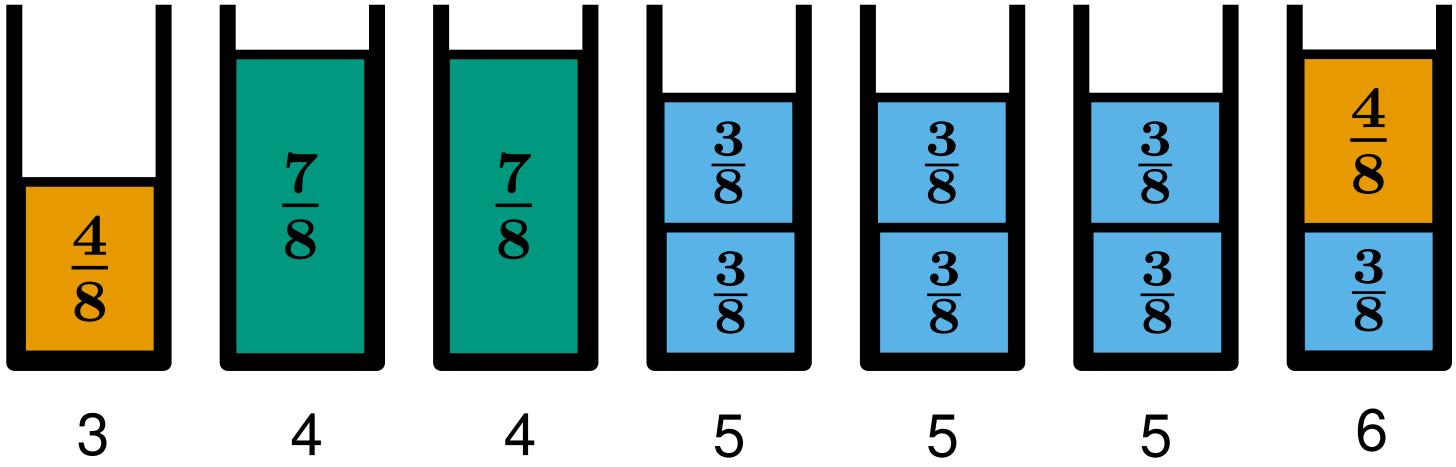

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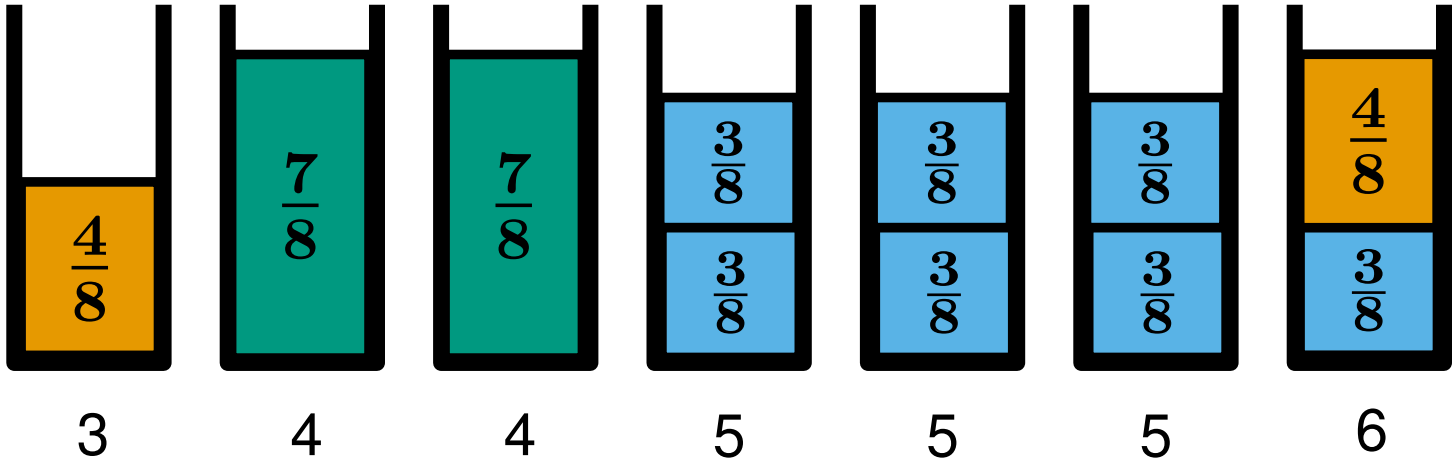

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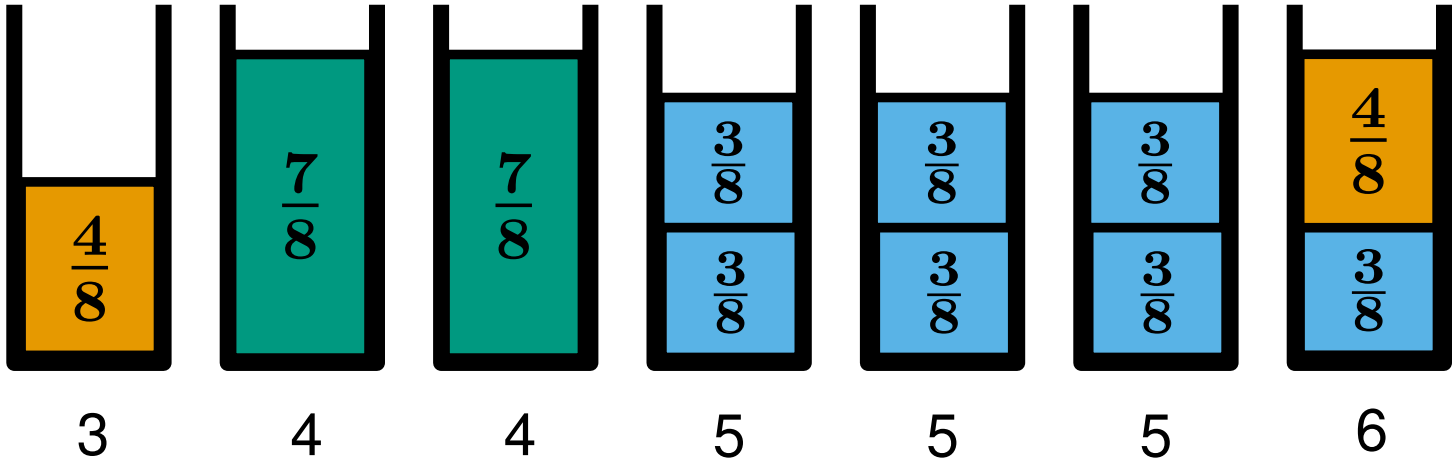

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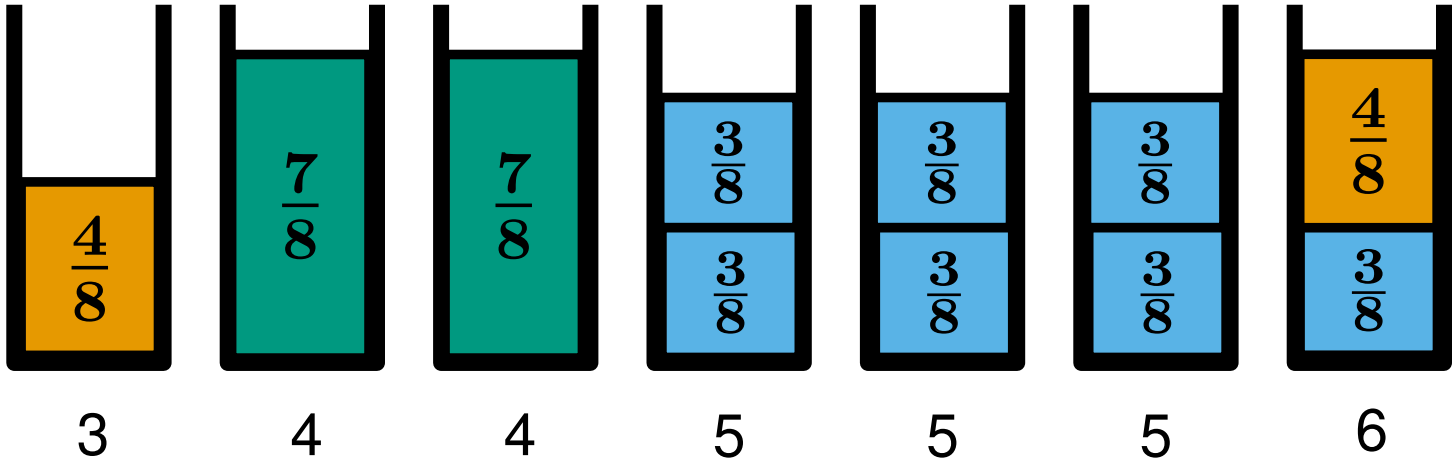

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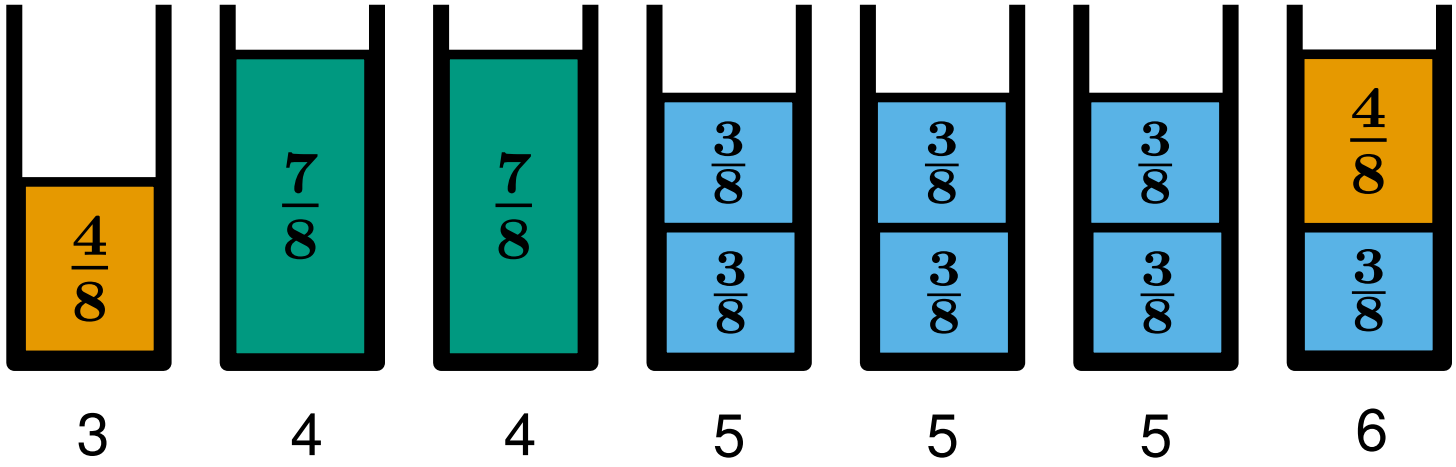

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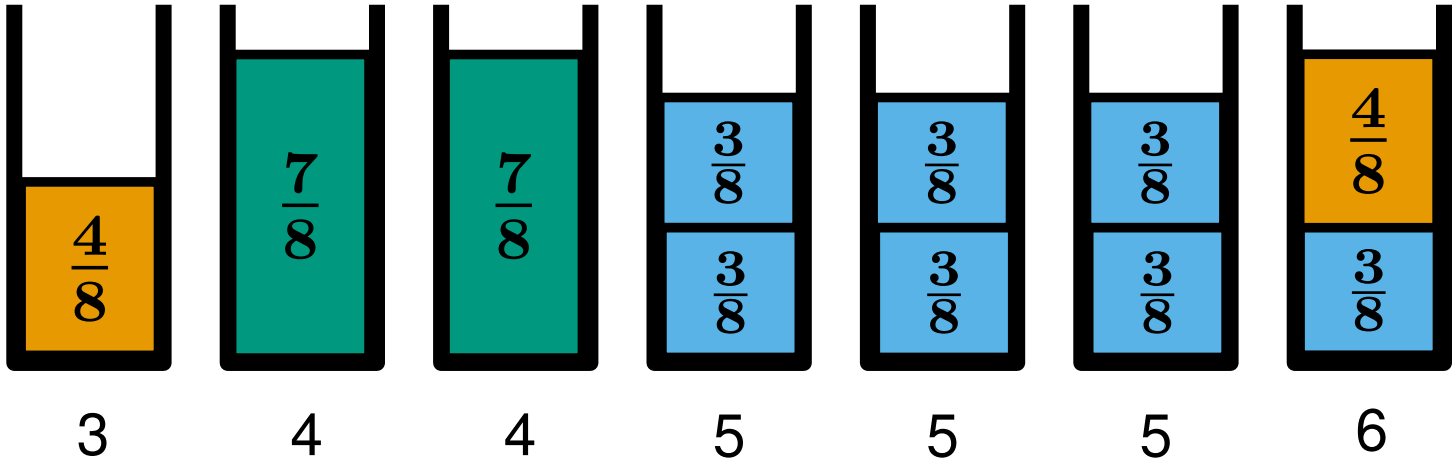

