Advanced Algorithms – COMS31900

Hashing part one

Chaining, true randomness and universal hashing

Raphaël Clifford

Slides by Benjamin Sach and Markus Jalsenius

Dictionaries

In a **dictionary** data structure we store $(\textit{key}, \textit{value})$-pairs

such that for any *key* there is at most one pair $(\textit{key}, \textit{value})$ in the dictionary.

Often we want to perform the following three operations:

add(x, v)	Add the the pair (x, v).
lookup(x)	Return v if (x, v) is in dictionary, or NULL otherwise.
delete(x)	Remove pair (x, v) (assuming (x, v) is in dictionary).

There are many data structures that will do this job, e.g.:

- Linked lists
- Binary search trees
- (2,3,4)-trees
- Red-black trees
- Skip lists
- van Emde Boas trees (later in this course)

these data structures all support extra operations beyond the three above

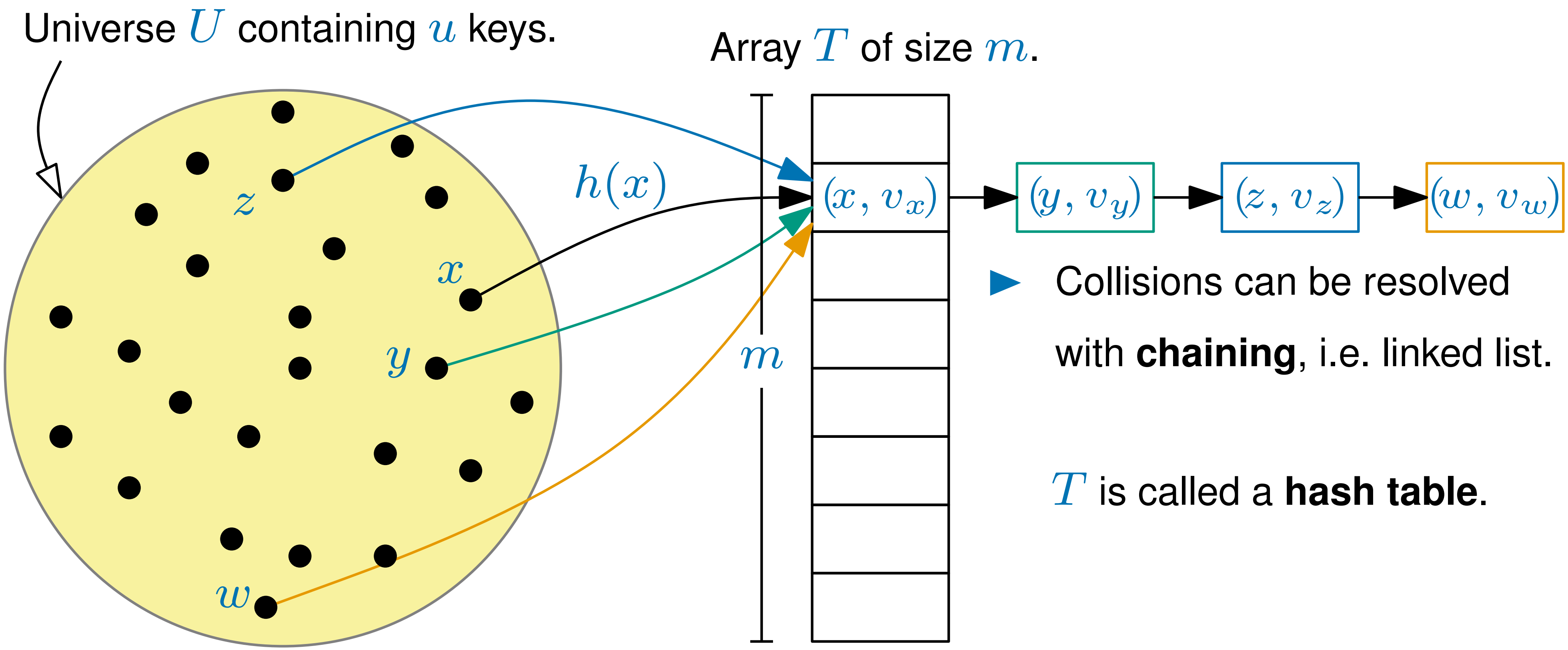
but none of them take $O(1)$ worst case time for all operations...

so *maybe* there is room for improvement?