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- This takes $O(n \cdot (n+1)^{(c_b+1)^{c_s}})$ time and exactly solves BinPacking

  (i.e. it outputs an optimal packing)

Towards an APTAS

The APTAS for BINPACKING will use a four step process:

Step 1 Remove all the *small* items

Only c_b of the remaining large items will fit into a single bin

Step 2 Divide the items into a constant number of groups

Sizes of items in each group are then rounded up

to match the size of the largest member

(and the largest group is removed)

This will leave only c_s different item sizes

Step 3 Use the poly-time algorithm to optimally pack the remaining special case

the *constants* c_b and c_s will depend on ϵ

Step 4 Reverse the grouping from **Step 2** and then

greedily pack all the small items removed in **Step 1**

Remember that the goal is to use b bins where $\mathrm{Opt} \leqslant b \leqslant (1+\epsilon) \cdot \mathrm{Opt} + c$ (for some c)

Removing the small items

Lemma Let $0 < \epsilon < 1$. Given a packing of the items $a \in S$ with size $a > \epsilon/2$ into b bins, in *polynomial time* we can find a packing of *all items* in S which either uses:

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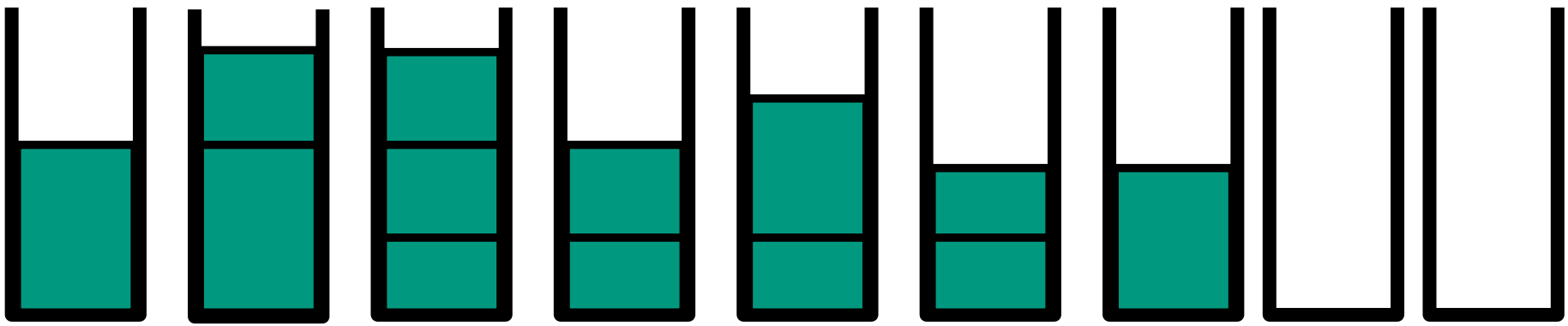

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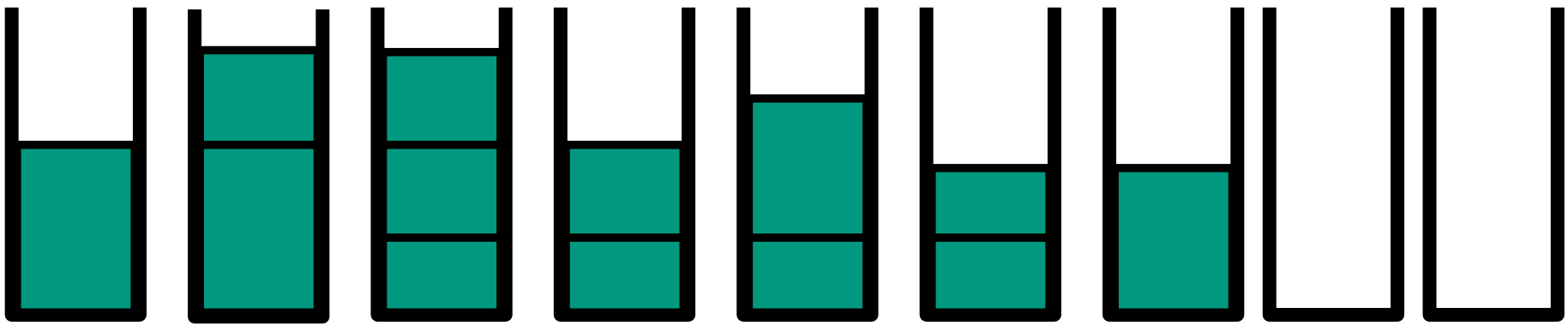

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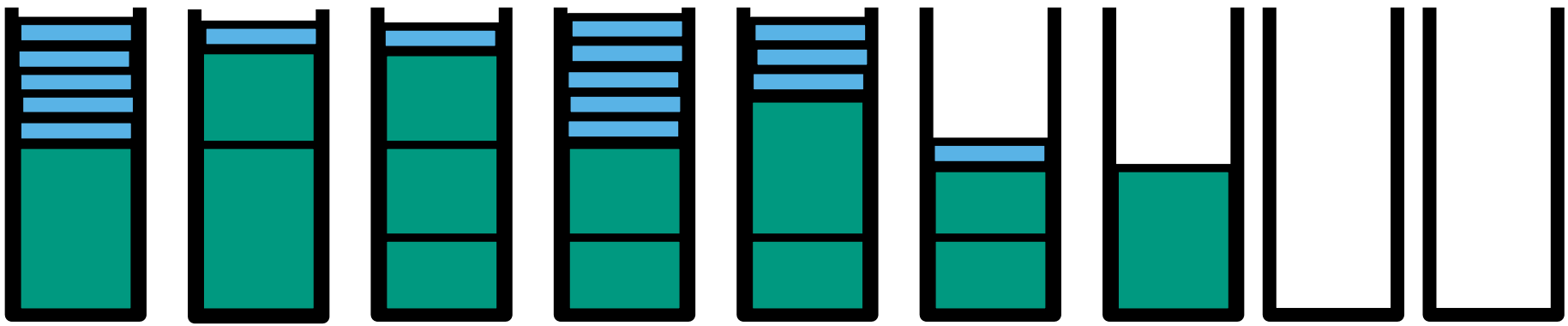

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pack the small items anywhere you like - just but don't use an extra bin unless you have to

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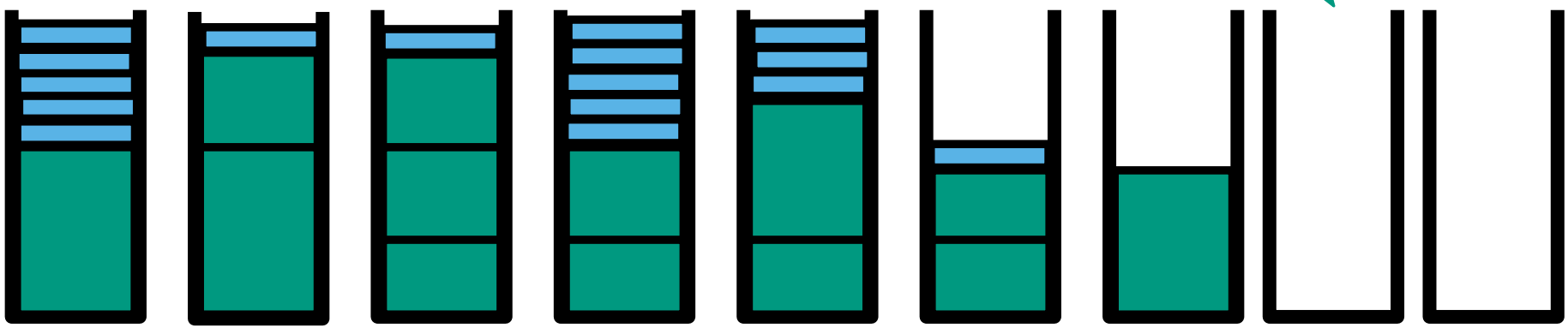

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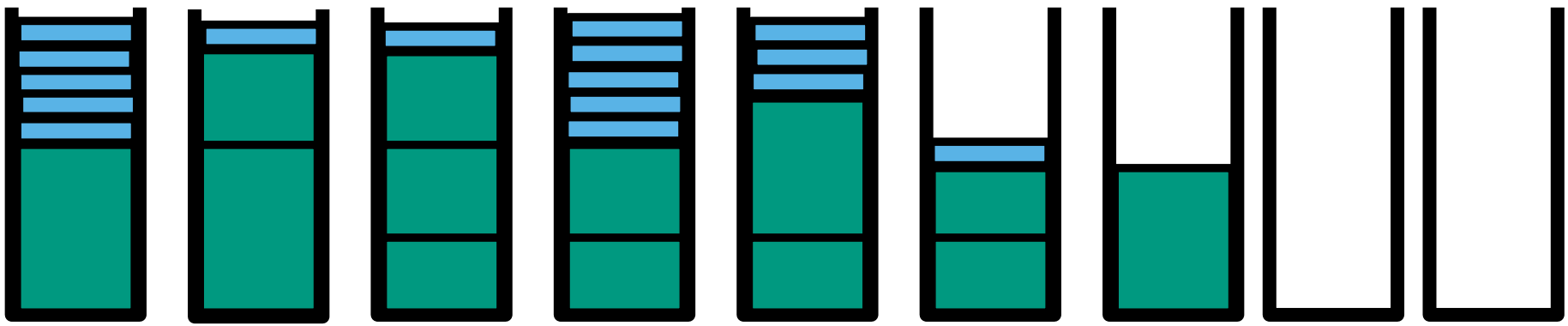

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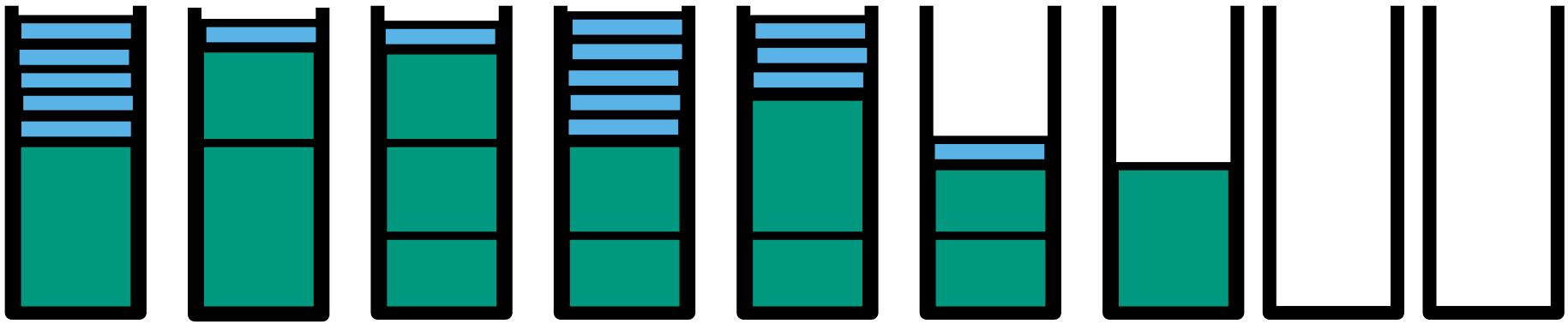

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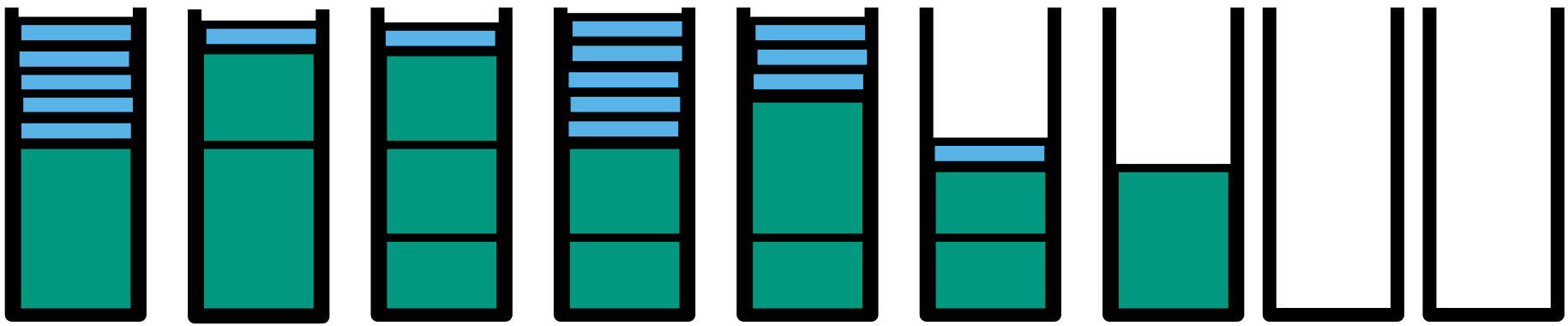

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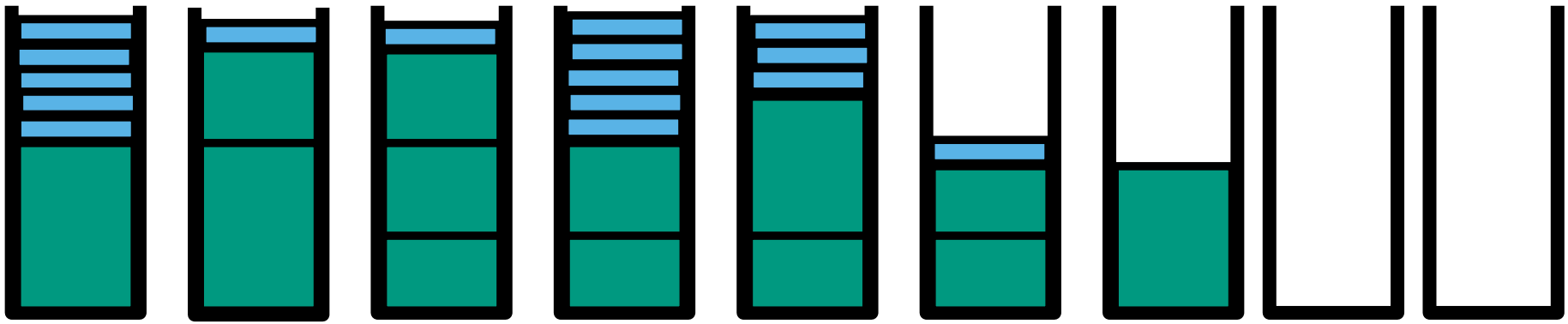

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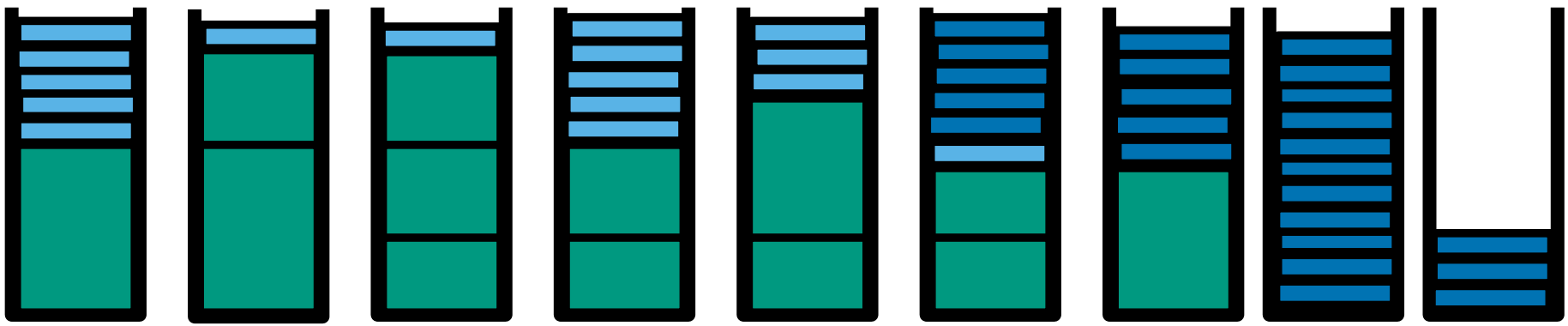

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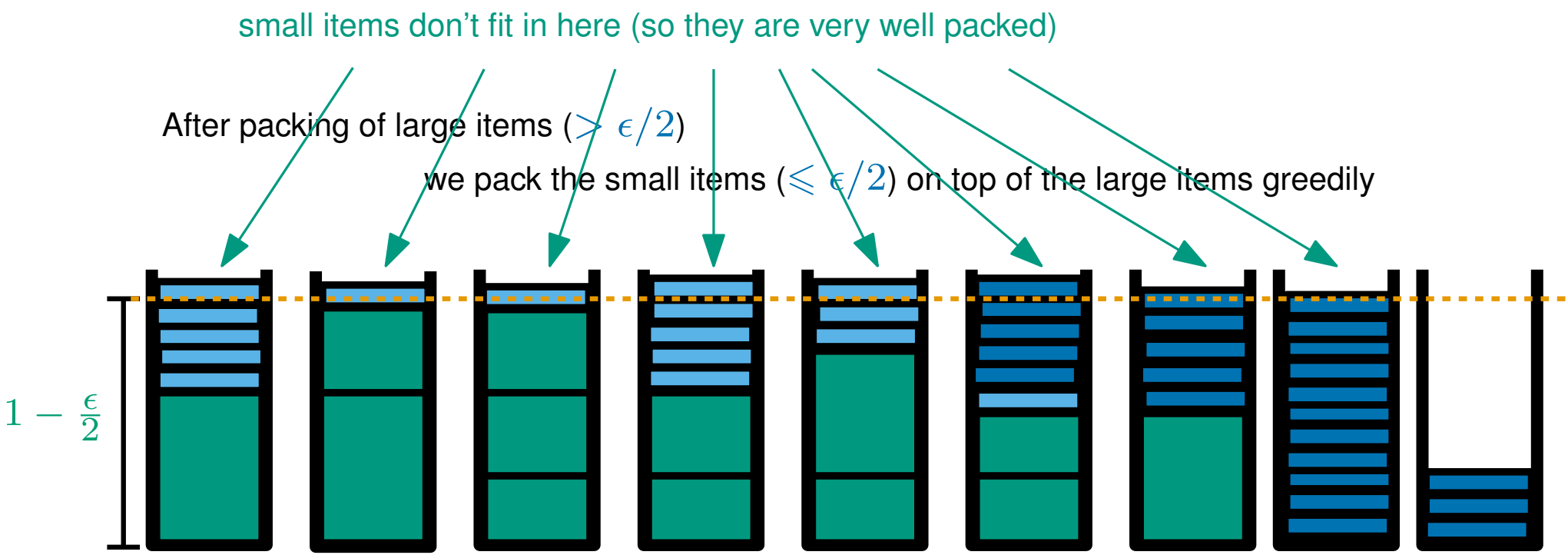

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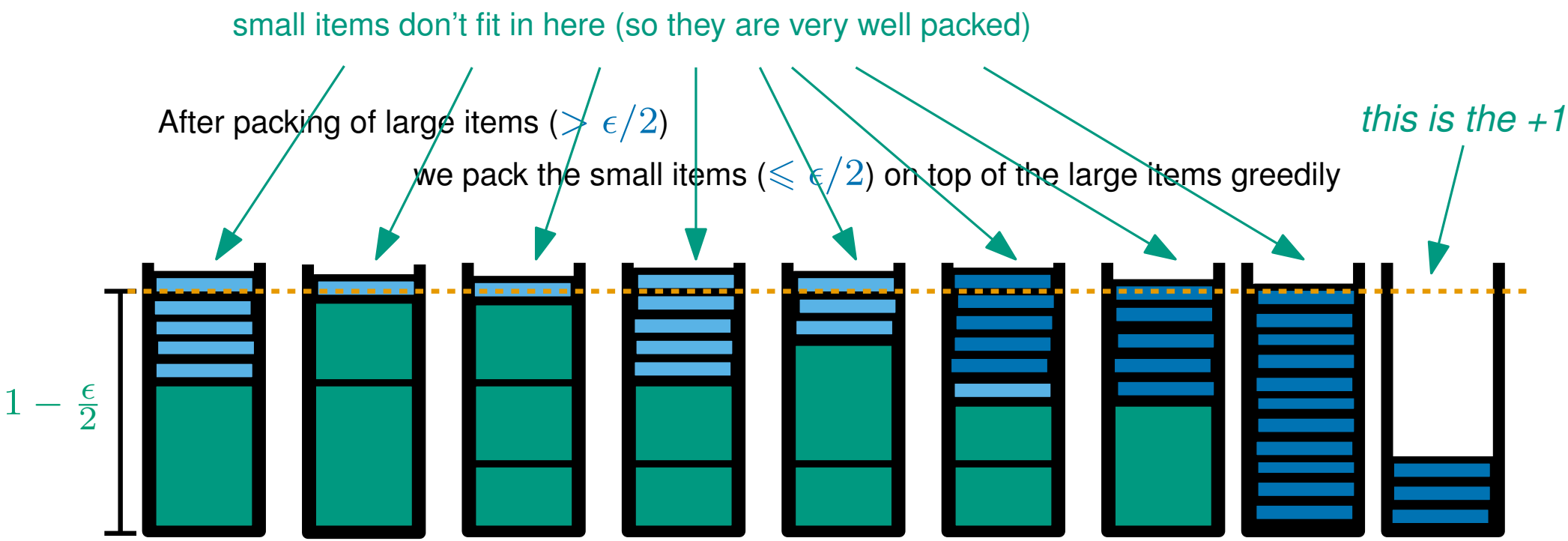

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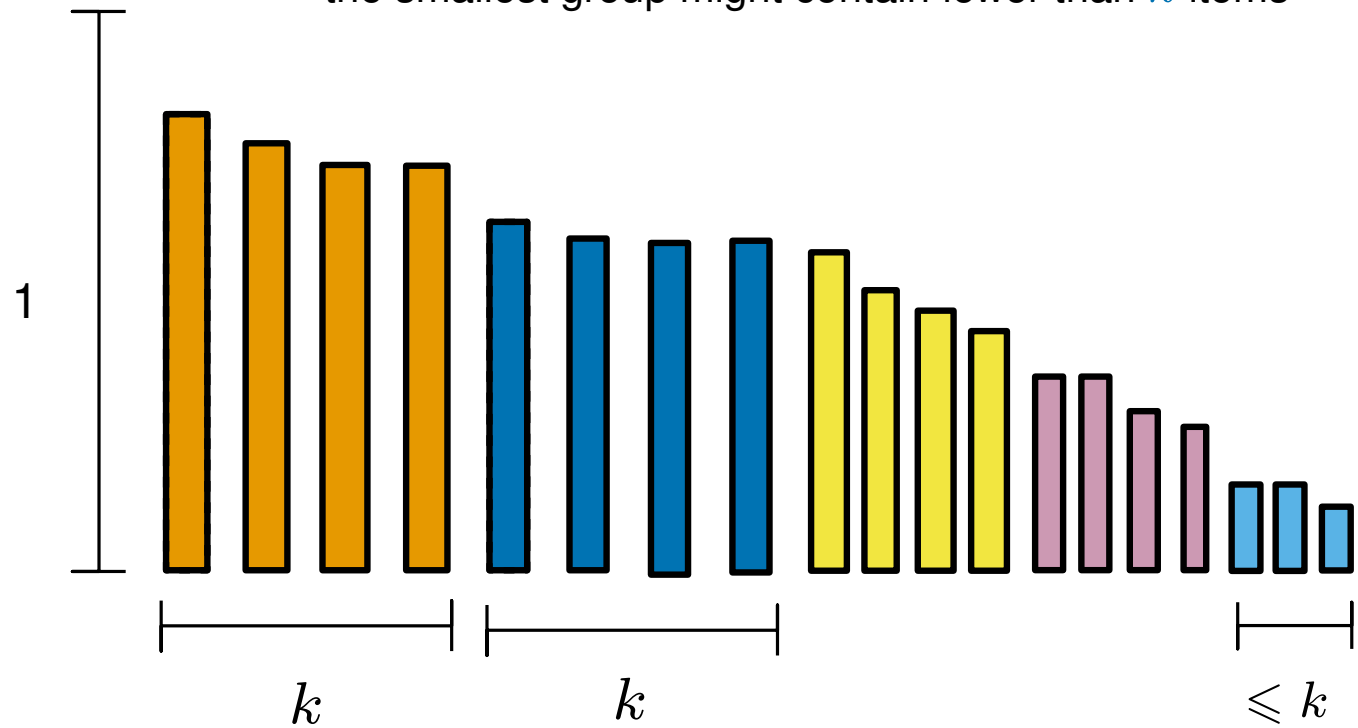
Reducing the number of item sizes