Hash tables

We want to store n elements from the universe, U in a dictionary.

Typically $u = |U|$ *is much, much larger than* n.



- Collisions can be resolved with **chaining**, i.e. linked list.

T is called a **hash table**.

A **hash function** $h : U \to [m]$ maps a key to a position in T.

We write $[m]$ to denote the set $\{0, \ldots, m-1\}$.

We want to avoid **collisions**, i.e. $h(x) = h(y)$ for $x \neq y$.

Time complexity

We cannot avoid collisions entirely since $u \gg m$;

some keys from the universe are bound to be mapped to the same position.

(remember u is the size of the universe and m is the size of the table)

By building a hash table with chaining, we get the following time complexities:

Operation	Worst case time	Comment
add(x, v)	$O(1)$	Simply add item to the list link if necessary.
lookup(x)	$O($length of chain containing $x)$	We might have to search through the whole list containing x.
delete(x)	$O($length of chain containing $x)$	Only $O(1)$ to perform the actual delete... but you have to find x first

So how long are these chains?

True randomness

THEOREM

Consider any n fixed inputs to the hash table *(which has size m)*,

i.e. any sequence of n add/lookup/delete operations.

Pick h uniformly at random from the set of *all* functions $U \to [m]$.

The expected run-time per operation is $O(1 + \frac{n}{m})$, or simply $O(1)$ if $m \geqslant n$.

PROOF

Let x, y be two distinct keys from U.

iff means *if and only if.*

Let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.

we have that, $$\Pr\left(h(x) = h(y)\right) = \frac{1}{m}$$

this is because $h(x)$ and $h(y)$ are chosen uniformly and independently from $[m]$.

Therefore, $\mathbb{E}(I_{x,y}) = \Pr(I_{x,y} = 1) = \Pr\left(h(x) = h(y)\right) = \frac{1}{m}$.

We have that, $\mathbb{E}(I_{x,y}) = \frac{1}{m}$.

True randomness

Theorem

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Proof

Let x, y be two distinct keys from U.

iff means if and only if.

Let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.

We have that, $\mathbb{E}(I_{x,y}) = \frac{1}{m}$.

Let N_x be the number of keys stored in T that are hashed to $h(x)$

so, in the worst case it takes N_x time to look up x in T.

Observe that $N_x = \sum_{y \in T} I_{x,y}$

the keys in T

Finally, we have that $\mathbb{E}(N_x) = \mathbb{E}\left(\sum_{y \in T} I_{x,y}\right) = \sum_{y \in T} \mathbb{E}(I_{x,y}) = n \cdot \frac{1}{m} = \frac{n}{m}$

linearity of expectation.

Specifying the hash function

Problem: how do we specify an *arbitrary* (e.g. a truly random) hash function?

For each key in U we need to specify an arbitrary position in T,

this is a number in $[m]$, so requires $\approx \log_2 m$ bits.

So in total we need $\approx u \log_2 m$ bits, which is a ridiculous amount of space!

(in particular, it's much bigger than the table :s)

Why not pick the hash function as we go?

Couldn't we generate $h(x)$ when we first see x?

Wouldn't we only use $n \log_2 m$ bits? (one per key we actually store)

The problem with this approach is recalling $h(x)$ *the next time we see* x

Essentially we'd need to build a dictionary to solve the dictionary problem!

This has become rather cyclic... let's try something else!

Specifying the hash function

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Instead, we define a set, or *family of hash functions*: $H = \{h_1, h_2, \dots\}$.

As part of initialising the hash table,

we choose the hash function h from H randomly.

How should we specify the hash functions in H and how do we pick one at random?

Weakly universal hashing

- A set H of hash functions is **weakly universal** if for any two distinct keys $x, y \in U$,

$$\Pr\left(h(x) = h(y)\right) \leqslant \frac{1}{m}$$

where h is chosen uniformly at random from H.

Observe

The randomness here comes from the fact that h is picked randomly.

Theorem

Consider any n fixed inputs to the hash table *(which has size m)*,

i.e. any sequence of n add/lookup/delete operations.

Pick h uniformly at random from a weakly universal set H of hash functions.

The expected run-time per operation is $O(1)$ if $m \geqslant n$.