- We divide the items by size, into groups of size k

the smallest group might contain fewer than k items



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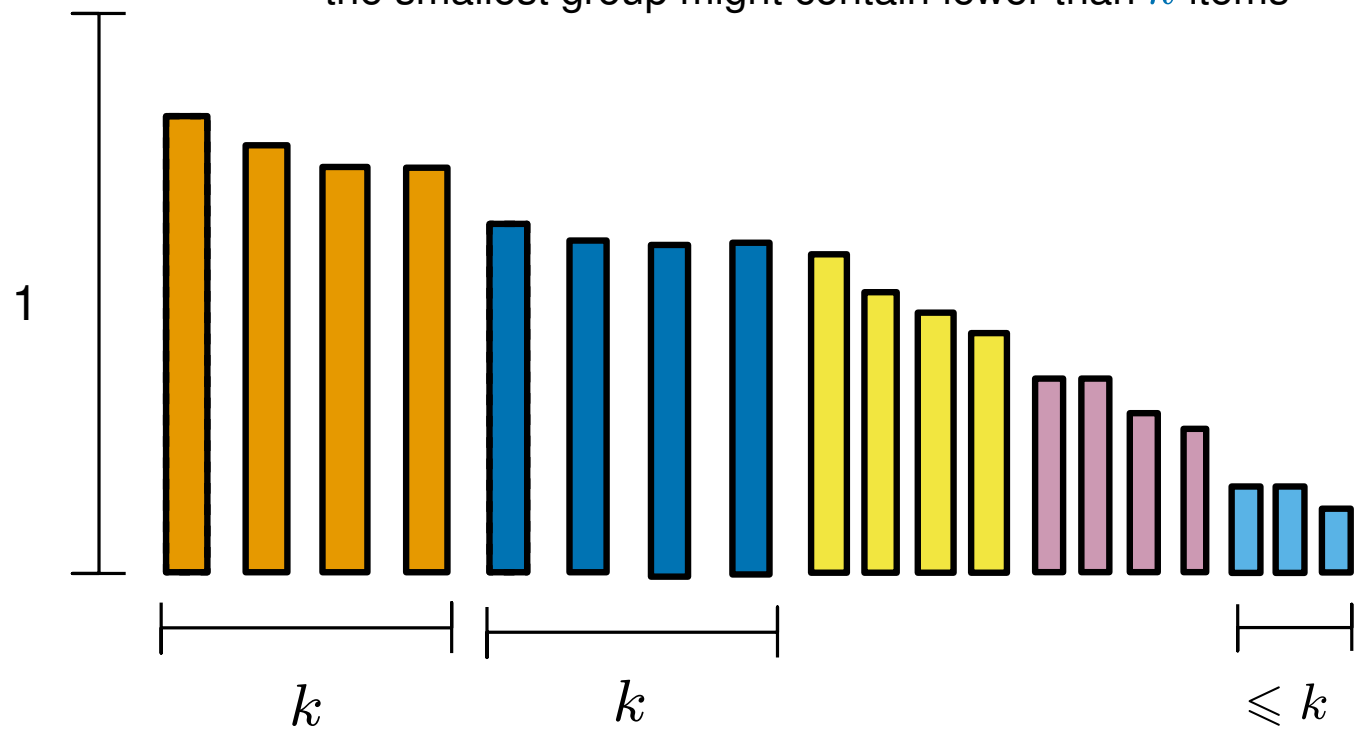

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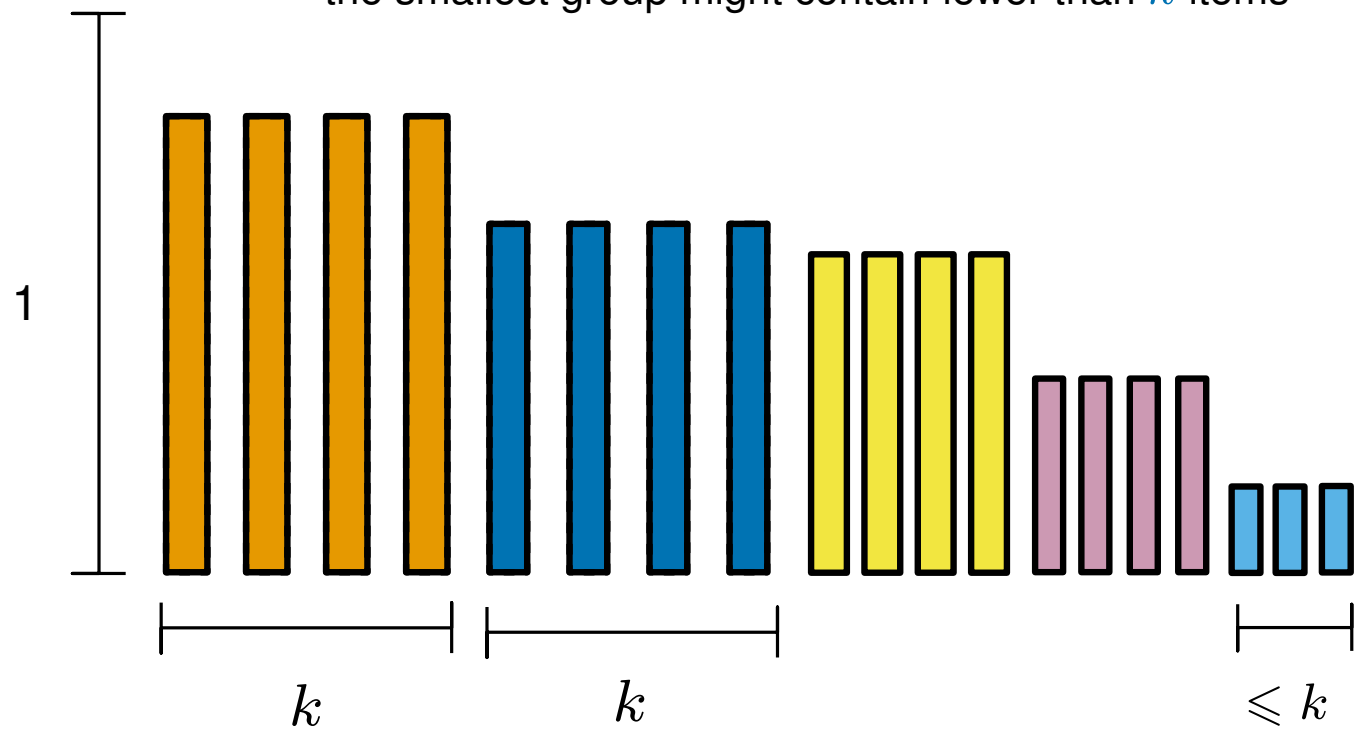

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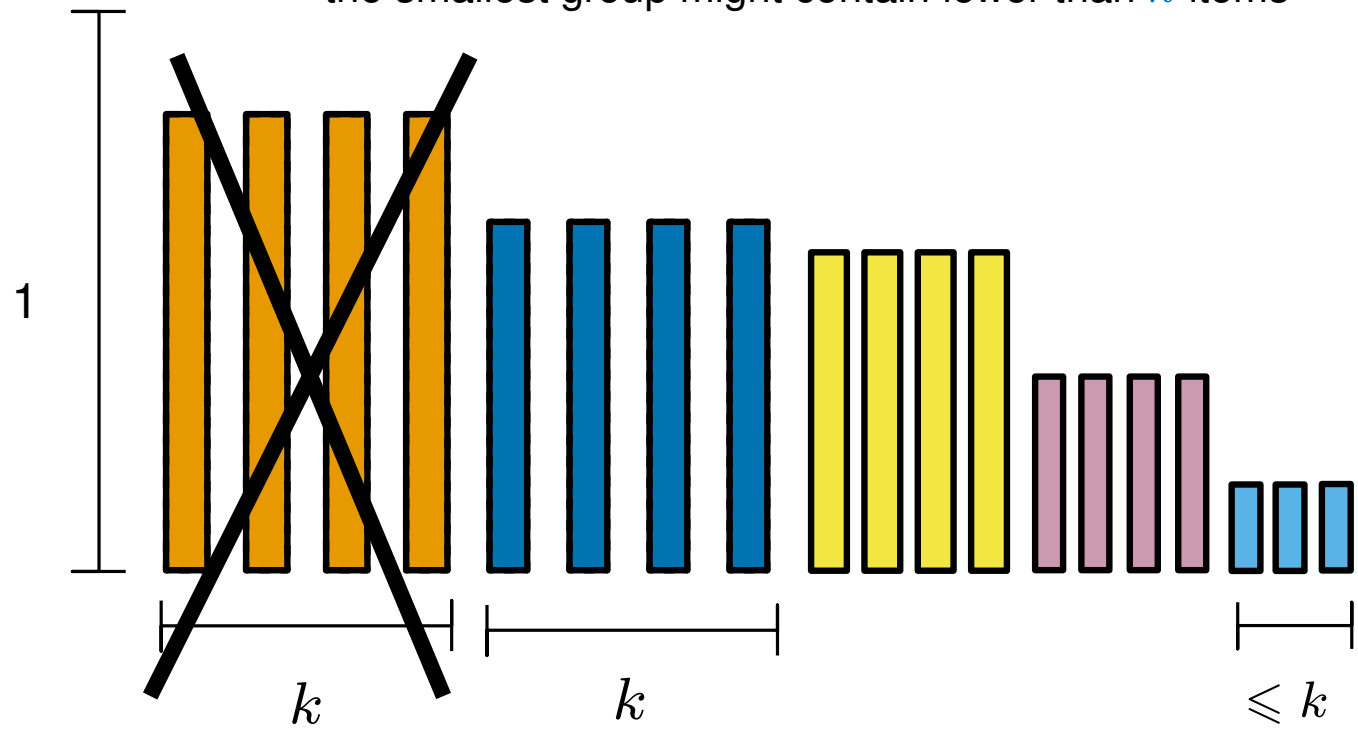

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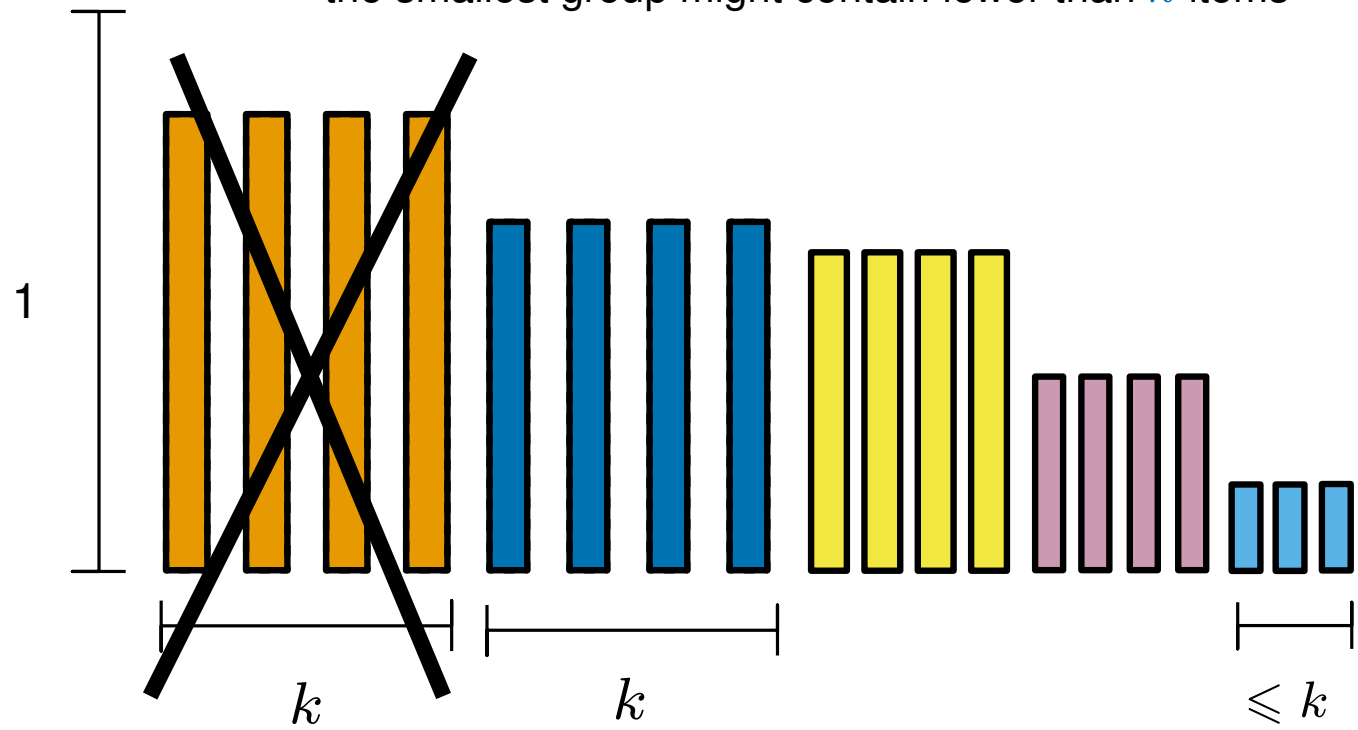

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- Notice that S' contains only n/k distinct item sizes

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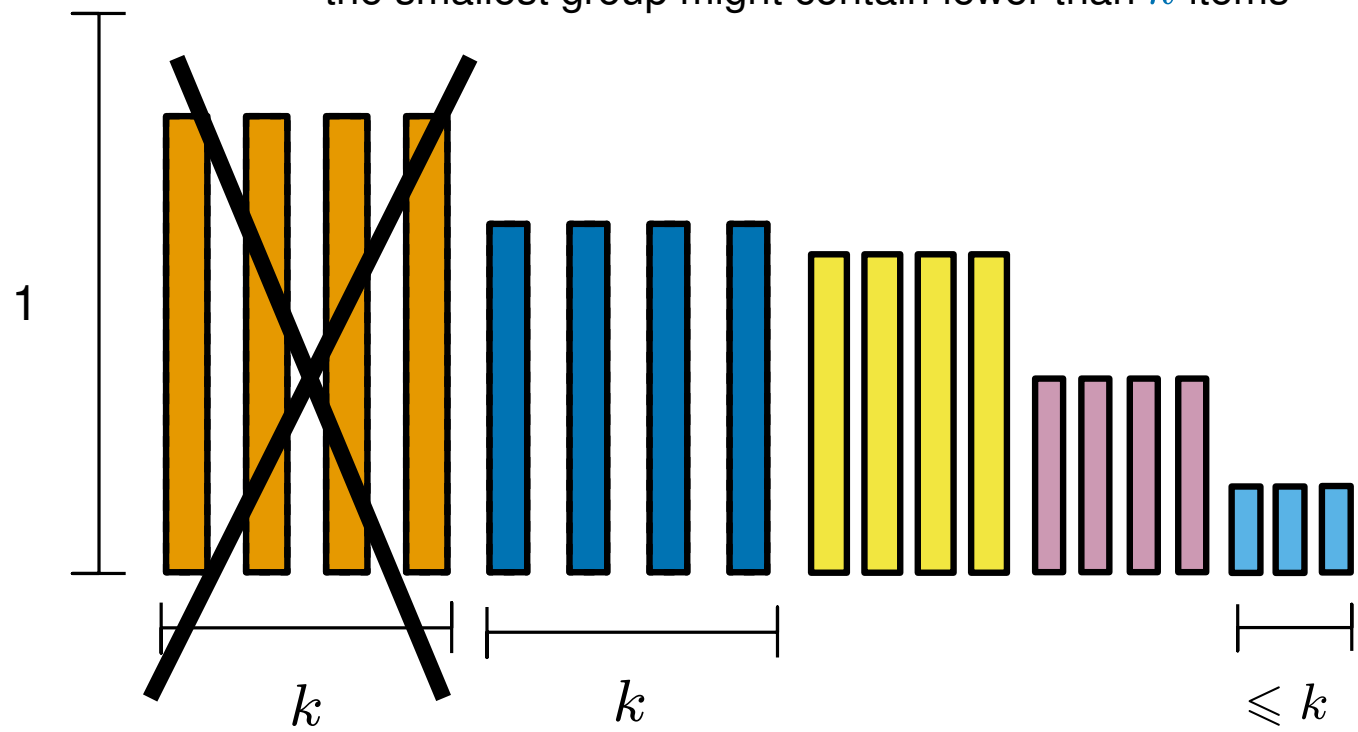

- We define a new set of items S' where each item is rounded up

  so each item in a group has the same size

  and the largest group is removed

- Notice that S' contains only n/k distinct item sizes

  (k will be big enough so that $n/k \leqslant 4/\epsilon^2$ - which is a constant)

Reducing the number of item sizes

Lemma Let S' be S after linear grouping (with groups of size k).

$$\mathrm{Opt}(S') \leqslant \mathrm{Opt}(S) \leqslant \mathrm{Opt}(S') + k\,.$$

. Further, any packing of S' can be converted into a packing of S in polynomial time using at most k extra bins

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Here $\mathrm{Opt}(S)$ is the optimal number of bins required to pack S

Similarly $\mathrm{Opt}(S')$ is the optimal number of bins required to pack S'

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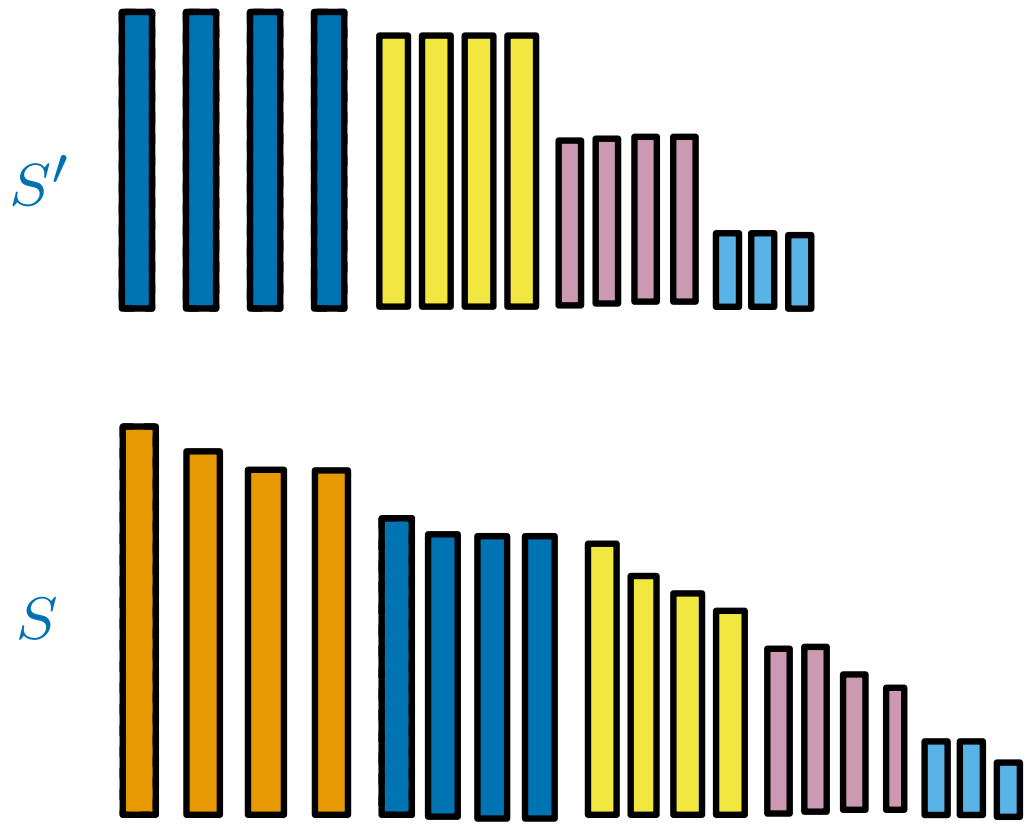

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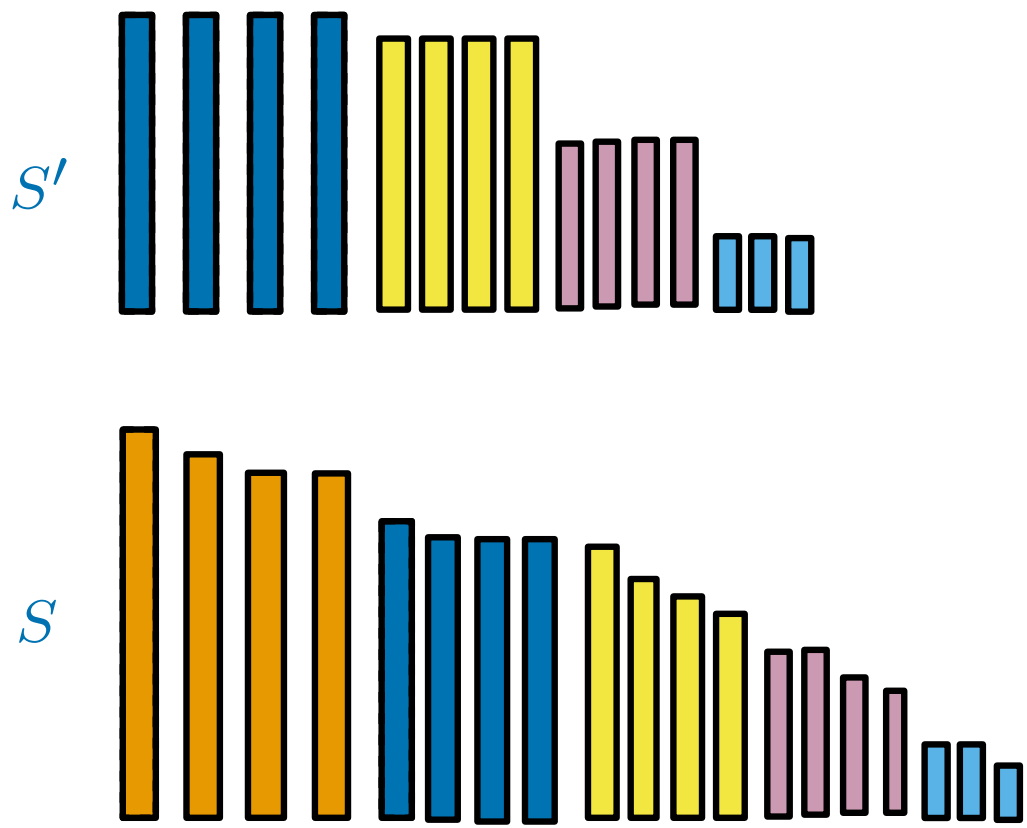

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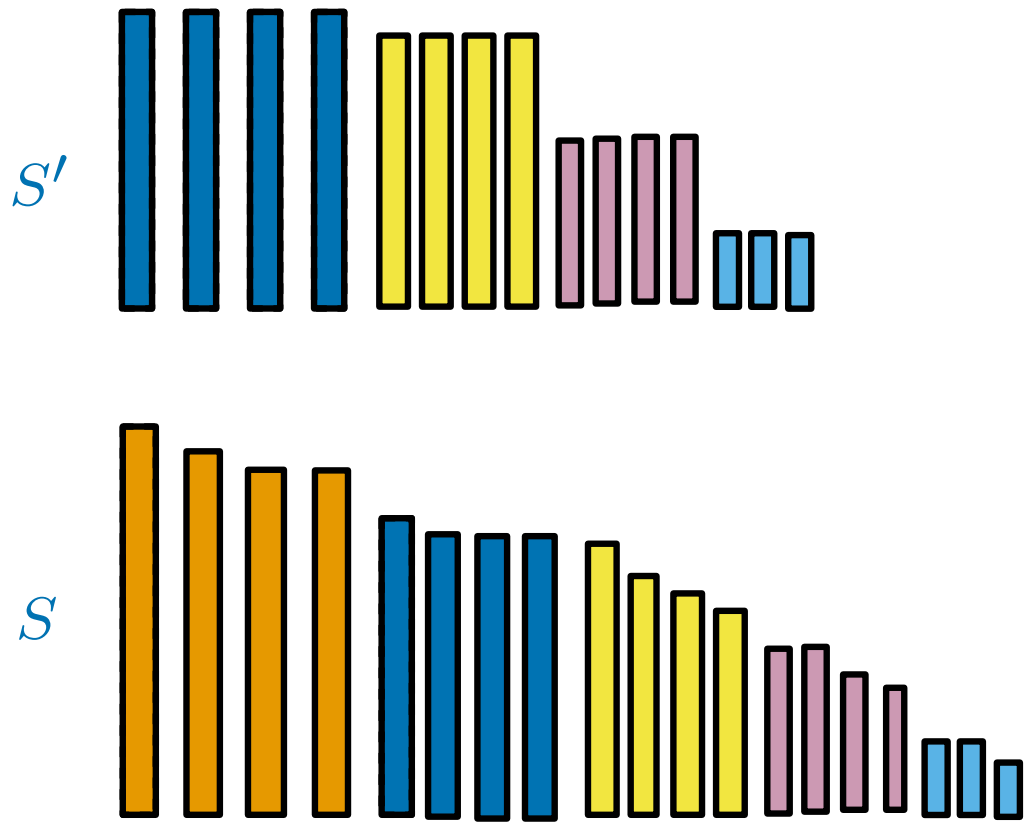