Proof

The proof we used for true randomness works here too (which is nice)

Constructing a weakly universal family of hash functions

- Suppose $U = [u]$, i.e. the keys in the universe are integers 0 to $u-1$.
- Let p be any prime bigger than u.
- For $a, b \in [p]$, let

$$h_{a,b}(x) = ((ax + b) \bmod p) \bmod m,$$

$$H_{p,m} = \{h_{a,b} \mid a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\}.$$

Theorem

$H_{p,m}$ is a weakly universal set of hash functions.

Proof

See CLRS, Theorem 11.5, (page 267 in 3rd edition).

Observe

- $ax + b$ is a linear transformation which "spreads the keys" over p values when taken modulo p. This does not cause any collisions.
- Only when taken modulo m do we get collisions.

True randomness vs. weakly universal hashing

For both,

true randomness

(h is picked uniformly from the set of all possible hash functions)

and **weakly universal hashing**

(h is picked uniformly from a weakly universal set of hash functions)

we have seen that when $m \geqslant n$,

the expected lookup time in the hash table is $O(1)$.

Since constructing a weakly universal set of hash functions seems much easier than obtaining true randomness, this is all good news!

isn't it?

What about the length of the *longest* chain? (the longest linked list)

If it is very long, some lookups could take a very long time...

Longest chain – true randomness

Lemma

If h is selected uniformly at random from all functions $U \to [m]$ then, over m fixed inputs,

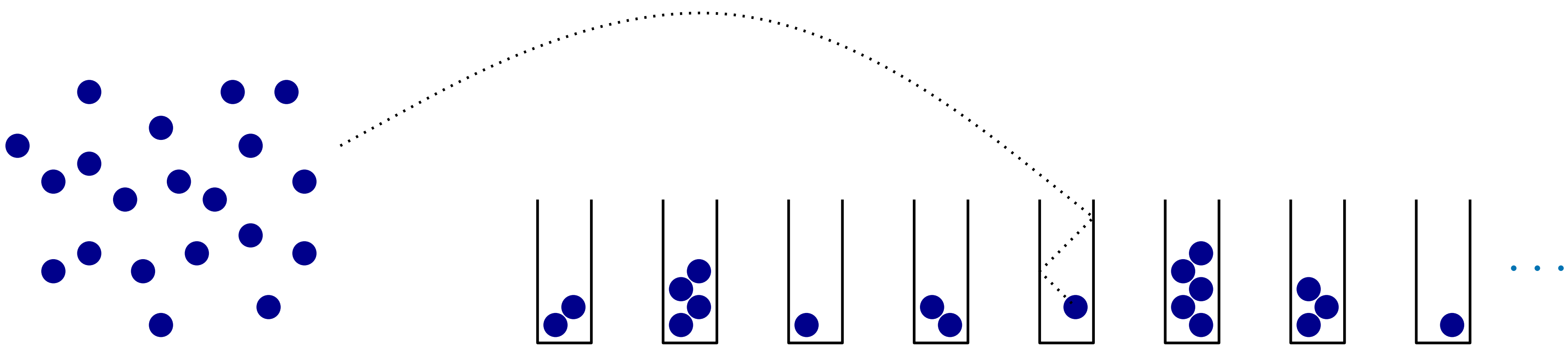
$$\Pr(\text{any chain has length} \geqslant 3\log m) \leqslant \frac{1}{m}.$$

Observe

In this lemma we insert m keys, i.e. $n = m$.

Proof

The problem is equivalent to showing that if we randomly throw m balls into m bins, the probability of having a bin with at least $3\log m$ balls is at most $\frac{1}{m}$.



Longest chain – true randomness

PROOF

continued...

Let X_1 be the number of balls in the first bin.

Choose any k of the m balls (we'll pick k in a bit)

the probability that all of these k balls go into the first bin is $\frac{1}{m^k}$.

So, the union bound gives us

Number of subsets of size k.

$$\Pr(X_1 \geqslant k) \leqslant \binom{m}{k} \cdot \frac{1}{m^k} \leqslant \frac{1}{k!}.$$

THEOREM

Let $V_1, \ldots, V_q$ be q events. Then

$$\Pr\left(\bigcup_{i=1}^{q} V_i\right) \leqslant \sum_{i=1}^{q} \Pr(V_i).$$

Longest chain – true randomness

PROOF

continued...