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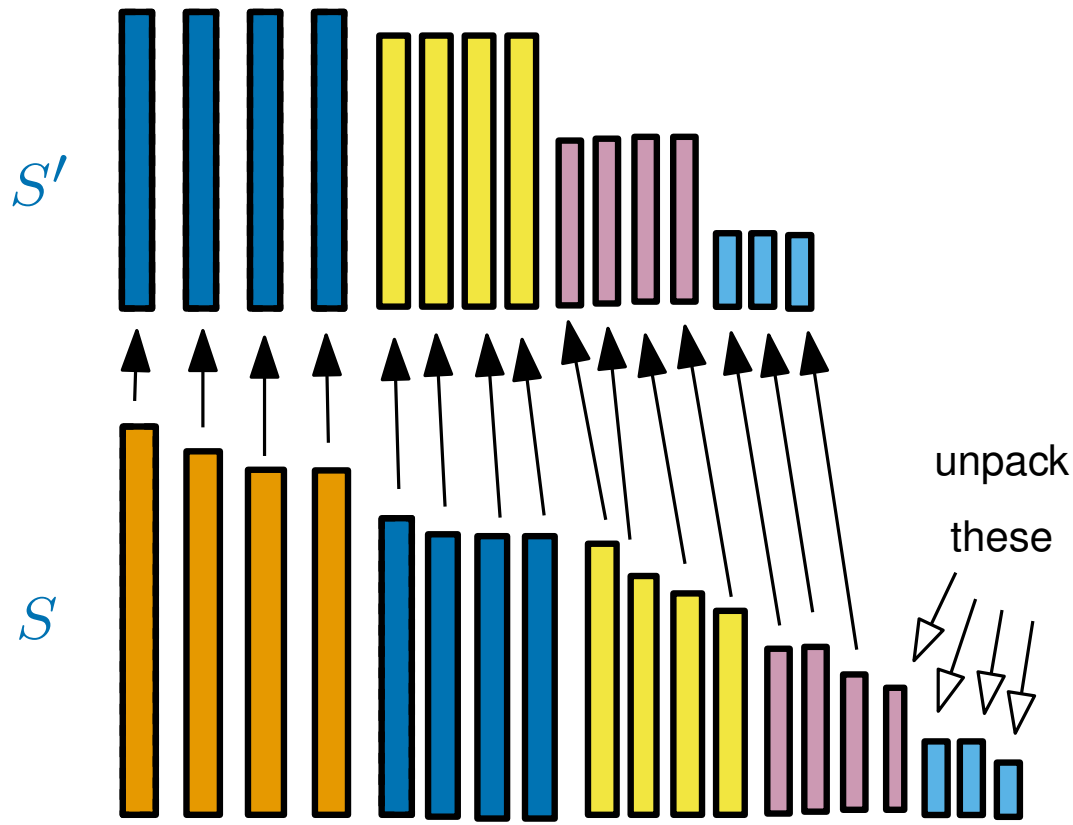
Take the packing of S

and replace each item as shown



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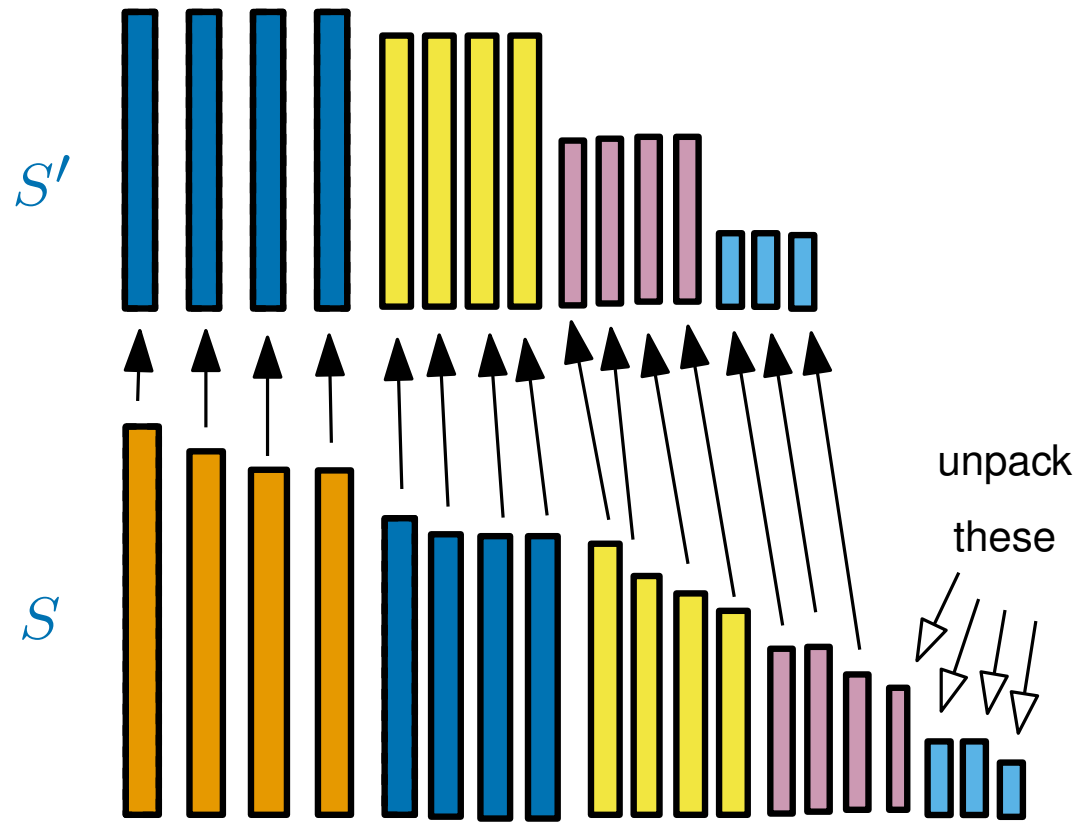
Each item from S is replaced

with one no larger from S'



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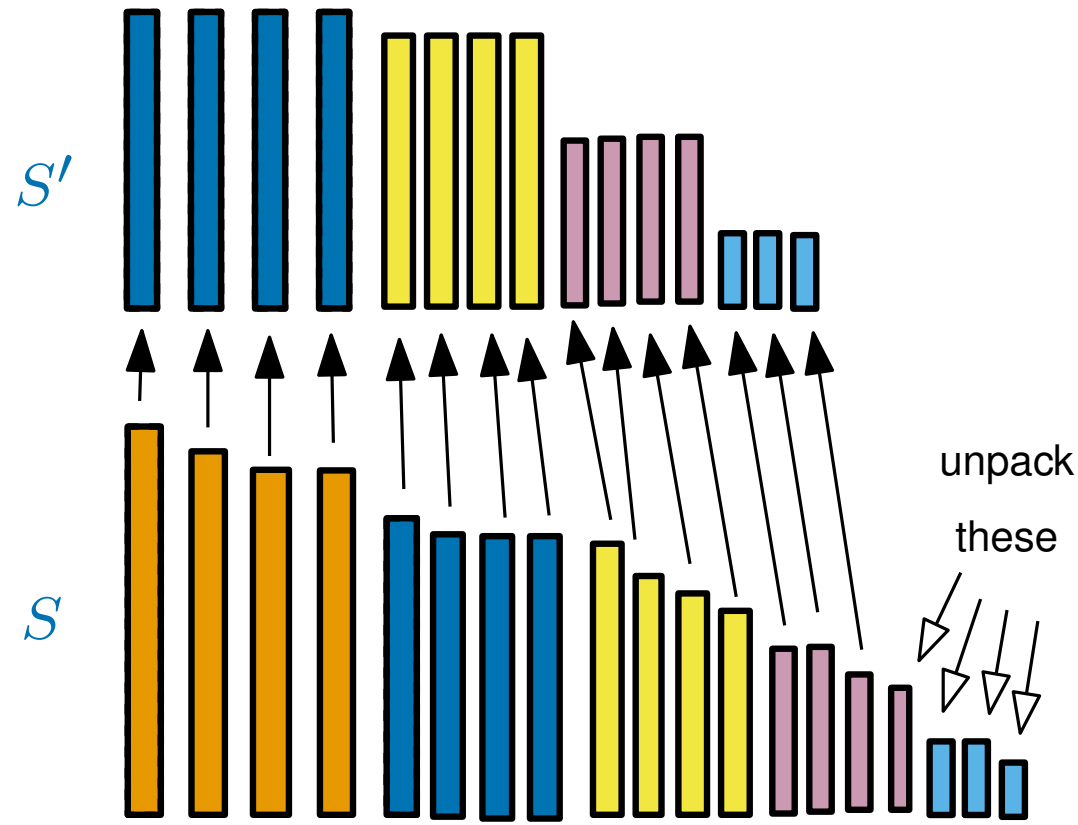
So the packing is valid and hence

$$\mathrm{Opt}(S') \leqslant \mathrm{Opt}(S)$$



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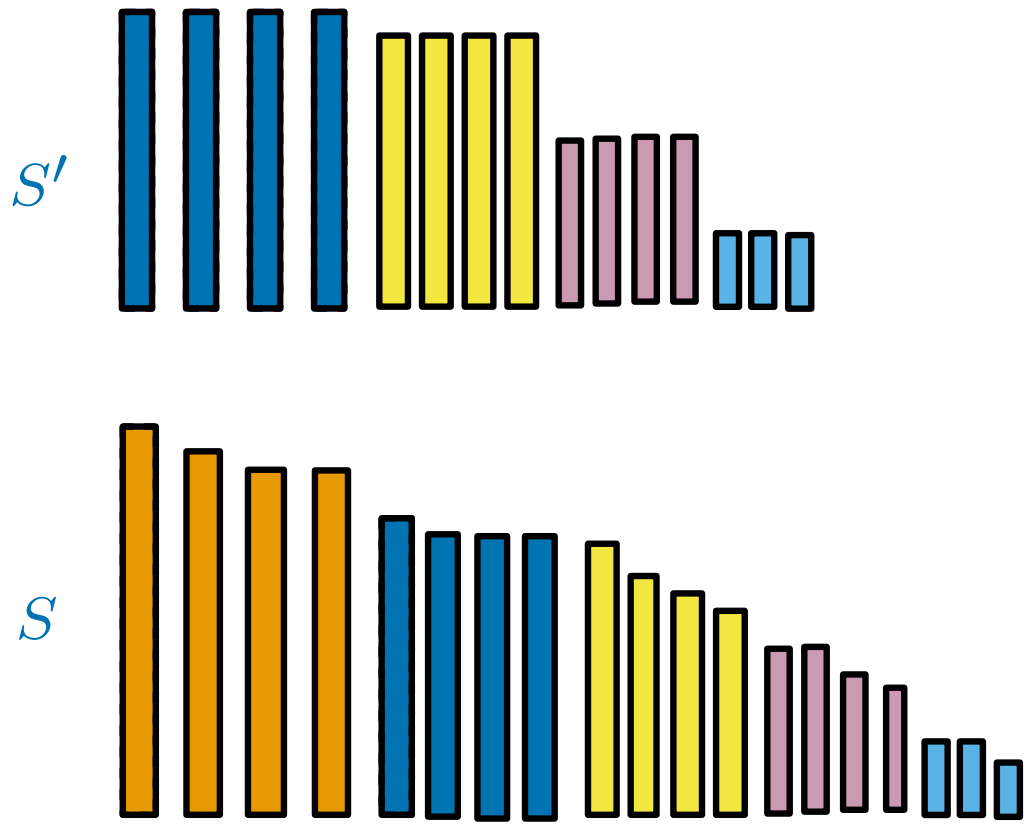
you can pack S into $b + k$ bins



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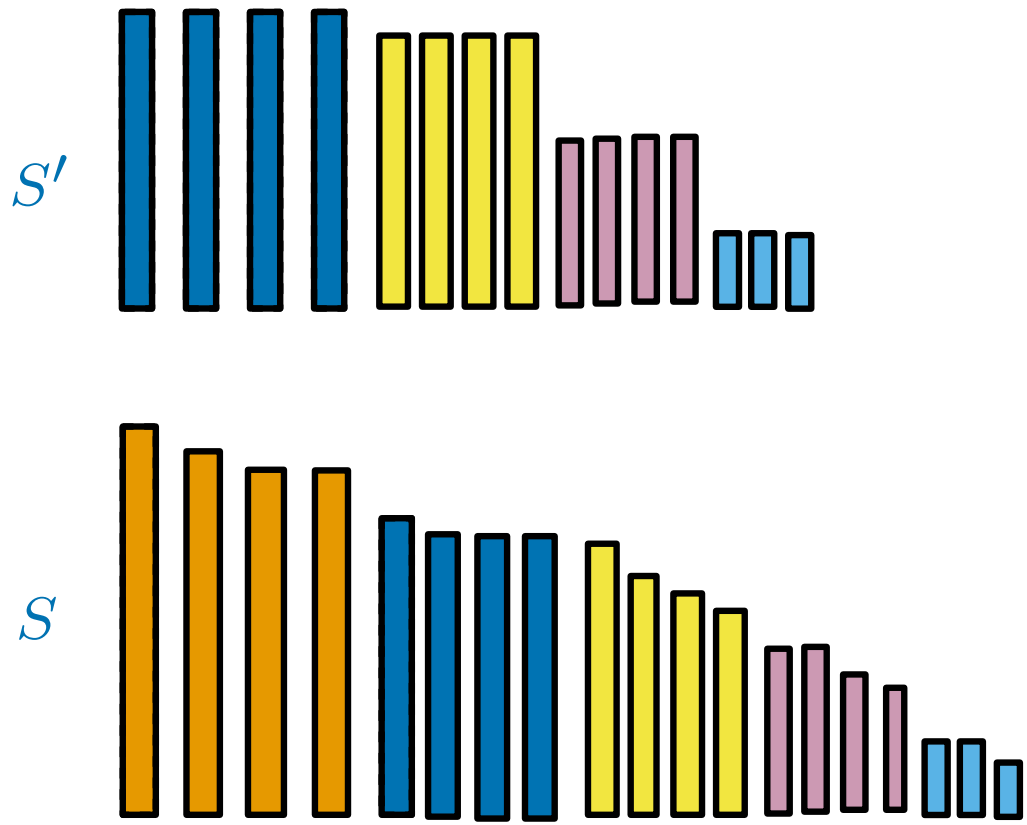

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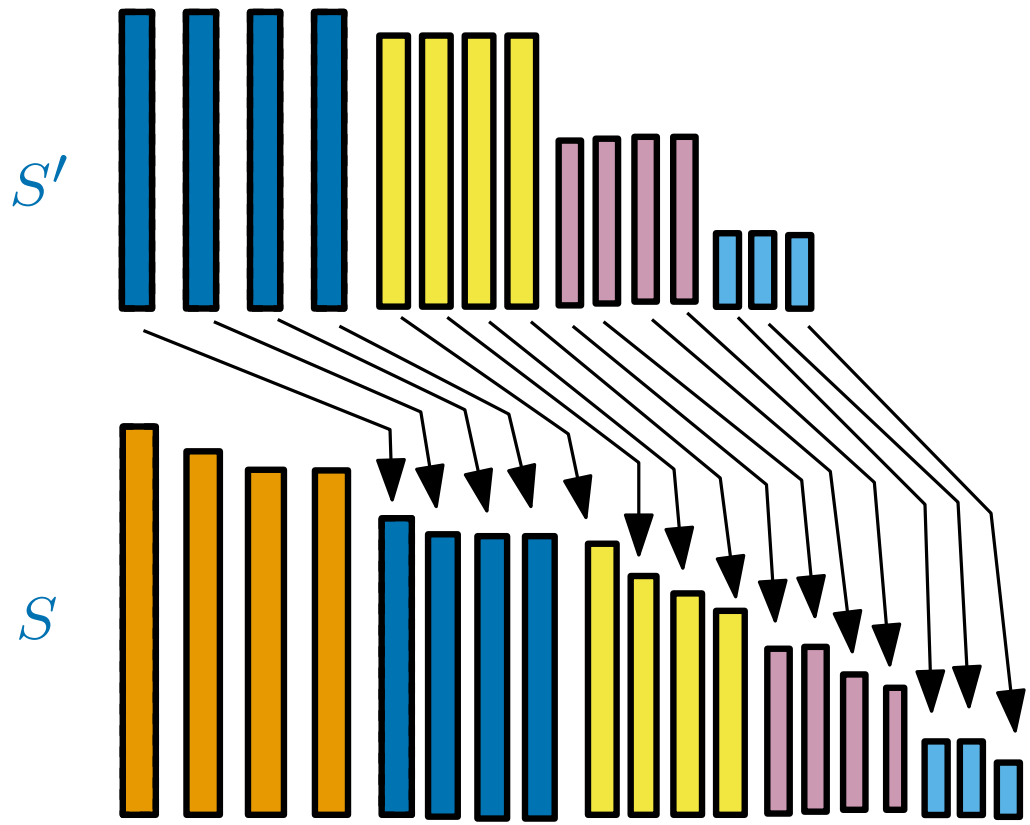

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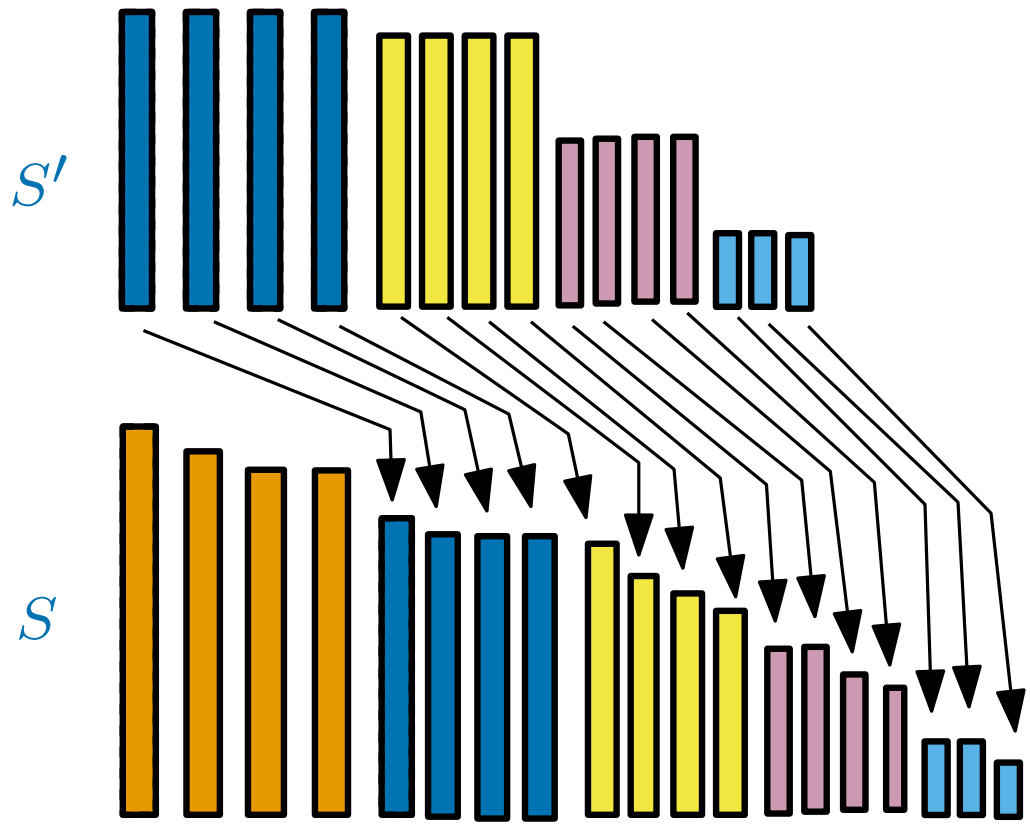

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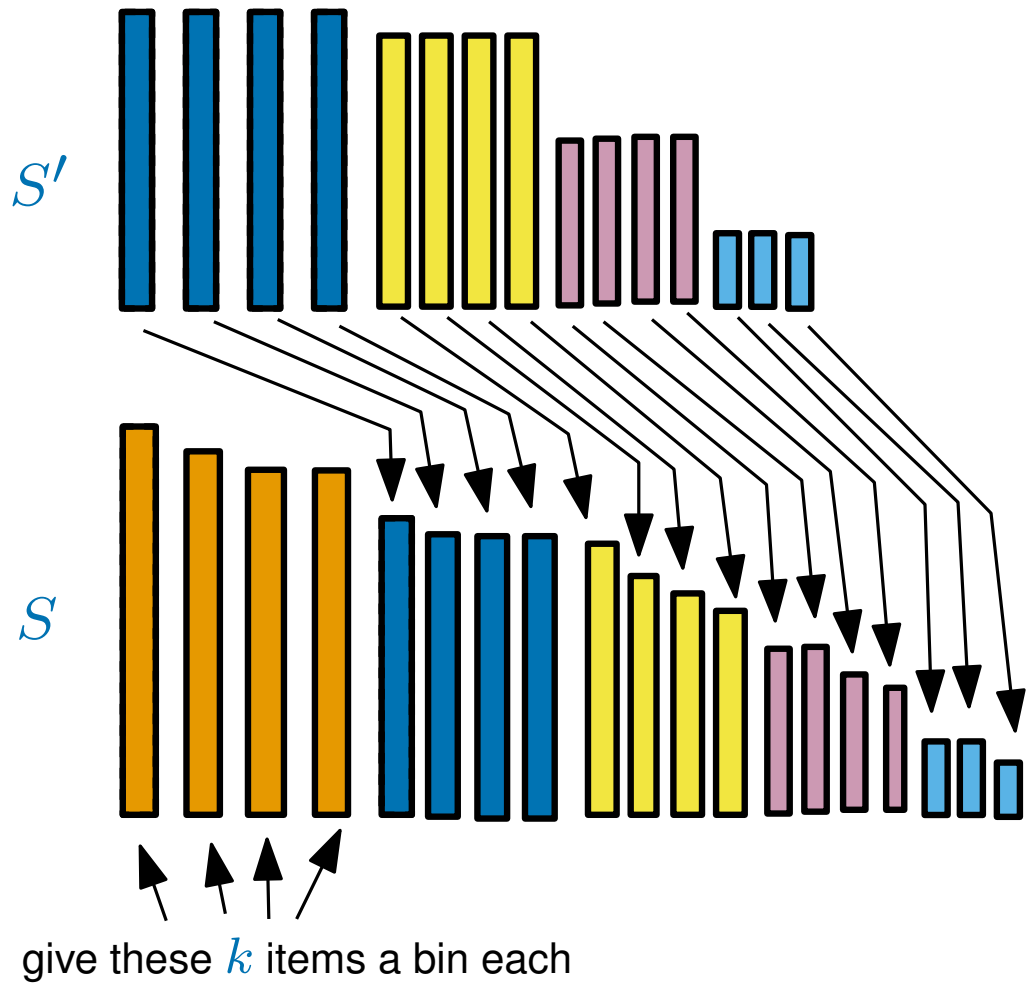
The k largest items are

given their own extra bins



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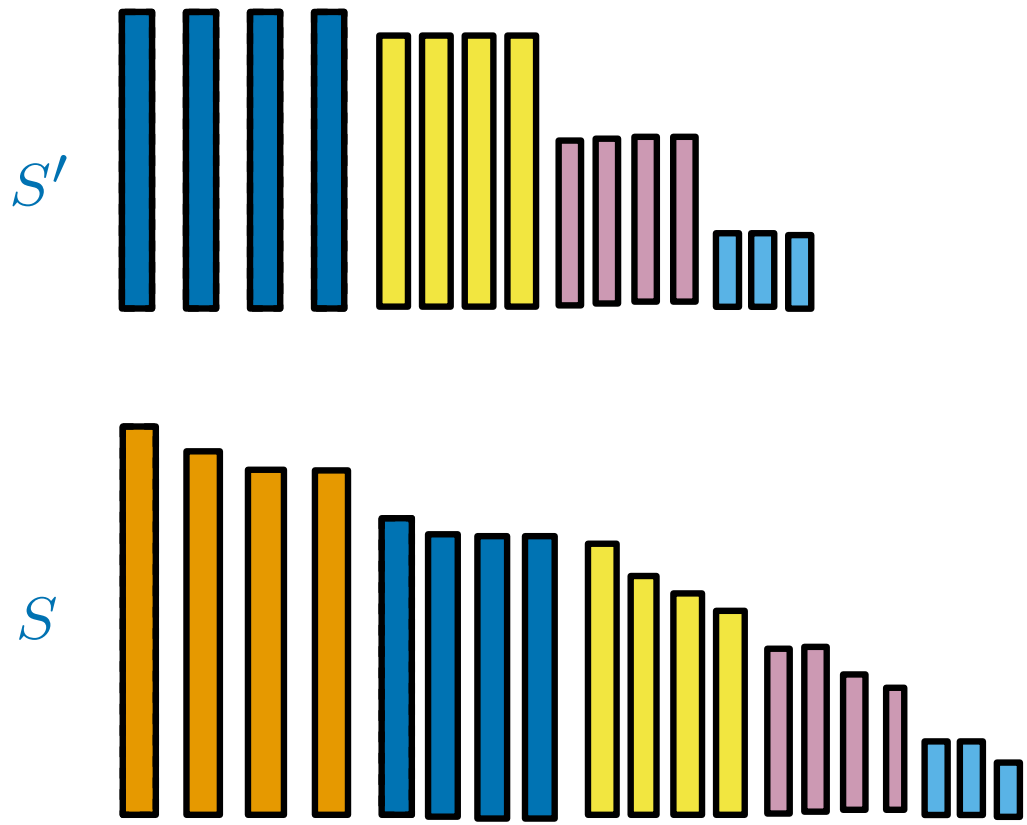