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$$\Pr(X_1 \geqslant k) \leqslant \binom{m}{k} \cdot \frac{1}{m^k} \leqslant \frac{1}{k!}.$$

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m \cdot (m-1) \cdot (m-2) \cdot \ldots (m-k+1) \cdot \cancel{(m-k)!}}{k!\cancel{(m-k)!}}$$

$$\leqslant \frac{\overbrace{m \cdot (m) \cdot (m) \cdot \ldots (m)}^{k}}{k!} \leqslant \frac{m^k}{k!}$$

Longest chain – true randomness

PROOF

continued...

Let X_1 be the number of balls in the first bin.

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the probability that all of these k balls go into the first bin is $\frac{1}{m^k}$.

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Number of subsets of size k.

$$\Pr(X_1 \geqslant k) \leqslant \binom{m}{k} \cdot \frac{1}{m^k} \leqslant \frac{1}{k!}.$$

By using the union bound *again*, we have that

$$\Pr(\text{at least one bin receives at least } k \text{ balls}) \leqslant m \cdot \Pr(X_1 \geqslant k) \leqslant \frac{m}{k!}.$$

- Now we set $k = 3 \log m$ and observe that $\frac{m}{k!} \leqslant \frac{1}{m}$ for $m \geqslant 2$,

and we are done.

Longest chain – true randomness

Proof

continued...

Let X_1 be the number of balls in the first bin.

Choose any k of the m balls (we'll pick k in a bit)

So, th

By us

$\Pr$ ... $\frac{m}{k!}$.

Why is $\frac{m}{k!} \leqslant \frac{1}{m}$? (when $k = 3\log m$)

$$k! = \overbrace{k \times (k-1) \times (k-2) \ldots \times 2 \times 1}^{k \text{ terms}}$$

$$k! > 2 \times 2 \times 2 \ldots \times 2 \times 1 = 2^{k-1}$$

Let $k = 3\log m \ldots$

$$k! > 2^{(3\log m - 1)} \geqslant 2^{2\log m} = (2^{\log m})^2 = m^2$$

so $\frac{m}{k!} \leqslant \frac{m}{m^2} = \frac{1}{m}$

- Now we set $k = 3\log m$ and observe that $\frac{m}{k!} \leqslant \frac{1}{m}$ for $m \geqslant 2$,

and we are done.

Longest chain – true randomness

Lemma

If h is selected uniformly at random from all functions $U \to [m]$ then,
over m fixed inputs,

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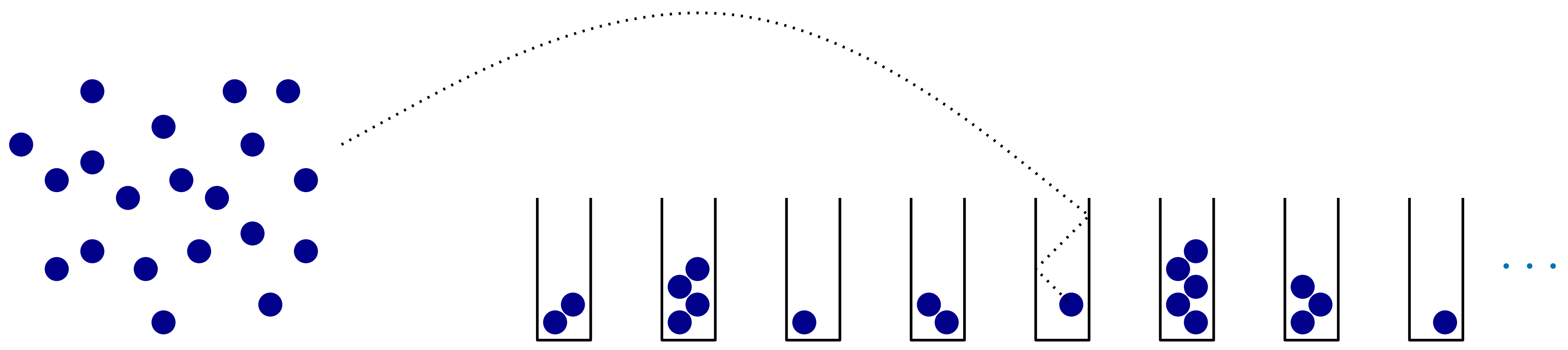
Observe

In this lemma we insert m keys, i.e. $n = m$.