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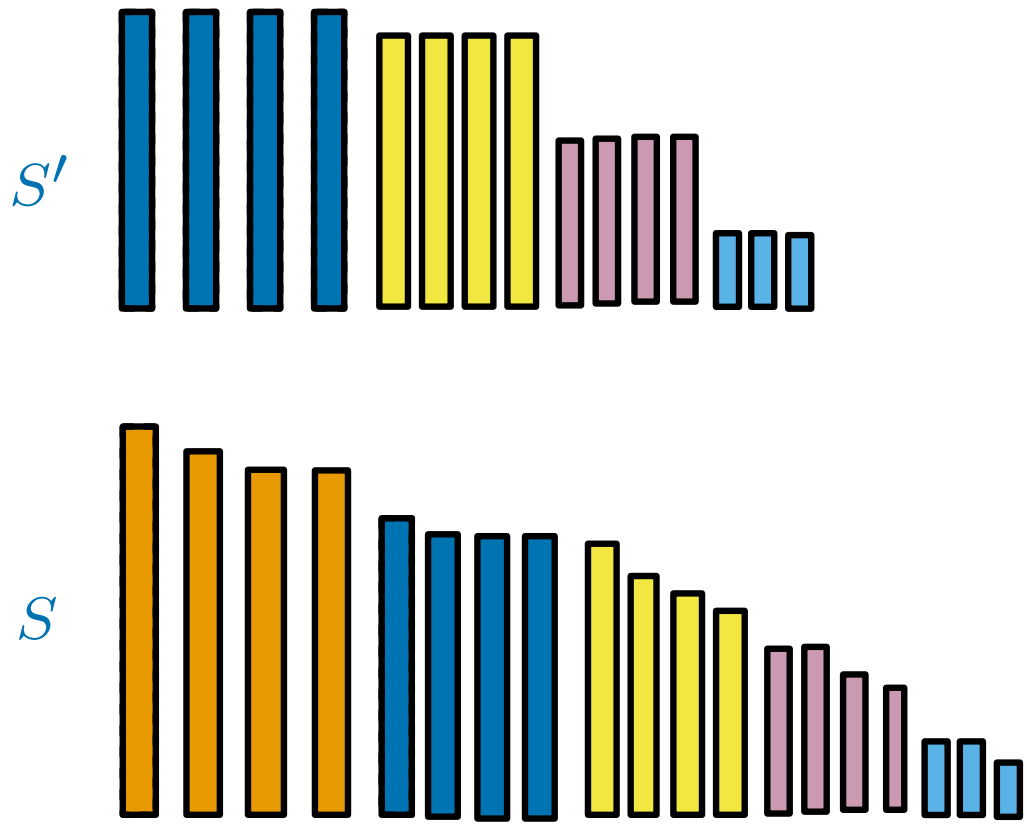
This gives a valid packing of S



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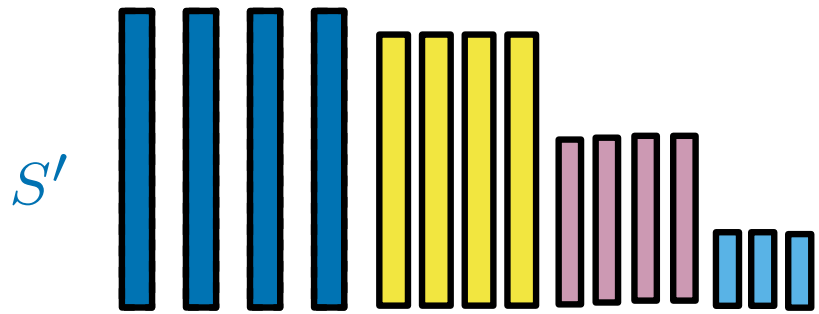

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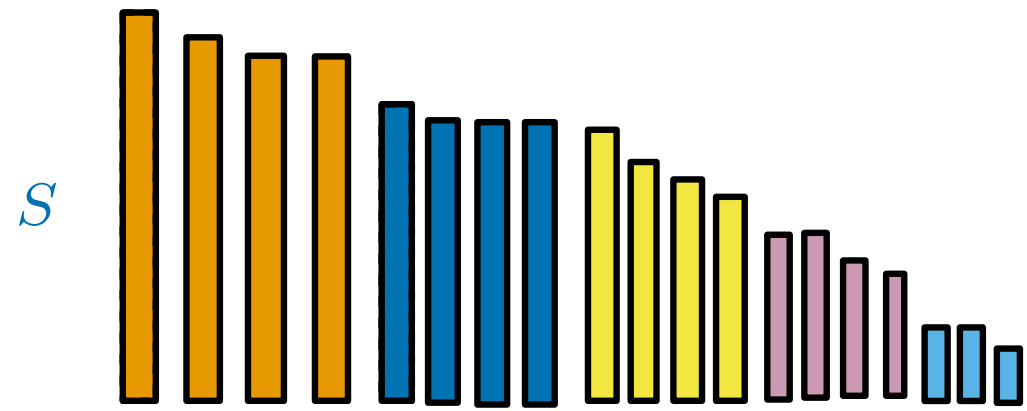
Hence, $\mathrm{Opt}(S) \leqslant \mathrm{Opt}(S') + k$



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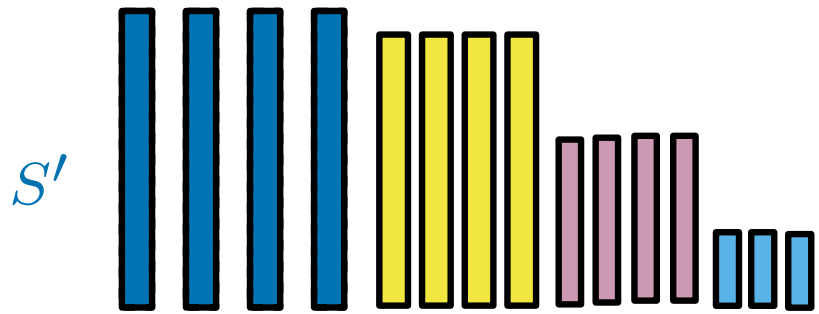

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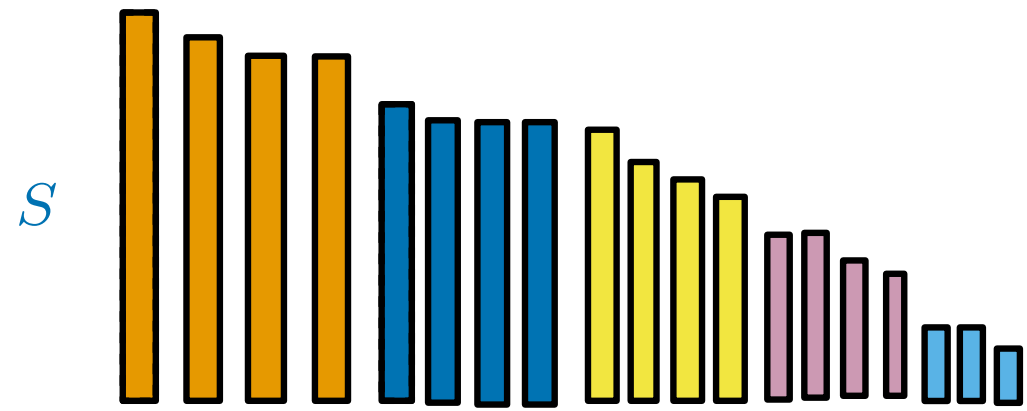
Note that both transformations
take polynomial time



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$n = |S|$

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We set $k = \lfloor n \cdot (\epsilon^2/2) \rfloor$ and let S be the set of large items which implies that...

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$$k \leqslant \epsilon \cdot \mathrm{Opt}(S)$$

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If we can find the optimal packing of S', which uses $\mathrm{Opt}(S')$ bins

we can convert it into a packing of S which uses

$\mathrm{Opt}(S') + k \leqslant (1+\epsilon)\mathrm{Opt}(S)$ bins

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S' contains $4/\epsilon^2$ distinct item sizes

and only $2/\epsilon$ items fit in each bin...

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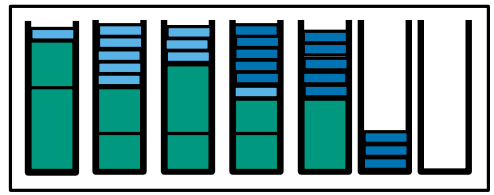
S' contains $4/\epsilon^2$ distinct item sizes

and only $2/\epsilon$ items fit in each bin...

I.e. we *can* optimally pack S' in polynomial time...

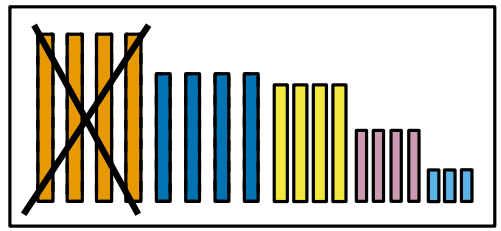
The overall APTAS

Step 1 Remove all the *small* items



Only $c_b = 2/\epsilon$ of the remaining large items will fit into a single bin

Step 2 Divide the items into $k = \lfloor n \cdot (\epsilon^2/2) \rfloor$ different groups



Sizes of items in each group are then rounded up

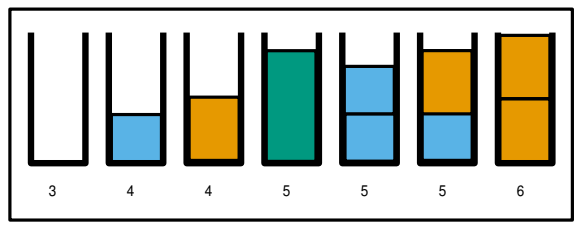
to match the size of the largest member

(and the largest group is removed)

This will leave only $c_s = 4/\epsilon^2$ different item sizes

Step 3 Use the poly-time algorithm to optimally pack the remaining special case

This takes $O\left(n \cdot (n+1)^{(4/\epsilon^2+1)^{2/\epsilon}}\right)$ time



Step 4 Reverse the grouping from **Step 2** and then

greedily pack all the small items removed in **Step 1**

Theorem For any $0 < \epsilon < 1$, the algorithm presented runs in polynomial time and returns a packing of any set of items using at most $(1+\epsilon)\mathrm{Opt} + 1$ bins

Conclusions

- There is no α-approximation for BinPacking with $\alpha < 3/2$ unless $\mathrm{P} = \mathrm{NP}$
- This in turn implies that there is no PTAS for BinPacking unless $\mathrm{P} = \mathrm{NP}$
- The First Fit Decreasing algorithm uses at most,

$$\frac{11}{9} \cdot \mathrm{Opt} + 1 \text{ bins}$$

- We saw an APTAS for BinPacking. (which uses at most $(1+\epsilon) \cdot \mathrm{Opt} + 1$ bins)

there is also an AFPTAS for bin packing

- There is a poly-time algorithm which outputs a solution using at most,

$$\mathrm{Opt} + O(\log^2 \mathrm{Opt}) = \left(1 + \frac{O(\log^2 \mathrm{Opt})}{\mathrm{Opt}}\right) \cdot \mathrm{Opt} \text{ bins}$$

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$$\mathrm{Opt} + O(\log^2 \mathrm{Opt}) = \left(1 + \frac{O(\log^2 \mathrm{Opt})}{\mathrm{Opt}}\right) \cdot \mathrm{Opt} \text{ bins}$$

Whether there is a poly-time algorithm which uses at most $\mathrm{Opt} + 1$ bins is an open problem