Proof

The problem is equivalent to showing that if we randomly throw m balls into m bins, the probability of having a bin with at least $3\log m$ balls is at most $\frac{1}{m}$.



Longest chain – weakly universal hashing

The conclusion from previous slides is that with true randomness, the longest chain is very short (at most $3 \log m$) with high probability.

LEMMA

If h is picked uniformly at random from a weakly universal set of hash functions then, over m fixed inputs,

$$\Pr\left(\text{any chain has length } \geqslant 1 + \sqrt{2m}\right) \leqslant \frac{1}{2}.$$

OBSERVE

This rubbish upper bound of $\frac{1}{2}$ does not necessarily rule out the possibility that the *tightest* upper bound is indeed very small. However, the upper bound of $\frac{1}{2}$ is in fact tight!

Longest chain – weakly universal hashing

PROOF

- For any two keys x, y, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. C be the total number of collisions: $C = \sum_{x,y \in T,\, x<y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),

$$\mathbb{E}(C) = \mathbb{E}\Big(\sum_{x,y \in T,\, x<y} I_{x,y}\Big) = \sum_{x,y \in T,\, x<y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leqslant \frac{m}{2}.$$

- by Markov's inequality, $\Pr(C \geqslant m) \leqslant \frac{\mathbb{E}(C)}{m} \leqslant \frac{1}{2}$.
- Let r.v. L be the length of the longest chain. Then $C \geqslant \binom{L}{2}$.

This is because a chain of length L causes $\binom{L}{2}$ collisions!

Longest chain – weakly universal hashing

PROOF

- For any two keys x, y, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. C be the total number of collisions: $C = \sum_{x,y \in T,\, x<y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),

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- by Markov's inequality, $\Pr(C \geqslant m) \leqslant \frac{\mathbb{E}(C)}{m} \leqslant \frac{1}{2}$.
- Let r.v. L be the length of the longest chain. Then $C \geqslant \binom{L}{2}$.
- Now, $\Pr\left(\frac{(L-1)^2}{2} \geqslant m\right) \leqslant \Pr\left(\binom{L}{2} \geqslant m\right) \leqslant \Pr\left(C \geqslant m\right) \leqslant \frac{1}{2}$.

this is because $\binom{L}{2} = \frac{L!}{2!(L-2)!} = \frac{L\cdot(L-1)}{2} \geqslant \frac{(L-1)^2}{2}$

Longest chain – weakly universal hashing

PROOF

- For any two keys x, y, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. C be the total number of collisions: $C = \sum_{x,y \in T,\, x<y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ (h is weakly universal),

$$\mathbb{E}(C) = \mathbb{E}\Big(\sum_{x,y\in T,\, x<y} I_{x,y}\Big) = \sum_{x,y\in T,\, x<y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leqslant \frac{m}{2}.$$

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- Let r.v. L be the length of the longest chain. Then $C \geqslant \binom{L}{2}$.
- Now, $\Pr\left(\frac{(L-1)^2}{2} \geqslant m\right) \leqslant \Pr\left(\binom{L}{2} \geqslant m\right) \leqslant \Pr\left(C \geqslant m\right) \leqslant \frac{1}{2}$.

By rearranging, we have that $\Pr\left(L \geqslant 1 + \sqrt{2m}\right) \leqslant \frac{1}{2}$, and we are done.

Conclusions

For both,

true randomness (h is picked uniformly from the set of all possible hash functions)

and **weakly universal hashing**

(h is picked uniformly from a weakly universal set of hash functions)

we have seen that when $m \geqslant n$,

the expected lookup time in a hash table with chaining is $O(1)$.

LEMMA

If h is selected uniformly at random from all functions $U \to [m]$ then,

$$\Pr\left(\text{any chain has length } \geqslant 3\log m\right) \leqslant \frac{1}{m}.$$

LEMMA

If h is picked uniformly at random from a weakly universal set of hash functions,

$$\Pr\left(\text{any chain has length } \geqslant 1 + \sqrt{2m}\right) \leqslant \frac{1}{2}.$$

(both Lemmas hold for m any fixed inputs)